STUDIA MATHEMATICA, T. LVIII. (1976)

On a.e. convergence of expansion with respect to a bounded orthonormal system of polygonals

by

F. SCHIPP (Budapest)

Abstract. It is proved that the bounded orthonormal system of polygonals introduced by Z. Ciesielski is an a.e. convergence system.

1. Introduction. In paper [1] by Z. Ciesielski was introduced a uniformly bounded orthonormal system of polygonals. This orthonormal set $\boldsymbol{C} = \{c_n: n \in \boldsymbol{N}\}$ ($\boldsymbol{N} = \{0, 1, 2, \ldots\}$) has some of the properties of the Walsh and trigonometric systems. The relation between the set $\boldsymbol{C}$ and the Franklin set is the same as between Walsh and Haar systems. In the present paper we prove that C is an a.e. convergence system. This follows from some property of the Franklin system and from the fact that the Walsh system is a convergence system. The method used here is the same as in [4].

2. Preliminaries and notation. The Walsh-Paley functions are defined as follows:

$$w_n(x) = \exp\Big(i\pi \sum_{k=0}^{\infty} n_k x_k\Big) \tag{1}$$

$$\Big(n = 1 + \sum_{k=0}^{\infty} n_k 2^k \in \boldsymbol{N},\ x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)} \in [0, 1),\ x_k, n_k \in \{0, 1\},\ i = \sqrt{-1}\Big).$$

The well-known relation between the Haar system $\{\chi_n: n \in \boldsymbol{N}\}$ and the Walsh–Paley system can be stated in the form

$$w_{2^m+k}(x) = 2^{-m/2} \sum_{j=1}^{2^m} w_k\big((j-1)2^{-m}\big)\chi_{2^m+j}(x) \tag{2}$$

$$(x \in [0, 1),\ 1 \leqslant k \leqslant 2^m,\ m \in \boldsymbol{N}).$$

In general, to every orthonormal set $F = \{F_n: n \in \boldsymbol{N}\}$ with elements defined on the interval $[0, 1]$ we can construct in this way a new orthonormal set $G = \{G_n \colon n \in \boldsymbol{N}\}$ as follows:

$$G_0 = F_0, \quad G_1 = F_1,$$

$$G_{2^m+k}(x) = 2^{-m/2}\sum_{j=1}^{2^m} w_k\big((j-1)2^{-m}\big)F_{2^m+j}(x), \tag{3}$$

where $m \in \boldsymbol{N}$, $1 \leqslant k \leqslant 2^m$ and $x \in [0, 1]$.

If $F_n = f_n$ $(n \in \boldsymbol{N})$ are the Franklin orthonormal functions (see [2]), then the system G is equal to the system C introduced by Ciesielski.

The following theorems will be used later:

Theorem A ([2], [3]). (a) *For all $m \in \boldsymbol{N}$ and $x \in [0, 1]$ we have*

$$\sum_{k=1}^{2^m} |f_{2^m+k}(x)| \leqslant 2^5\sqrt{3}\,\sqrt{2^m}. \tag{4}$$

(b) *For $1 \leqslant k \leqslant 2^m$ and $m \in \boldsymbol{N}$ the inequality*

$$\|f_{2^m+k}\|_1 \leqslant 6\sqrt{3}\,\sqrt{2^{-m}} \tag{5}$$

holds.

(c) *The Fourier–Franklin series of integrable functions converges a.e.* (See [2], Theorem 5, Lemma 5 and Lemma 7, and [3], Theorem 4.)

Theorem B [5]. *Let a_n $(n \in \boldsymbol{N})$ real numbers. Then*

$$\int_0^1 \sup_n \Big|\sum_{k=1}^{n} a_k w_k(x)\Big|^2 dx \leqslant A^2\Big(\sum_{k=1}^{\infty}|a_k|^2\Big), \tag{6}$$

where the constant A is independent of $(a_n, n \in \boldsymbol{N})$.

3. The main inequality. Let

$$M_m(x) = \max_{1 \leqslant k \leqslant 2^m}\Big|\sum_{j=2^m+1}^{2^m+k} a_j G_j(x)\Big| \qquad (m \in \boldsymbol{N},\ x \in [0, 1]),$$

where the a_j's are real numbers. We shall prove the following

Theorem 1. *If the system $F = \{F_n \colon n \in \boldsymbol{N}\}$ satisfies the conditions*

$$\sum_{j=1}^{2^m}|F_{2^m+j}(x)| \leqslant C\sqrt{2^m}, \qquad \|F_{2^m+k}\|_1 \leqslant C\sqrt{2^m} \tag{7}$$

$$(x \in [0, 1],\ 1 \leqslant k \leqslant 2^m,\ m \in \boldsymbol{N}),$$

then

$$\|M_m\|_2 \leqslant CA\Big(\sum_{k=2^m+1}^{2^{m+1}}|a_k|^2\Big) \qquad (m \in \boldsymbol{N}). \tag{8}$$

Proof. Let us introduce the functions

$$K_m(x, t) = 2^{m/2}F_{2^m+k}(x) \qquad ((k-1)2^{-m} \leqslant t < k2^{-m},\ 1 \leqslant k \leqslant 2^m,\ m \in \boldsymbol{N}).$$

Then by (7)

$$\sup_t \int_0^1 |K_m(x, t)|\,dx \leqslant C, \qquad \sup_x \int_0^1 |K_m(x, t)|\,dt \leqslant C \tag{9}$$

and from (3) it follows that

$$G_{2^m+k}(x) = \int_0^1 w_k(t)K_m(x, t)\,dt. \tag{10}$$

Let

$$N_m(t) = \max_{1 \leqslant n \leqslant 2^m}\Big|\sum_{k=1}^{n} a_{2^m+k}w_k(t)\Big| \qquad (m \in \boldsymbol{N},\ t \in [0, 1]).$$

Then by (10)

$$M_m(x) \leqslant \int_0^1 N_m(t)\,|K_m(x, t)|\,dt \qquad (m \in \boldsymbol{N})$$

and for an arbitrary $g \in L^2(0, 1)$ with $\|g\|_2 \leqslant 1$ by Hölder's inequality we have

$$\int_0^1 gM_m \leqslant \int_0^1 N_m(t)\Big(\int_0^1 g(x)\,|K_m(x, t)|\,dx\Big)\,dt \leqslant \|N_m\|_2 I,$$

where

$$I = \Big(\int_0^1\Big(\int_0^1 g(x)\,|K_m(x, t)|\,dx\Big)^2 dt\Big)^{1/2}.$$

We apply the well-known equality

$$I = \sup_{\|h\|_2 \leqslant 1}\int_0^1 h(t)\Big(\int_0^1 g(x)\,|K_m(x, t)|\,dx\Big)\,dt.$$

Using the inequality $uv \leqslant (u^2+v^2)/2$ by (9) for $\|h\|_2 \leqslant 1$ we have

$$\int_0^1\int_0^1 h(t)g(x)\,|K_m(x, t)|\,dt\,dx$$
$$\leqslant \tfrac{1}{2}\int_0^1 h^2(t)\Big(\int_0^1|K_m(x, t)|\,dx\Big)\,dt + \tfrac{1}{2}\int_0^1 g^2(x)\Big(\int_0^1|K_m(x, t)|\,dt\Big)\,dx \leqslant C,$$

thus $I \leqslant C$ and $\int_0^1 gM_m \leqslant C\|N_m\|_2$ for every $g \in L^2(0, 1)$ with $\|g\|_2 \leqslant 1$. This and Theorem B imply (8).

4. Convergence theorems. Denote by $S_n(f; F)$ and $S_n(f; G)$ the nth partial sum of f with respect to the systems F and G, respectively. Since the matrices $\left(2^{-m/2}w_k\left((l-1)2^{-m}\right)\right)_{k,l=1}^{2^m}$ $(m \in \boldsymbol{N})$ are orthogonal, we have

$$\sum_{n=0}^{2^m} F_n(t)F_n(x) = \sum_{n=0}^{2^m} G_n(t)G_n(x),$$

thus $S_{2^m}(f; F) = S_{2^m}(f; G)$.

From inequality (8) it follows that

$$\sum_{n=0}^{\infty} a_n^2 < \infty \tag{11}$$

implies $\lim M_m = 0$ a.e. This gives

THEOREM 2. *If the orthonormal system F satisfies conditions* (7) *and if for every $f \in L^2(0,1)$ $S_{2^m}(f; F)$ converges a.e., then G is a convergence system, i.e., for every sequence $(a_n, n \in \boldsymbol{N})$ with property* (11) *the series $\sum_{n=0}^{\infty} a_n G$ converges a.e.*

Since by Theorem A for the Franklin system the conditions of Theorem 2 are satisfied, we have

THEOREM 3. *The system C is an a.e. convergence system.*

References

[1] Z. Ciesielski, *A bounded orthonormal system of polygonals*, Studia Math. 31 (1968), pp. 339–346.
[2] — *Properties of the orthonormal Franklin system*, ibid. 23 (1963), pp. 141–157.
[3] — *Properties of the orthonormal Franklin system*, II, ibid. 27 (1966), pp. 289–323.
[4] F. Schipp, *Über die Konvergenz von Reihen nach Productsystemen*, Acta Sci. Math. 35 (1973), pp. 13–16.
[5] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh–Fourier series*, Arkiv för Math. 7 (1968), pp. 551–570.

MATHEMATICAL INSTITUTE OF EÖTVÖS LARÁND UNIVERSITY
BUDAPEST

Received June 10, 1975 (1026)

STUDIA MATHEMATICA, T. LVIII. (1976)

On maximal ideals in commutative m-convex algebras

by
W. ŻELAZKO (Warszawa)

Abstract. We give a characterization of commutative complete unital m-convex algebras in which all maximal ideals are of codimension one. We describe also situations in which there exist dense maximal ideals (of finite or infinite codimension).

All algebras in this paper are commutative complex complete locally convex and multiplicatively convex algebras (shortly: m-convex algebras). We shall also assume the existence of the unit element, denoted by e. If A is such an algebra, then its topology is given by means of a family $(\|x\|_a)$ of submultiplicative seminorms, i.e., homogeneous seminorms satisfying

$$\|xy\|_a \leqslant \|x\|_a \|y\|_a$$

or all $x, y \in A$ and all indexes a, and

$$\|e\|_a = 1$$

for all a. Moreover, if (x_ι) is a Cauchy net, i.e., if $\|x_\iota \to x_{\iota'}\|_a \to 0$ for each fixed a, then there exists an $x_0 \in A$ such that $\lim \|x_\iota - x_0\|_a = 0$ for each a. Every such algebra is an inverse limit of a directed system of Banach algebras. We shall describe shortly some facts on these algebras. The details can be found in paper [2].

Let A be a commutative m-convex algebra. We denote by $\mathfrak{M}(A)$ its maximal ideal space, i.e., the space of all non-zero multiplicative linear continuous functionals on A, provided with the weak star topology. We denote by $\mathfrak{M}^{\#}(A)$ the space of all non-zero multiplicative linear functionals on A, also provided with the weak star topology, so that $\mathfrak{M}(A)$ is a subspace of $\mathfrak{M}^{\#}(A)$. Let us remark that the topology of $\mathfrak{M}^{\#}(A)$ depends only upon algebraic (linear) structure of A and remains unchanged under any modification of the topology of A, though, of course the space $\mathfrak{M}(A)$ depends upon this topology. If $x \in A$, then its Gelfand transform is given by

$$x^{\wedge}(f) = f(x), \quad f \in \mathfrak{M}(A).$$

It is a continuous function on $\mathfrak{M}(A)$. The same formula defines also a continuous function on $\mathfrak{M}^{\#}(A)$, being an extension of the Gelfand transform