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Theorems on lacunary sets, especially p-Sidon sets

by

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Abstract. Let G be a compact abelian group with dual group Γ and suppose that $E \subset \Gamma$. Several known results say that certain types of lacunary sets E can contain only a restricted part of certain kinds of sets. (An early example of such a theorem says that if $\Gamma = Z$ and E is a $\Lambda(p)$ set, then E cannot contain arbitrarily long arithmetic progressions.) We give a general theorem from which these results as well as some new results follow as corollaries. A corollary of particular interest gives new information on p-Sidon sets and is related to the problem of distinguishing the various p-Sidon classes.

1. Introduction and terminology. Let G be a compact abelian group with discrete dual group Γ. $M(G)$ denotes as usual the measure algebra on G ([13], p. 265). If $E \subset \Gamma$, $M_E(G)$ will consist of those μ in $M(G)$ such that $\hat{\mu}$ is supported by E. Similar notation will be used for other spaces. In particular, T denotes the class of complex-valued trigonometric polynomials on G and T_E denotes those f in T with $\hat{f}$ supported by E.

$E \subset \Gamma$ is called *p-Sidon* ($1 \leqslant p < 2$) if there is a constant $B = B(E, p)$ such that for each f in T_E

$$\|\hat{f}\|_p \leqslant B\|f\|_\infty. \tag{1.1}$$

We denote the class of p-Sidon sets by $\mathscr{S}_p$. Clearly if $1 \leqslant p_1 < p_2 < 2$, $\mathscr{S}_{p_1} \subset \mathscr{S}_{p_2}$. p-Sidon sets have recently been extensively studied by Edwards and Ross [5]; see also [1], [9], [11]. When p is definitely 1, we follow the usual terminology and call E a Sidon set rather than a 1-Sidon set. Sidon sets have been extensively studied for many years.

Let $0 < p < \infty$. E is called a $\Lambda(p)$ set if there exists q in $(0, p)$ and a constant $B = B(E, p, q)$ such that for each f in T_E

$$\|f\|_p \leqslant B\|f\|_q. \tag{1.2}$$

$\Lambda(p)$ sets were introduced and studied by Rudin [14] in the case $\Gamma = Z$. Later results on $\Lambda(p)$ sets particularly relevant to us are in [4] and [5].

There are several theorems in the literature saying that Sidon sets or $\Lambda(p)$ sets or p-Sidon sets can contain only a restricted part of certain kinds of sets. We establish a general theorem out of which these results as well as some new results of the same nature can be obtained as corollaries. The development shows quite clearly what these results have in common and what separate considerations are necessary in the different cases. It also indicates some circumstances sufficient for proving further theorems of the same type.

Next we describe three of the known results which come from our general theorem. Throughout this paper $\nu(E)$ will denote the cardinality of a subset E of Γ.

Let $M \geqslant 1$ be a real number. A family $\mathfrak{F}$ of non-void finite subsets of Γ will be called a *test family of order M* ([4], p. 790) if

$$\sup\{\nu(\Phi+\Phi-\Phi)/\nu(\Phi)\colon \Phi\in\mathfrak{F}\} \leqslant M. \tag{1.3}$$

If $\Gamma = Z$, the family of all finite arithmetic progressions is an example of a test family of order 3. For any abelian group Γ, the family of all finite subgroups is a test family of order 1. For more examples and some discussion of test families, see [4], pp. 790–792.

One of the known results ([4], p. 790) that we obtain says that if $E \subset \Gamma$ is a $\Lambda(p)$ set $(p > 2)$ with associated constant $B = B(E, p, 2)$ and if $\mathfrak{F}$ is a test family of order M, then for every Φ in $\mathfrak{F}$,

$$\nu(E\cap\Phi) \leqslant B^2 M \nu(\Phi)^{2/p}. \tag{1.4}$$

Another known result ([5], 2.6 Corollary) that we obtain says that if E is a p-Sidon set and $\mathfrak{F}$ is as above, then for every Φ in $\mathfrak{F}$,

$$\nu(E\cap\Phi) \leqslant BM^{a'/2}\big(\log\nu(\Phi)\big)^{a'/2}, \tag{1.5}$$

where B, a' are fixed constants to be described later. When $p = 1$, we will see that $a' = 2$ and the result just given includes the result known for Sidon sets ([4], p. 791).

The other known result that we mention here is the following: Let $n \geqslant 2$ be a positive integer. Suppose $E \subset \Gamma$ is p-Sidon for some $1 \leqslant p < 2n/(n+1)$. Then there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying

$$\text{(1.6)}\quad \begin{array}{ll} \text{(i)} & \nu(A_i) = k, \quad i = 1, \ldots, n, \\ \text{(ii)} & A_1 + \ldots + A_n \subset E, \end{array}$$

then $k \leqslant K$. For $n = 2$, this result is in [5], 2.7 Corollary; for general n, see [11], Lemma 1.

We finish this introduction by stating the new result obtainable from our theorem which seems most interesting to us and then discussing very briefly its relationship to the theorem on sums of sets immediately above and to the problem of distinguishing the p-Sidon classes; the last section of this paper contains a more detailed discussion of these relationships and some related combinatorial questions.

THEOREM. *Let E be in $\mathscr{S}_p$ and let $0 < \delta \leqslant 1$. Suppose that we have a positive integer $n \geqslant 2$ and $1 < s \leqslant n$ such that $1 \leqslant p < 2s/(s+1)$. Under these hypotheses there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying*

$$\text{(1.7)}\quad \begin{array}{ll} \text{(i)} & \nu(A_i) = k, \quad i = 1, \ldots, n, \\ \text{(ii)} & \nu[(A_1 + \ldots + A_n)\cap E] \geqslant \delta k^s, \\ \text{(iii)} & A_i \cap A_j = \varphi, \quad i \neq j, \\ \text{(iv)} & \sum_{i=1}^{nk} \delta_i\gamma_i = 0 \text{ with } \delta_i \in \{-1, 0, 1\} \text{ and } A_1\cup\ldots\cup A_n = \{\gamma_1, \ldots \\ & \ldots, \gamma_{nk}\} \text{ implies } \delta_1 = \ldots = \delta_{nk} = 0, \end{array}$$

then $k \leqslant K$. [This result will appear as part (1) of Corollary 6 in Section 2.]

It is shown in [11] that there exist n infinite sets $A_1, \ldots, A_n$ such that $A_1 + \ldots + A_n \in \mathscr{S}_{2n/(n+1)}$. On the other hand, by the theorem on sums of n sets discussed above (circa (1.6)), $A_1 + \ldots + A_n \notin \mathscr{S}_p$ for $1 \leqslant p < 2n/(n+1)$. It follows that if $1 \leqslant p_1 < 2n/(n+1) \leqslant p_2 < 2$ for some n in $\{2, 3, 4, \ldots\}$, then $\mathscr{S}_{p_1} \nsubseteq \mathscr{S}_{p_2}$. It seems likely that $\mathscr{S}_{p_1} \nsubseteq \mathscr{S}_{p_2}$ whenever $1 \leqslant p_1 < p_2 < 2$, but this has not been shown. For example, if $1 \leqslant p_1 < p_2 < 4/3$, no examples are known of p_2-Sidon sets that are not p_1-Sidon. However, sets of the type suggested by our new result above seem like reasonable possibilities for members of $\mathscr{S}_{p_2}\setminus\mathscr{S}_{p_1}$ since the theorem involves the numbers $2s/(s+1)$ for any $s > 1$ and not just the values $\{2, 3, 4, \ldots\}$ encountered in the earlier theorem [11] on sums of n sets.

2. The general theorem and its corollaries. Let T^+ denote the trigonometric polynomials with non-negative coefficients. Given a number $l \geqslant 1$, l and l' will always be related by the equation $1/l + 1/l' = 1$.

THEOREM. *Let E be a subset of Γ and suppose that there exist numbers $2 \leqslant r < \infty$, $1 < l < \infty$ and $C = C(E, r, l) \geqslant 0$ such that if $g = \sum\chi$ is a finite sum of characters from E, then*

$$\|g\|_r \leqslant C\|\hat{g}\|_l. \tag{2.1}$$

Then given Φ, a finite subset of Γ, $\eta > 0$ and f in T^+ satisfying (a) $\hat{f} \geqslant \eta$ *on Φ and* (b) $\|f\|_1 \leqslant 1$, *we have*

$$\nu(E\cap\Phi) \leqslant (C/\eta)^{l'}\|f\|_2^{2l'/r}. \tag{2.2}$$

Proof. Let $E \subset \Gamma$ satisfy the hypothesis. Let appropriate Φ, η and f be given. Let $g = \sum_{E\cap\Phi}\chi$. Then

$$\eta\nu(E\cap\Phi) \leqslant \sum_{E\cap\Phi}\hat{f} = \sum_{E\cap\Phi}\hat{f}\hat{g} = \sum_{\Gamma}\hat{f}\hat{g} = (f*g)(0) \leqslant \|f\|_{r'}\|g\|_r, \tag{2.3}$$

where the last inequality follows from Hölder's inequality. Now $1 < r' \leqslant 2$ and so we apply an interpolation inequality ([10], Theorem 13.19) to $\|f\|_{r'}$ obtaining

$$\|f\|_{r'} \leqslant \|f\|_1^{1-2/r} \|f\|_2^{2/r} \leqslant \|f\|_2^{2/r}.$$

Combining this with (2.3) and the hypothesis of the theorem, we obtain

$$\eta\nu(E\cap\Phi) \leqslant \|f\|_2^{2/r}\, C\big(\nu(E\cap\Phi)\big)^{1/l}.$$

(2.2) now follows.

Remark. To obtain useful corollaries from the general theorem above, one of course needs a hypothesis on E which insures an inequality of the form (2.1), but also one must have appropriate Φ, η and f such that $\|f\|_2^{2l'/r} < \nu(\Phi)$.

The first corollary we give is Rudin's result [14], pp. 213–214, on $\Lambda(p)$ sets and arithmetic progressions. This result is a special case of Corollary 2 below as is known ([4], p. 791). We include it because it gives a good illustration of how to apply the general theorem in a concrete case.

COROLLARY 1. *Let $\Gamma = Z$, the group of integers, and suppose that $E \subset Z$ is a $\Lambda(p)$ set for some $p > 2$ with appropriate constant $B = B(E, p, 2)$ as in (1.2). Then for any N-term arithmetic progression Φ, we have*

$$\nu(E\cap\Phi) \leqslant 4B^2 N^{2/p}.$$

Proof. Let $\Phi = \{a+b, a+2b, \ldots, a+Nb\}$ be any N-term arithmetic progression. To apply the theorem, take $r = p$, $l = 2$, $C = B$ and $f(x) = \exp(imx)K_N(bx)$, where m equals $bN/2 + a$ if N is even and $b(N+1)/2 + a$ if N is odd and where $K_N(x) = \sum_{|n|\leqslant N}(1-|n|/N)\exp(inx)$ is the Fejér kernel. Note that when $l = 2$ the right-hand side of (2.1) equals $C\|g\|_2$.

Now $\|f\|_1 = 1$ and $\hat{f} \geqslant 1/2$ on Φ and so, applying (2.2),

$$\begin{aligned}\nu(E\cap\Phi) &\leqslant (2B)^2\|f\|_2^{4/p} = 4B^2\|K_N\|_2^{4/p}\\ &= 4B^2[(2N^2+1)/3N]^{2/p} \leqslant 4B^2 N^{2/p}.\end{aligned}$$

The next corollary is a result due to Edwards, Hewitt and Ross [4] pp. 790–791.

COROLLARY 2. *Let $E \subset \Gamma$ be a $\Lambda(p)$ set for some $p > 2$ with appropriate constant $B = B(E, p, 2)$ as in (1.2). Let $\mathfrak{F}$ be a test family of order M. Then for every Φ in $\mathfrak{F}$ we have*

$$\nu(E\cap\Phi) \leqslant B^2 M\big(\nu(\Phi)\big)^{2/p}. \tag{2.4}$$

Proof. Let Φ be in $\mathfrak{F}$. Take $r = p$, $l = 2$ and $C = B$. In this case f is obtained by applying a theorem from Rudin [13], p. 48, on local units in the transform space. Interpreted in the present setting it insures the existence of f_0 in T^+ such that $0 \leqslant \hat{f}_0 \leqslant 1$, $\hat{f}_0 = 1$ on Φ, $\hat{f}_0 = 0$ off $\Phi+\Phi-\Phi$ and $\|f_0\|_1 \leqslant [\nu(\Phi-\Phi)/\nu(\Phi)]^{1/2} \leqslant M^{1/2}$. Take $f = M^{-1/2}f_0$. Using the assumption that Φ is in $\mathfrak{F}$ and applying the theorem we obtain

$$\begin{aligned}\nu(E\cap\Phi) &\leqslant (BM^{1/2})^2\|f\|_2^{4/p} = B^2M(M^{-1/2})^{4/p}\|\hat{f}_0\|_2^{4/p}\\ &\leqslant B^2MM^{-2/p}\big(\nu(\Phi+\Phi-\Phi)\big)^{2/p}\\ &\leqslant B^2MM^{-2/p}M^{2/p}\big(\nu(\Phi)\big)^{2/p} = B^2M\big(\nu(\Phi)\big)^{2/p}.\end{aligned}$$

For the remaining corollaries, it will be convenient to introduce a class of sets which we will denote by $\mathcal{T}_p$. Given $1 \leqslant p < 2$, let $a = 2p/(3p-2)$. Note that as p varies from 1 to 2, a varies from 2 to 1. Also $1/p + 1/a = 3/2$. Now let p in $[1,2)$ be fixed. $E \subset \Gamma$ is said to be in $\mathcal{T}_p$ if there exists a constant $B = B(E, p)$ such that if $g = \sum\chi$ is a finite sum of characters from E, then for all $2 \leqslant r < \infty$,

$$\|g\|_r \leqslant Br^{1/2}\|\hat{g}\|_a. \tag{2.5}$$

The class $\mathcal{T}_p$ may be worthy of study in itself but we use it essentially just as a notational convenience. However, the following lemma will be useful.

LEMMA 1. *$\mathcal{T}_p$ is closed under finite unions.*

Proof. It suffices to show that E, F in $\mathcal{T}_p$ implies $E\cup F$ is in $\mathcal{T}_p$. Further, since subsets of sets in $\mathcal{T}_p$ are clearly in $\mathcal{T}_p$, we may consider the case where E and F are disjoint. Let B_E and B_F be appropriate constants for E and F respectively and let $B_0 = \max(B_E, B_F)$. Now all the l_q-norms are equivalent on R^2 and so there exists a constant D such that $|t_1| + |t_2| \leqslant D[|t_1|^a + |t_2|^a]^{1/a}$ for all (t_1, t_2) in R^2.

Now let $g = \sum\chi$ be a finite sum of characters from $E\cup F$. We may write $g = g_1 + g_2$, where g_1 is a finite sum of characters from E and g_2 is a finite sum of characters from F. Then for any $2 \leqslant r < \infty$

$$\begin{aligned}\|g\|_r &= \|g_1+g_2\|_r \leqslant \|g_1\|_r + \|g_2\|_r \leqslant B_E r^{1/2}\|\hat{g}_1\|_a + B_F r^{1/2}\|\hat{g}_2\|_a \leqslant B_0 r^{1/2}[\|\hat{g}_1\|_a + \|\hat{g}_2\|_a]\\ &\leqslant DB_0 r^{1/2}[\|\hat{g}_1\|_a^a + \|\hat{g}_2\|_a^a]^{1/a} = DB_0 r^{1/2}[\|\hat{g}_1 + \hat{g}_2\|_a^a]^{1/a} = DB_0 r^{1/2}\|\hat{g}\|_a.\end{aligned}$$

The next lemma will also be useful; it identifies some types of sets that are contained in $\mathcal{T}_p$. Parts (1) and (3) of the lemma are essentially contained in Edwards and Ross [5]; part (2) follows from Lemma 1.

LEMMA 2. *Let $E \subset \Gamma$. (1) If E is a p-Sidon set, then E is in $\mathcal{T}_p$. (2) If E is a finite union of p-Sidon sets, then E is in $\mathcal{T}_p$. (3) Suppose that E is a $\Lambda(q)$ set for all $1 < q < \infty$. For $q > 2$, let $A(q) = A(E, q, 2)$ denote* *the smallest number $A(q)$ such that $\|f\|_q \leqslant A_q \|f\|_2$ for all f in T_E. Suppose further that $A(q) = O(q^{d/2})$ for some positive number d, where* $\max(1, 2d/(d+1)) \leqslant p < 2$. *Then E is in $\mathcal{T}_p$.*

Proof. (1) Theorem 2.4 (i) of [5] assures us that there exists a constant B such that if μ is in $M_E(G)$ with $\hat{\mu}$ in l^a, then μ is in $L^r(G)$ and

$$\|\mu\|_r \leqslant Br^{1/2}\|\hat{\mu}\|_a. \tag{2.6}$$

Hence $\mathcal{S}_p \subset \mathcal{T}_p$.

(2) follows immediately from (1) and Lemma 1.

(3) In the first part of the proof of Theorem 3.3 (i) [5] Edwards and Ross show that (2.6) holds under the hypotheses of (3). It follows that E is in $\mathcal{T}_p$.

Remark. In Corollaries 3, 5, 7, 9 below results will be established for $\mathcal{T}_p$. In each case Lemma 2 will tell us that the conclusions of these corollaries hold for the three types of sets described in Lemma 2. When $p = 1$, (2) of Lemma 2 is already contained in (1) since Drury's theorem [3] insures that $\mathcal{S}_1$ is closed under finite unions. However, for $1 < p < 2$, it is not known if $\mathcal{S}_p$ is closed under finite unions.

Corollary 3. *Suppose that E is in $\mathcal{T}_p$ and that $\mathfrak{F}$ is a test family of order M. Then if Φ is an element of $\mathfrak{F}$ with $\nu(\Phi) \geqslant 3$, we have*

$$\nu(E\cap\Phi) \leqslant (2eB^2M)^{a'/2}\big(\log\nu(\Phi)\big)^{a'/2}. \tag{2.7}$$

Proof. Let Φ be in $\mathfrak{F}$ with $\nu(\Phi) \geqslant 3$. Take $r = 2\log\nu(\Phi)$, $l = a$, $C = Br^{1/2}$ and choose f just as in the proof of Corollary 2. Applying the theorem we obtain

$$\begin{aligned}
\nu(E\cap\Phi) &\leqslant (Br^{1/2}M^{1/2})^{a'}\|f\|_2^{2a'/r}\\
&\leqslant (B^2M)^{a'/2}r^{a'/2}M^{-a'/r}\big(\nu(\Phi+\Phi-\Phi)\big)^{a'/r}\\
&\leqslant (B^2M)^{a'/2}r^{a'/2}M^{-a'/r}M^{a'/r}\big(\nu(\Phi)\big)^{a'/r}\\
&= (B^2M)^{a'/2}2^{a'/2}\big(\log\nu(\Phi)\big)^{a'/2}e^{a'/2}\\
&= (2eB^2M)^{a'/2}\big(\log\nu(\Phi)\big)^{a'/2}.
\end{aligned}$$

Corollary 3 and Lemma 2 yield immediately the following corollary. Parts (1) and (3) are due to Edwards and Ross [5]; Corollary 2.6, Theorem 3.3 (i) (the last assertion).

Corollary 4. *Let $E \subset \Gamma$. Suppose that the hypotheses of either* (1) *or* (2) *or* (3) *of Lemma 2 holds. Then if $\mathfrak{F}$ is a test family of order M and if Φ is an element of $\mathfrak{F}$ with $\nu(\Phi) \geqslant 3$, we have* (2.7) *holding.*

Next we give a corollary from which the theorem stated formally in the introduction will follow as one part of a further corollary.

Corollary 5. *Let E be in $\mathcal{T}_p$ and let $0 < \delta \leqslant 1$. Suppose that we have a positive integer $n \geqslant 2$ and $1 < s \leqslant n$ such that $1 \leqslant p < 2s/(s+1)$. Under these hypotheses, there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying* (i)–(iv) *of* (1.7), *then $k \leqslant K$.*

Proof. Suppose there is no bound on the k's that can appear in (i)–(iv) of (1.7). Let one such k be fixed for now and let $A_1, \ldots, A_n$ be appropriate sets. Let Φ be a subset of $(A_1+\ldots+A_n)\cap E$ such that $\delta k^s \leqslant \nu(\Phi) < \delta k^s+1$. To apply the general theorem take $r = k$, $l = a$, $C = Bk^{1/2}$ and take f to be the following Riesz polynomial:

$$f(x) = \prod_{i=1}^{nk}\big[1+1/2\big(\gamma_i(x)+\overline{\gamma_i(x)}\big)\big]. \tag{2.8}$$

(iii) and (iv) of (1.7) and standard arguments with Riesz polynomials yield the following facts: $f \geqslant 0$, $\|f\|_1 = \hat{f}(0) = 1$, $\|f\|_2 \leqslant \|f\|_\infty = 2^{nk}$, and f is in T^+ with $\hat{f} \geqslant 2^{-n}$ on $A_1+\ldots+A_n \supset \Phi$. Now we apply the general theorem obtaining

$$\delta k^s \leqslant \nu(\Phi) \leqslant (Bk^{1/2}2^n)^{a'}(2^{nk})^{2a'/k} = (2^{3n}B)^{a'}k^{a'/2}.$$

Since this inequality holds for unboundedly large k's we must have $s \leqslant a'/2$. But $a' = 2p/(2-p)$ and so we get $2s/(s+1) \leqslant p$ which contradicts our assumption.

Corollary 5 and Lemma 2 now yield immediately the following corollary.

Corollary 6. *Let $E \subset \Gamma$. Suppose that the hypotheses of either* (1) *or* (2) *or* (3) *of Lemma 2 hold. Let $0 < \delta \leqslant 1$. Suppose that we have a positive integer $n \geqslant 2$ and $1 < s \leqslant n$ such that $1 \leqslant p < 2s/(s+1)$. Under these hypotheses, there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying* (i)–(iv) *of* (1.7), *then $k \leqslant K$.*

Remark. In the case where $\Gamma = Z$, one can alternately prove Corollary 6 (1) by applying Kahane's powerful theorem [12], Theorem 4, p. 57. One first uses Kahane's Theorem 4 to prove a p-Sidon version of [12], Theorem 5, p. 58, and then applies this result to Corollary 6 (1). Similar remarks hold for Corollaries 8 (1) and 10 (1) further on.

Corollary 7. *Let E be in $\mathcal{T}_p$. Let $n \geqslant 2$ be a positive integer such that $1 \leqslant p < 2n/(n+1)$. Under these hypotheses, there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying* (1.6), *then $k \leqslant K$.*

Proof. Taking $s = n$ and $\delta = 1$ we apply Corollary 5 to obtain a positive integer K_0 such that if $C_1, \ldots, C_n$ are n sets satisfying (i)–(iv) of (1.7) (except that, for notational convenience, use k_0 here instead of k) then $k_0 < K_0$. We claim that $K \equiv 3^{K_0 n-1}$ is such that if $A_1, \ldots, A_n$ are n sets satisfying (1.6), then $k \leqslant K$. Suppose, on the contrary, that there exist n sets $A_1, \ldots, A_n$ satisfying (1.6) with $k > K$. We finish the proof by constructing sets $C_1, \ldots, C_n$ satisfying (i)–(iv) of (1.7) with $\nu(C_1) = \ldots \ldots = \nu(C_n) = K_0$.

Let $\gamma_1 \neq 0$ be in A_1. We inductively pick K_0 elements each from $A_1, \ldots, A_n$ as follows. Suppose $1 \leqslant t < K_0 n$ and $\gamma_1, \ldots, \gamma_t$ satisfy

$$\gamma_l \in A_j \quad \text{if} \quad (j-1)K_0 < l \leqslant jK_0 \quad (j \in \{1, \ldots, n\}) \tag{2.9}$$

and

$$\gamma_l \notin D_{l-1} \equiv \Big\{\sum_{i=1}^{l-1} \delta_i \gamma_i \colon (\delta_1, \ldots, \delta_{l-1}) \in \{-1, 0, 1\}^{l-1}\Big\}. \tag{2.10}$$

Then there is a γ_{t+1} satisfying (2.9) and (2.10) since $\nu(D_t) \leqslant 3^t < \nu(A_j)$ $(1 \leqslant j \leqslant n)$. Let $C_j \equiv \{\gamma_l\colon (j-1)K_0 < l \leqslant jK_0\}$. It follows from (2.10) that for $(\delta_1, \ldots, \delta_{K_0 n})$ in $\{-1, 0, 1\}^{K_0 n}$,

$$\sum_{i=1}^{K_0 n} \delta_i \gamma_i = 0 \quad \text{if and only if} \quad \delta_1 = \ldots = \delta_{K_0 n} = 0. \tag{2.11}$$

In particular note that if $i \neq j$, then γ_i is not in $\{0, \gamma_j, -\gamma_j\}$. To complete the proof we just need to show that (ii) of (1.7) is satisfied (with 1, n and K_0 playing the roles of δ, s and k, respectively). Since $C_1 + \ldots + C_n \subset A_1 + \ldots$ $\ldots + A_n \subset E$, we only need to show that $\nu(C_1 + \ldots + C_n) = K_0^n$. If not, there must be elements $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$ in $C_1 \times \ldots \times C_n$ such that

$$0 = \sum_1^n a_j - \sum_1^n b_j = \sum_1^{K_0 n} \delta_i \gamma_i$$

for some $(\delta_1, \ldots, \delta_{K_0 n}) \neq 0$ in $\{-1, 0, 1\}^{K_0 n}$, which is impossible.

The following corollary is an immediate consequence of Corollary 7 and Lemma 2. Part (1) is in Edwards and Ross ([5], Corollary 2.7) for $n = 2$ and in Johnson and Woodward ([11], Lemma 1) for $n \geqslant 2$. Part (3) is in Edwards and Ross ([5], Theorem 3.3 (ii)) for $n = 2$.

COROLLARY 8. *Let $E \subset \Gamma$. Suppose that the hypotheses of either* (1) *or* (2) *or* (3) *of Lemma* 2 *hold. Let $n \geqslant 2$ be a positive integer such that $1 \leqslant p < 2n/(n+1)$. Under these hypotheses, there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying* (1.6), *then $k \leqslant K$.*

As discussed briefly in the introduction, (1) of Corollary 6 provides candidates for members of $\mathscr{S}_{p_2} \setminus \mathscr{S}_{p_1}$, where $1 \leqslant p_1 < p_2 < 2$. In fact, as we will discuss in Section 3, (1) of Corollary 6 provides candidates for $\mathscr{S}_{p_2} \setminus \bigcup_{1 \leqslant p < p_2} \mathscr{S}_p$. The following is a related question. Suppose $1 \leqslant p_1 < 2$ is fixed. Certainly $\mathscr{S}_{p_1} \subset \bigcap_{p_1 < p < 2} \mathscr{S}_p$. Is the containment proper? Part (1) of the second corollary to follow provides reasonable candidates for members of $\bigcap_{p_1 < p < 2} \mathscr{S}_p \setminus \mathscr{S}_{p_1}$.

COROLLARY 9. *Let E be in $\mathscr{T}_p$ and let $0 < \delta \leqslant 1$. Suppose that we have a positive integer $n \geqslant 2$ and $1 \leqslant s < n$ such that $1 \leqslant p \leqslant 2s/(s+1)$. Under these hypotheses, there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying* (i), (iii), *and* (iv) *of* (1.7) *along with*

$$\text{(ii)}' \qquad \nu[(A_1 + \ldots + A_n) \cap E] \geqslant \delta k^s \log k, \tag{2.12}$$

then $k \leqslant K$.

Proof. The proof proceeds just like the proof of Corollary 5 except that the extra $\log k$ factor in (ii)′ allows one to conclude that $s < a'/2$ (rather than $s \leqslant a'/2$) and hence $2s/(s+1) < p$, but this contradicts our present assumption.

Corollary 9 and Lemma 2 yield immediately the following corollary.

COROLLARY 10. *Let $E \subset \Gamma$. Suppose that the hypotheses of either* (1) *or* (2) *or* (3) *of Lemma* 2 *hold. Let $0 < \delta \leqslant 1$. Suppose that we have a positive integer $n \geqslant 2$ and $1 \leqslant s < n$ such that $1 \leqslant p \leqslant 2s/(s+1)$. Under these hypotheses, there exists a positive integer K such that if $A_1, \ldots, A_n$ are n sets satisfying* (i), (iii) *and* (iv) *of* (1.7) *along with* (ii)′ *of* (2.12), *then $k \leqslant K$.*

3. Corollary 6 (1) and the problem of distinguishing p-Sidon classes. As we observed earlier, if $1 \leqslant p_1 < p_2 < 2$, $\mathscr{S}_{p_1} \subset \mathscr{S}_{p_2}$. A basic problem is to determine whether this containment is proper. For $1 < s < \infty$, let

$$\mathscr{R}_s \equiv \mathscr{S}_{2s/(s+1)} \setminus \bigcup_{1 \leqslant p < 2s/(s+1)} \mathscr{S}_p. \tag{3.1}$$

Corollary 8 (1) suggests candidates for members of $\mathscr{R}_n$, where $n \geqslant 2$, is a positive integer. Specifically, if one can find a set E in $\mathscr{S}_{2n/(n+1)}$ such that for unboundedly large k there are sets $A_1, \ldots, A_n$ satisfying (i) and (ii) of (1.6), then, by Corollary 8 (1), E is in $\mathscr{R}_n$. Indeed, for each $n \geqslant 2$, such sets have been found ([5] and [11]).

In similar fashion, Corollary 6 (1) suggests candidates for members of $\mathscr{R}_s$ $(1 < s < \infty)$. If one could find a set E in $\mathscr{S}_{2s/(s+1)}$ such that for unboundedly large k there are sets $A_1, \ldots, A_n$ satisfying (i)–(iv) of (1.7), then it would follow from Corollary 6 (1) that E is in $\mathscr{R}_s$. We describe below a construction which for $n = 2$ and $s = 3/2$ produces a class of sets among which the desired set E in $\mathscr{R}_{3/2}$ may possibly be found, perhaps after some modification of the construction. The author has not yet been able to prove that any such set E is actually in $\mathscr{S}_{6/5}$. We also indicate briefly below how the construction may be modified for some other choices of n and s. The question of which n and s are possible in the construction reduces to an essentially combinatorial question. While the discussion to follow raises more questions than it answers, the questions and a few of the facts brought out seem quite interesting.

We need the concept of a "dissociate" set. $A \subset \Gamma$ is said to be *dissociate* if $\sum_{i=1}^{N} \delta_i \gamma_i = 0$ with δ_i in $\{-2, -1, 0, 1, 2\}$ and γ_i in A $(i = 1, \ldots, N)$

implies $\delta_1 = \ldots = \delta_N = 0$. The set $\{3^j: j = 1, 2, \ldots\}$ is an example of a dissociate subset of Z.

Let A be a countably infinite dissociate subset of Γ and partition A into two infinite subsets A_1 and A_2. E will turn out to be a subset of $A_1 + A_2$. (We remark that $A_1 + A_2$ itself is known to be in $\mathscr{S}_{4/3}$ ([5] and [10]).) The map $\varrho\colon A_1 \times A_2 \to A_1 + A_2$ defined by $\varrho((a_1, a_2)) = a_1 + a_2$ is onto and, since $A_1 \cup A_2$ is dissociate, it is also one-to-one. For convenience we index each of A_1 and A_2 using the positive integers. Let $A_{1,1}$ consist of the first $k(1) = 2^2 + 2 + 1$ elements of A_1 and let $A_{2,1}$ consist of the first $k(1)$ elements of A_2. Having picked $k(j) = (2^j)^2 + 2^j + 1$ elements for A_{1j} and $A_{2,j}$ out of A_1 and A_2, respectively, let $A_{1,j+1}$ and $A_{2,j+1}$ consist of the next $k(j+1) = (2^{j+1})^2 + 2^{j+1} + 1$ elements from A_1 and A_2, respectively. In this manner $A_{1,j}$ and $A_{2,j}$ are inductively defined for $j = 1, 2, \ldots$ Note that since ϱ is one-to-one, $\nu(A_{1,j} + A_{2,j}) = \nu(A_{1,j} \times A_{2,j}) = [k(j)]^2$. E will turn out to be a subset of $\bigcup_{j=1}^{\infty} (A_{1,j} + A_{2,j})$. Now by Corollary 6 (1) and the properties of the construction so far, if E is chosen so that $\nu[(A_{1,j} + A_{2,j}) \cap E] \geqslant [k(j)]^{3/2}$, we will have $E \notin \bigcup_{1 \leqslant p < 6/5} \mathscr{S}_p$. Since $\varrho\colon A_{1,j} \times A_{2,j} \to A_{1,j} + A_{2,j}$ is one-to-one, instead of thinking of choosing at least $[k(j)]^{3/2}$ elements for E from $A_{1,j} + A_{2,j}$, we may think of choosing elements from $A_{1,j} \times A_{2,j}$. There is of course no problem in choosing that many elements from $A_{1,j} \times A_{2,j}$. However, if E is to have any chance of being in $\mathscr{S}_{6/5}$, we must make sure that the subsets D_j of $A_{1,j} \times A_{2,j}$ are chosen so that $D \equiv \bigcup_{j=1}^{\infty} D_j$ does not contain arbitrarily large "squares" $D' \times D''$ from $A_1 \times A_2$ ($D' \times D''$ is a "square" from $A_1 \times A_2$ if $D' \subset A_1$, $D'' \subset A_2$ and $\nu(D_1) = \nu(D_2)$); for, if D did contain arbitrarily large squares from $A_1 \times A_2$, then Corollary 8 (1) would imply that E is in $\mathscr{R}_2$ and, in particular, E is not in $\mathscr{S}_{6/5}$. It is possible to choose D_j so that $\nu(D_j) \geqslant [k(j)]^{3/2}$ but D does not contain arbitrarily large squares; in fact, it can be done so that D contains no 2 by 2 squares. However, this fact and related facts and questions appear to lie deeper than one might at first think.

The key to the choice of the D_j's is the following combinatorial theorem guaranteeing the existence of finite projective planes of unboundedly large orders ([15], Theorem 4.2, p. 93):

Let $m = q^a$, where q is a prime and a is a positive integer. Then there exists a finite projective plane Π of order m. (We will use the case $q = 2$.)

It is known ([15], Theorem 3.2, p. 91) that the projective plane of order m has $m^2 + m + 1$ points and $m^2 + m + 1$ lines and that each line contains exactly $m + 1$ points. One may associate an $m^2 + m + 1$ square incidence matrix (C_{ij}) with such a projective plane by letting $C_{ij} = 1$ or 0 according as the jth point is on the ith line or not. $(m^2 + m + 1)(m + 1)$ 1's will appear in the incidence matrix. Further one of the axioms for projective planes ([15], p. 89) insures that any two distinct points are on exactly one line. Translating this in terms of our incidence matrix, we see that the incidence matrix contains no 2 by 2 submatrix (or "square") consisting entirely of 1's.

Now $A_{1,j} \times A_{2,j}$ is in one-to-one correspondence with the incidence matrix of the projective plane of order 2^j. We choose for D_j the members of $A_{1,j} \times A_{2,j}$ which are associated via this correspondence with the 1's of the incidence matrix. This completes the construction.

In summary, we formed E by choosing $[2^j + 1][(2^j)^2 + 2^j + 1] \geqslant [k(j)]^{3/2}$ elements out of each of $A_{1,j} + A_{2,j}$ $(1 \leqslant j < \infty)$. The choice of elements was made using the incidence matrix of the projective plane of order 2^j and the fact that this incidence matrix is in one-to-one correspondence with $A_{1,j} \times A_{2,j}$ which, in turn, is in one-to-one correspondence with $A_{1,j} + A_{2,j}$.

Remarks. 1. The above construction was done for $n = 2$ and $s = 3/2$. For $1 < s < 3/2$ one may simply use the same construction but pick fewer points.

2. What about the case $n = 2$ and $3/2 < s < 2$? Results of W. G. Brown [2] imply that for unboundedly large k there are finite graphs with k vertices and with greater than or equal to $\frac{1}{4}k^{5/3}$ edges which contain no "Thomsen subgraph". The incidence matrix of a graph with k vertices is a k by k 0–1 matrix. It is easy to show that if a graph contains no Thomsen subgraph then its incidence matrix contains no 3 by 3 submatrix (or "square") of 1's. Hence for $n = 2$ and $s = 5/3$ the earlier construction can be modified using the incidence matrices from Brown's graphs instead of the incidence matrices of the projective plane. For $n = 2$ and $5/3 < s < 2$ it seems likely that appropriate 0–1 matrices exist to allow one to carry out a construction similar to the earlier one, but the author has not been able to show this. Specifically, the problem is the following: Given $5/3 < s < 2$, does there exist a positive integer C and a $\delta > 0$ such that there exists an infinite number of positive integers k for which there are k by k 0–1 matrices containing greater than or equal to δk^s 1's but containing no C by C submatrix consisting entirely of 1's? This problem is related to, but seems simpler than, the *problem of Zarankiewicz* ([7] and [8]), an unsolved problem of some standing in combinatorics. The above problem and the problem of Zarankiewicz have natural extentions to the case $n > 2$ [1].

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[1] The problem stated here has been solved in [6], p. 60. See also the thesis of S. Roman, University of Washington, 1975.

3. When $n > 2$ and $s = n-1/2$, the results of the earlier construction can be "stacked" to produce a suitable set E. Specifically, in the earlier construction partition A into n infinite subsets $A_1, \ldots, A_n$. For each j, choose $A_{1,j}, \ldots, A_{n,j}$ just as $A_{1,j}$ and $A_{2,j}$ were chosen before. Pick D_j from $A_{1,j} \times A_{2,j}$ as above but then let $D = \bigcup_{j=1}^{\infty} [D_j \times A_{3,j} \times \ldots \times A_{n,j}]$. We have $\nu(D) \geqslant [k(j)]^{n-1/2}$ but D contains no $2 \times 2 \times \ldots \times 2$ hypercube.

4. Let E be the set that comes out of the initial construction above. The missing link is a proof that E (or some modified version of E) is in $\mathscr{S}_{6/5}$. The proofs that worked in the analogous place in the earlier setting ([5] and [11]) seem to fail miserably here. On the brighter side, the projective planes that are the key to the development are constructively given [15] so that one knows, in some sense, exactly where the points are.

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