$C_X(T)$ has the DP property from Theorem 3 and Theorem 8 of [15]. If T is locally compact, let T^* be the one-point compactification of T with ∞ denoting the point at infinity. Then $C_0(T, X)$ is isometrically isomorphic to the closed subspace Γ of $C_X(T^*)$ consisting of those functions which vanish at ∞. But Γ is complemented in $C_X(T^*)$ via the projection P: $f \to f - f(\infty)$ and $C_X(T^*)$ has the DP property, so Γ, and hence $C_0(T, X)$, has the DP property ([10], 9.4.3).

Remark 5. Partial solutions to this problem were given in [8], [2], and [16]; for the scalar version see [9], VI, 7.4.

It also follows from Theorem 4 that if Z is a complemented subspace of a space $C(S)$, then $Z \otimes_\varepsilon X$ ([14], 7.1.1) has the DP property when X has the DP property for $Z \otimes_\varepsilon X$ is then a complemented subspace of $C(S) \otimes_\varepsilon X = C_X(S)$. This suggests the conjecture that if X and Y have the DP property, then $X \otimes_\varepsilon Y$ also has the DP property.

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On the Vitali covering properties of a differentiation basis

by

ANTONIO CORDOBA (Princeton, N. J.)

Abstract. A functional analysis technique is introduced to relate differentiation and covering properties of a basis.

A. Let $\mathscr{B}$ be a Busemann–Feller differentiation basis in $\mathbf{R}^n$. That is, for each $x \in \mathbf{R}^n$ we have a collection of bounded open sets $\mathscr{B}(x)$ containing x, such that there exists at least one sequence $\{R_k\} \subset \mathscr{B}(x)$ with diameter $(R_k) \to 0$, and if $x \in R \in \mathscr{B}$, then $R \in \mathscr{B}(x)$.

Given a mesurable set E in $\mathbf{R}^n$, we say that $V \subset \mathscr{B}$ is a *$\mathscr{B}$-Vitali covering of E* if for every $x \in E$ there is a sequence $\{R_k\} \subset V$ such that $R_k \in \mathscr{B}(x)$ for each k and $R_k \to x$ as $k \to \infty$.

DEFINITION 1. The differentiation basis $\mathscr{B}$ has the *covering property V_q* if there exists a constant C such that for every measurable bounded set E, every $\mathscr{B}$-Vitali covering V of E and any $\varepsilon > 0$, one can select a sequence $\{R_k\} \subset V$ with the properties:

(i) $|E - \bigcup R_k| = 0$, $|\bigcup R_k - E| \leqslant \varepsilon$,

(ii) $\|\sum \chi_{R_k}\|_q \leqslant C |E|^{1/q}$.

Given a locally integrable function f, we define the upper derivative $\overline{D}(\int f, x)$ with respect to $\mathscr{B}$ as follows:

$$\overline{D}\left(\int f, x\right) = \sup \limsup_{k \to \infty} \frac{1}{|R_k|} \int_{R_k} f(y)\, dy,$$

where the "sup" is taken over all the sequences $\{R_k\} \subset \mathscr{B}(x)$ such that $R_k \to x$ as $k \to \infty$. The lower derivative $\underline{D}(\int f, x)$ is defined by setting inf liminf above.

DEFINITION 2. We say that $\mathscr{B}$ *differentiates* $\int f$ if

$$\overline{D}\left(\int f, x\right) = \underline{D}\left(\int f, x\right) = f(x) \quad \text{at almost every point } x \in \mathbf{R}^n.$$

The purpose of this paper is to relate the following two properties of a differentiation basis:

(1) $\mathscr{B}$ differentiates $\int f$ for all $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$,

(2) $\mathscr{B}$ has the covering property V_q, $1/p+1/q=1$.

For the particular case $q=1$, $p=\infty$ the equivalence of (1) and (2) is due to de Possel [7]. The implication (2) $\Rightarrow$ (1) is well known, and Hayes and Pauc [4] proved that if $\mathscr{B}$ differentiates $\int f$ for all $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$, then $\mathscr{B}$ has the covering property V_{q_1} for all $q_1 < q$. In Theorem 1, we prove that for a basis $\mathscr{B}$ invariant by translations, the properties (1) and (2) are equivalent. For more detailed information about this problem see de Guzman [2] and [3].

B. Suppose that $\mathscr{B}$ is a differentiation basis invariant by translations.(¹) That is, there exists a family $\mathscr{B}(0)$ of bounded open sets containing the origin such that the fiber of $\mathscr{B}$ at the point x is given by $\mathscr{B}(x) = \{x \in R, R \in \mathscr{B}(0)\}$. Then we have:

THEOREM 1. *$\mathscr{B}$ differentiates $\int f$ for all $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ if and only if it has the covering property V_q, $1/q+1/p=1$, $1 \leqslant q < \infty$.*

Proof. (1) $\Rightarrow$ (2). Associated to the basis $\mathscr{B}$ we can consider the maximal function M_r ($r > 0$), defined on locally integrable functions f by the formula

$$M_r f(x) = \sup_{\substack{R \in \mathscr{B}(x) \\ \mathrm{diameter}(R) \leqslant r}} \frac{1}{|R|} \int_R |f(y)|\,dy$$

The fact that $\mathscr{B}$ is translation invariant and differentiates $\int f$ for $f \in L^p(\mathbf{R}^n)$, allows us to apply the theorems of Stein [9] and Sawyer [1], to conclude that there exists $r > 0$ such that the maximal function M_r is of weak type (p,p). Further generalizations of this argument have been obtained by B. Rubio [9] and I. Peral [6].

Given a measurable bounded set E and given $\varepsilon > 0$, we pick an open set Ω s.t. $\Omega > E$ and $|\Omega - E| \leqslant \varepsilon$. From now on, we shall consider only the elements of the Vitali covering of E which are contained in Ω and have diameter less than r. Obviously they constitute another Vitali covering of E; we shall denote by V that covering.

Since the measures of the elements of V are bounded, we can choose an element R_1 such that $|R_1| \geqslant \frac{1}{2}\sup\{|R|, R \in V\}$.

Suppose that we have chosen $R_1, \ldots, R_k$. Then we divide the family V in two classes:

1) Elements R s.t. $\left|R \cap \bigcup_{j \leqslant k} R_j\right| \leqslant \frac{1}{2}|R|$;

2) Elements R s.t. $\left|R \cap \bigcup_{j \leqslant k} R_j\right| > \frac{1}{2}|R|$.

We eliminate the second class and observe that the first class constitutes a Vitali covering of $E - \bigcup_{j \leqslant k} R_j$.

(¹) A Busemann-Feller lifferentation basis.

Now we choose R_{k+1} to be an element of the first class such that

$$|R_{k+1}| \geqslant \tfrac{1}{2}\sup\{|R|;\ R \text{ is in the first class}\}.$$

By induction we get a sequence $\{R_k\}$ such that

$$|E_k| \geqslant \tfrac{1}{2}|R_k| \quad \text{where} \quad E_k = R_k - \bigcup_{j<k} R_j$$

and furthermore $|R_k|$ is of the order of the biggest possible. From this, and using the fact that $\mathscr{B}$ differentiates integrals of functions in L^p, it is easy to see that $|E - \bigcup R_k| = 0$. The relation $|\bigcup R_k - E| \leqslant \varepsilon$ is an immediate consequence of the fact that $R_k \subset \Omega$ for every k.

Next we consider the linear operator

$$Tf(x) = \sum \frac{1}{|R_k|} \int_{R_k} f(y)\,dy \cdot \chi_{E_k}(x)$$

and its formal adjoint

$$Sf(x) = \sum \frac{1}{|R_k|} \int_{E_k} f(y)\,dy \cdot \chi_{R_k}(x).$$

Observe that $|Tf(x)| \leqslant M_r f(x)$ and $S(\chi_{\bigcup R_k}) \geqslant \frac{1}{2}\sum \chi_{R_k}$.

Since M_r is of weak type (p,p), we have that the family of operators like T (corresponding to different sequences $\{R_k\}$) is a uniformly bounded family of linear operators from $L^p(\mathbf{R}^n)$ to the Lorentz space $L(p,\infty)$. Therefore their duals T^* are uniformly bounded operators from $\big(L(p,\infty)\big)^*$ to L^q. But since $L(p,\infty)$ is the dual Banach space of $L(q,1)$ $(1/p+1/q=1)$, it follows that the operators S are uniformly bounded from the Lorentz space $L(q,1)$ to L^q. That is, there exists a constant C independent of E, ε and the sequence $\{R_k\}$, such that

$$\Big\|\sum \chi_{R_k}\Big\|_q \leqslant \tfrac{1}{2} C \|\chi_{\bigcup R_k}\|^*_{q,1} \leqslant C|E|^{1/q}.$$

(This is true because $\|\chi_F\|^*_{q,r} = |F|^{1/q}$ for every measurable set F, and every r, $1 \leqslant r < \infty$, see [5].)

The implication (2) $\Rightarrow$ (1) is straightforward. Q.E.D.

Remark. The same linearization technique also allows us to prove the following two results:

1° If $\mathscr{B}$ differentiates integrals of functions in L^1 then it has a covering property of exponential type, i.e. there exists a constant $C > 0$ such that given a $\mathscr{B}$-Vitali covering of the set E, we can find a subcovering $\{R_k\}$ satisfying

$$\Big\|\exp\Big(C \sum \chi_{R_k}(x)\Big)\Big\|_1 \leqslant |E|$$

2° If $\mathscr{B}$ differentiates integrals of functions in $L\overset{+}{\log}L$ (for example the basis of intervals in $\boldsymbol{R}^2$), then there exists $C > 0$ such that, under the same conditions of 1°, we have

$$\left\|\exp\left(C\sum\chi_{R_k}(x)\right)^{1/2}\right\|_1 \leqslant |E|.$$

However, these two covering properties are far from being the best possible for the corresponding situations.

C. The halo problem. Let $\mathscr{B}$ be a differentiation basis in $\boldsymbol{R}^n$ (not necessarily invariant by translations) and let $\varphi(u)$ be its halo function, that is

$$\varphi(u) = \sup\left\{\frac{1}{|A|}|\{x\colon M\chi_A(x) > u^{-1}\}|,\quad A \text{ bounded and with positive measures}\right\},\quad u \geqslant 1.$$

We can extend φ to $[0, \infty)$ by setting $\varphi(u) = u$ for $u \in [0, 1]$ (see [2]). Theorem 2 gives us an alternative proof of some results of Hayes and de Guzman.

THEOREM 2. *Suppose that $\varphi(u) = O(u^p)$ as $u \to \infty$ for some $1 < p < \infty$, then $\mathscr{B}$ differentiates integrals of functions in $L_{\text{loc}}(p, 1)$.*

Proof. We shall show that $\mathscr{B}$ has the Vitali covering property V_q(weak), $1/p + 1/q = 1$. That is, there exists $C > 0$ such that given a bounded measurable set E, $\varepsilon > 0$, and a $\mathscr{B}$-Vitali covering of E, we can select a sequence $\{R_k\}$ satisfying $|\bigcup R_k \Delta E| \leqslant \varepsilon$ and

$$\left|\left\{x\colon \sum\chi_{R_k}(x) > \lambda\right\}\right| \leqslant C\frac{|E|}{\lambda^q} \text{ for every } \lambda > 0.$$

To see this we select a sequence $\{R_k\}$ as in Theorem 1 and we consider the linear operators T and T^*.

Then

$$\begin{aligned}|E_\lambda| &= \left|\left\{x\colon \sum\chi_{R_k}(x) > \lambda\right\}\right| \\ &\leqslant \frac{2}{\lambda}\int_{E_\lambda} T^*\chi_E(x)\,dx = \frac{2}{\lambda}\int T\chi_{E_\lambda}(x)\chi_E(x)\,dx \\ &\leqslant \frac{2}{\lambda}\|\chi_E\|_{q,1}\|T\chi_{E_\lambda}\|_{p,\infty} \leqslant \frac{C^{1/q}}{\lambda}|E|^{1/q}|E_\lambda|^{1/p}\end{aligned}$$

and therefore $|E_\lambda| \leqslant C\dfrac{|E|}{\lambda^q}$.

(The same argument shows that T^* is a bounded linear operator from $L(q, 1)$ to $L(q, \infty)$.)

The proof of the fact that V_q (weak) implies differentiation of integrals of functions in $L_{\text{loc}}(p, 1)$ is straightforward. Q.E.D.

COROLLARY. *If $\varphi(u) = O(u^1)$ then $\mathscr{B}$ differentiates integrals of functions in $L(1+\log^+ L)^1$.*

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PRINCETON UNIVERSITY

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