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A weighted norm inequality for singular integrals

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Abstract. Let T be a singular integral in $\boldsymbol{R}^n$. Given $s > 1$, consider $A_s(f) = \big(((|f|^s)^*)^{1/s}\big)^*$, where g^* denotes the Hardy–Littlewood maximal function of g. The main result of this paper is the inequality

$$\int |Tf(x)|^p g(x)\,dx \leqslant C_{p,s}\int |f(x)|^p A_s g(x)\,dx, \qquad 1 < p,\ s < \infty,\ g \geqslant 0.$$

This result is used in Proposition 2, to get an alternative proof of the theorem of Benedek, Calderón and Panzone about vector valued singular integrals.

Let $\tilde{f} = Tf$ and let f^* be the Hardy–Littlewood maximal function of f. Given any $s > 1$, we shall consider $A_s(f) = \big(((|f|^s)^*)^{1/s}\big)^*$. The purpose of this paper is to prove the inequality

$$\int |\tilde{f}(x)|^p \omega(x)\,dx \leqslant C_{p,s}\int |f(x)|^p A_s(\omega)(x)\,dx$$

where $p > 1$ and the constant $C_{p,s}$ depends only upon p and s. This estimate is, of course, a possible formulation of the general principle of the Calderón–Zygmund decomposition, that is, that the maximal function controls the Hilbert transform. It turns out that this and similar inequalities can be used in order to reduce the study of some multipliers to the study of corresponding maximal functions (see [2], [3]). Proposition 2 gives us another application, namely an alternative proof of the results of Benedek, Calderón and Panzone about vector-valued singular integrals [1]. In this connection see also Herz [6] and Herz and Rivière [7].

Throughout this paper C will denote a constant, not necessarily the same at each occurrence and $|E|$ will denote the Lebesgue measure of a set E.

Suppose that T is a singular integral in $\boldsymbol{R}^n$, i.e. suppose that K is a locally integrable function in $\boldsymbol{R}^n$ such that:

(a) The Fourier transform of K is essentially bounded: $|\hat{K}(x)| \leqslant B$.

(b) K is of class C^1 outside of the origin and $|\nabla K(x)| \leqslant B|x|^{-n-1}$.

Let

$$Tf(x) = \text{P.V.}\int_{\boldsymbol{R}^n} K(x-y)f(y)\,dy \quad \text{for integrable functions } f.$$

PROPOSITION 1. *Given any $p, s > 1$, there exists a constant $C_{p,s}$ depending only upon p, s and B such that*

(1) $$\int |Tf(x)|^p \omega(x)\,dx \leqslant C_{p,s} \int |f(x)|^p A_s(\omega)(x)\,dx$$

for every locally integrable $\omega \geqslant 0$ and every $f \in \bigcup_{1<q<\infty} L^q(\mathbf{R}^n)$.

Proof. (1) (a) Given a locally integrable function f let $f^{\#}$ be defined by (Fefferman–Stein [5]):

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy$$

where Q is a cube and $f_Q = \frac{1}{|Q|}\int_Q f(z)\,dz$.

We have that $(Tf)^{\#}(x) \leqslant C_p[(f^{p_1})^*(x)]^{1/p_1}$ with C_{p_1} independent of f, $1 < p_1 < \infty$.

To see this, let Q be a cube containing the point x and denote by Q^* the result of expanding Q by a factor of two. Let $f = f_1 + f_2$ where $f_1 = f$ in Q^* and $f_1 = 0$ elsewhere; therefore, $Tf = Tf_1 + Tf_2$. Now

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Tf_1(y)|\,dy &\leqslant \left(\frac{1}{|Q|}\int_Q |Tf_1(y)|^{p_1}\,dy\right)^{1/p_1}\\ &\leqslant C_{p_1}\left(\frac{1}{|Q|}\int |f_1(y)|^{p_1}\,dy\right)^{1/p_1} \leqslant 2^{n/p_1} C_{p_1}\left(\frac{1}{|Q^*|}\int_{Q^*} |f(y)|^{p_1}\,dy\right)^{1/p_1}\\ &\leqslant 2^{n/p_1} C_{p_1}[(|f|^{p_1})^*(x)]^{1/p_1}.\end{aligned}$$

Thus

$$\frac{1}{|Q|}\int_Q |Tf_1(y) - Tf_{1_Q}|\,dy \leqslant \bar{C}_{p_1}[(|f|^{p_1})^*(x)]^{1/p_1}.$$

On the other hand,

$$\begin{aligned}Tf_2(y) - Tf_{2_Q} &= \frac{1}{|Q|}\int_Q \big(Tf_2(y) - Tf_2(z)\big)\,dz\\ &= \frac{1}{|Q|}\int_Q dz\left\{\int f_2(t)\,[K(y-t) - K(z-t)]\,dt\right\}.\end{aligned}$$

(1) In this proof we shall assume that $A_s(f)$ is defined as before but using the dyadic Hardy–Littlewood-maximal function f^*. By a cube Q we always mean a dyadic one. The result for the ordinary maximal function follows immediately if we show (1) for the dyadic case.

From this formula, using the fact that f_2 vanishes in Q^* and

$$|K(y-t) - K(z-t)| \leqslant B|y-z|\,|z-t|^{-n-1},$$

we get

$$|Tf_2(y) - Tf_{2_Q}| \leqslant 2Bf^*(x) \quad \text{for} \quad y \in Q.$$

Thus

$$\frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_{2_Q}|\,dy \leqslant 2Bf^*(x).$$

Combining this with the inequality for f_1, we get

$$(Tf)^{\#}(x) \leqslant C_{p_1}[(|f|^{p_1})^*(x)]^{1/p_1},$$

with C_{p_1} depending on p_1 and B.

(b) By the above result (with $p_1 < p$) it is obvious that in order to prove the proposition, it is enough to show the estimate

(2) $$\int |F(x)|^p \omega(x)\,dx \leqslant C_{p,s}\int |F^{\#}(x)|^p[(\omega^s)^*(x)]^{1/s}\,dx$$

because then

$$\begin{aligned}\int |Tf(x)|^p \omega(x)\,dx &\leqslant C_{p,s}\int |Tf^{\#}(x)|^p[(\omega^s)^*(x)]^{1/s}\,dx\\ &\leqslant C_{p,p_1,s}\int [(f^{p_1})^*(x)]^{p/p_1}[(\omega^s)^*(x)]^{1/s}\,dx\\ &\leqslant C_{p,p_1,s}\int |f(x)|^p A_s(\omega)(x)\,dx\end{aligned}$$

by using Lemma 1 of Fefferman–Stein [4].

To prove (2) we need first to show that the measure $d\mu = V(x)\,dx$ where $V(x) = [(\omega^s)^*]^{1/s}(x)$ satisfies Muckenhoupt's condition A_∞. That is, for every $\varepsilon > 0$ we can find $\delta > 0$ such that if $E \subset Q$ and $|E|/|Q| \leqslant \delta$, then $\mu(E)/\mu(Q) \leqslant \varepsilon$.

Let $1 < r < s$ and suppose that the following estimate is true:

(3) $$\left(\frac{1}{|Q|}\int_Q V^r(x)\,dx\right)^{1/r} \leqslant C_{s,r}\left(\frac{1}{|Q|}\int_Q V(x)\,dx\right) \quad \text{for every cube } Q.$$

Then

$$\begin{aligned}\frac{1}{|Q|}\int_E V(x)\,dx &\leqslant \left(\frac{|E|}{|Q|}\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q V^r(x)\,dx\right)^{1/r}\\ &\leqslant C_{s,r}\left(\frac{|E|}{|Q|}\right)^{1/r'}\frac{1}{|Q|}\int_Q V(x)\,dx, \quad \frac{1}{r} + \frac{1}{r'} = 1.\end{aligned}$$

Therefore

$$\frac{\mu(E)}{\mu(Q)} \leqslant C_{p,r}\left(\frac{|E|}{|Q|}\right)^{1/r'}.$$

That is μ satisfies the condition A_∞.

Now, to show (3), let Q be a cube and modify the function V in the following way: Let $V_1(x) = \left((\omega^s)^+(x)\right)^{1/s}$. Where g^+ is the dyadic maximal function of g but with respect to dyadic subcubes of Q.

Then

$$\left(\frac{1}{|Q|}\int_Q V_1(x)^r\,dx\right)^{1/r} = \frac{1}{|Q|}\left(\int_Q [(\omega^s)^+(x)]^{r/s}\,dx\right)^{1/r}$$

$$\leqslant C_{r,s}\left(\frac{1}{|Q|}\int_Q \omega^s(x)\,dx\right)^{1/s} \leqslant C_{r,s}\inf_{x\in Q}[(\omega^s)^+(x)]^{1/s}$$

$$\leqslant C_{r,s}\frac{1}{|Q|}\int_Q V_1(x)\,dx$$

With this estimate (3) follows directly from the fact that

$$V(x) = \sup(V_1(x), \text{constant}) \quad \text{for} \quad x\in Q.$$

(c) We are now in position to use the technique of [5], Part III, to obtain an inequality of the form:

$$\mu\{x: f^*(x) > a\} \leqslant \mu\left\{x: f^\#(x) > \frac{a}{A}\right\} + \frac{C}{A}\mu\left\{x: f^*(x) > \frac{a}{2^{n+1}}\right\}$$

with C independent of A. (In fact, [5] proves the result when μ = Lebesgue measure, and the only property of Lebesgue measure used there is (A_∞).) This inequality implies Proposition 1. Q.E.D.

Let $\{K_j\}_{j=1,2,\ldots}$ be a collection of kernels like the one in Proposition 1 with common bounds. That is:

$$|\hat{K}_j(x)| \leqslant B, \qquad |\nabla K_j(x)| \leqslant B|x|^{-n-1} \quad \text{for every } j.$$

Given p, $r > 1$, consider the space

$$L^p(l^r) = \left\{(f_j):\ \|(f_j)\| = \left(\int_{\mathbf{R}^n}\left(\sum|f_j(x)|^r\right)^{p/r}dx\right)^{1/p} < \infty\right\}.$$

PROPOSITION 2. *The operator* $T: L^p(l^r)\to L^p(l^r)$ *defined by* $T(f_j) =$ P.V.$(K_j * f_j)$ *is bounded* $(1 < p < \infty,\ 1 < r < \infty)$.

Proof. (a) Consider first the case $p \geqslant r$. Since the result for $p = r$ is immediate, we can assume $p > r$. Let q be defined by the formula $1/q + r/p = 1$; then:

$$\int_{\mathbf{R}^n}\left(\sum|T_j(f_j)(x)|^r\right)^{p/r}dx = \sup_{\|\omega\|_q=1}\left[\int_{\mathbf{R}^n}\left(\sum|T_j(f_j)(x)|^r\right)\omega(x)\,dx\right]^{p/r}$$

$$\leqslant C_{r,s}\sup_{\|\omega\|_q=1}\left[\int_{\mathbf{R}^n}\sum_j|f_j(x)|^r A_s(\omega)(x)\,dx\right]^{p/r}$$

$$\leqslant C_{r,s}\int\left(\sum_j|f_j(x)|^r\right)^{p/r}dx\sup_{\|\omega\|_q=1}\left[\int A_s(\omega)^q(x)\right]^{p/qr}.$$

But taking $s < q$ we have that

$$\int A_s(\omega)^q(x)\,dx = \int\left[\left((\omega^s)^*\right)^{1/s*}(x)\right]^q dx$$

$$\leqslant C_q\int[(\omega^s)^*(x)]^{q/s}dx \leqslant C_{q,s}\int\omega^q(x)\,dx \leqslant C_{q,s},$$

as we wished to show.

(b) In the case $p < r$ we can use a duality argument because the dual of T is essentially equal to T, and if $p < r$ then $p' > r'$ where $1/p + 1/p' = 1$, $1/r + 1/r' = 1$. Q.E.D.

Added in proof: R. Coifman has kindly pointed out to us that $V(x) = [(\omega^s)^{*1/s}(x)]$ also satisfies the condition A_1. This fact can be used to improve Proposition 1.

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