4. Convergence theorems. Denote by $S_n(f;\,F)$ and $S_n(f;\,G)$ the nth partial sum of f with respect to the systems F and G, respectively. Since the matrices $\left(2^{-m/2}w_k\left((l-1)2^{-m}\right)\right)_{k,l=1}^{2^m}$ $(m\in \boldsymbol{N})$ are orthogonal, we have

$$\sum_{n=0}^{2^m} F_n(t)F_n(x) = \sum_{n=0}^{2^m} G_n(t)G_n(x),$$

thus $S_{2^m}(f;\,F) = S_{2^m}(f;\,G)$.

From inequality (8) it follows that

$$\sum_{n=0}^{\infty} a_n^2 < \infty \tag{11}$$

implies $\lim M_m = 0$ a.e. This gives

THEOREM 2. *If the orthonormal system F satisfies conditions* (7) *and if for every $f\in L^2(0,1)$ $S_{2^m}(f;\,F)$ converges a.e., then G is a convergence system, i.e., for every sequence $(a_n,\, n\in \boldsymbol{N})$ with property* (11) *the series $\sum_{n=0}^{\infty} a_n G$ converges a.e.*

Since by Theorem A for the Franklin system the conditions of Theorem 2 are satisfied, we have

THEOREM 3. *The system C is an a.e. convergence system.*

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On maximal ideals in commutative m-convex algebras

by

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Abstract. We give a characterization of commutative complete unital m-convex algebras in which all maximal ideals are of codimension one. We describe also situations in which there exist dense maximal ideals (of finite or infinite codimension).

All algebras in this paper are commutative complex complete locally convex and multiplicatively convex algebras (shortly: m-convex algebras). We shall also assume the existence of the unit element, denoted by e. If A is such an algebra, then its topology is given by means of a family $(\|x\|_\alpha)$ of submultiplicative seminorms, i.e., homogeneous seminorms satisfying

$$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha$$

or all $x, y\in A$ and all indexes α, and

$$\|e\|_\alpha = 1$$

for all α. Moreover, if (x_ι) is a Cauchy net, i.e., if $\|x_\iota \to x_{\iota'}\|_\alpha \to 0$ for each fixed α, then there exists an $x_0\in A$ such that $\lim \|x_\iota - x_0\|_\alpha = 0$ for each α. Every such algebra is an inverse limit of a directed system of Banach algebras. We shall describe shortly some facts on these algebras. The details can be found in paper [2].

Let A be a commutative m-convex algebra. We denote by $\mathfrak{M}(A)$ its maximal ideal space, i.e., the space of all non-zero multiplicative linear continuous functionals on A, provided with the weak star topology. We denote by $\mathfrak{M}^{\#}(A)$ the space of all non-zero multiplicative linear functionals on A, also provided with the weak star topology, so that $\mathfrak{M}(A)$ is a subspace of $\mathfrak{M}^{\#}(A)$. Let us remark that the topology of $\mathfrak{M}^{\#}(A)$ depends only upon algebraic (linear) structure of A and remains unchanged under any modification of the topology of A, though, of course the space $\mathfrak{M}(A)$ depends upon this topology. If $x\in A$, then its Gelfand transform is given by

$$x^{\wedge}(f) = f(x), \quad f\in \mathfrak{M}(A).$$

It is a continuous function on $\mathfrak{M}(A)$. The same formula defines also a continuous function on $\mathfrak{M}^{\#}(A)$, being an extension of the Gelfand transform which will be also denoted by $x^\wedge$. One proves that

$$x^\wedge(\mathfrak{M}(A)) = x^\wedge(\mathfrak{M}^{\#}(A)), \quad x \in A, \tag{1}$$

and this set is equal to the algebraic spectrum $\sigma(x)$ of the element $x \in A$, i.e., to the set

$$\sigma(x) = \{\lambda \in C\colon x - \lambda e \text{ is non-invertible in } A\}.$$

If U is an open subset of the complex plane containing the spectrum $\sigma(x)$ of an element $x \in A$, and if φ is a function holomorphic in U, then φ operates on x in the sense that there exists a (unique) continuous unital homomorphism h_x from the algebra of all functions holomorphic in U (it is an m-convex algebra with the compact open topology and pointwise algebra operations) into A, sending the function $\varphi(z) \equiv z$ onto the element x. If we denote by $\varphi(x)$ the value $h_x(\varphi)$ we have

$$[\varphi(x)]^\wedge(f) = \varphi(x^\wedge(f)) \tag{2}$$

for each $f \in \mathfrak{M}(A)$. In particular, all entire function soperate on all elements of A, and if for such a function its Taylor expansion is $\varphi(z) = \sum_0^\infty a_n z^n$, then $\varphi(x) = \sum_0^\infty a_n x^n$ for each $x \in A$.

The Gelfand–Mazur theorem for m-convex algebras implies that every closed maximal ideal of A is of codimension 1 and it is the kernel of a functional $f \in \mathfrak{M}(A)$. Since for any closed ideal $I \subset A$ the quotient algebra is also an m-convex algebra, it follows that every closed ideal is contained in a closed maximal ideal. An m-convex algebra is said to be a *Q-algebra* if the set of all its invertible elements is open, or equivalently, if it has a non-void interior. If A is a Q-algebra, then every its maximal ideal is closed and so it is of codimension 1. So $\mathfrak{M}(A) = \mathfrak{M}^{\#}(A)$, moreover, this is a compact space. In the case of a barrelled algebra A (but not in general) the converse is also true, i.e., if $\mathfrak{M}(A)$ is compact, then A is a Q-algebra. The latter is equivalent also to the compactness of the spectra $\sigma(x)$ of all elements $x \in A$ (again provided A is a barrelled space).

We say that A is an *m-convex B_0-algebra*, if A is, moreover, metrizable. Since in many examples of non-Q-algebras we found dense maximal ideals of infinite codimension, we asked in [3] the following question:

Is it true that an m-convex B_0-algebra has a dense maximal ideal of infinite codimension if and only if it is not a Q-algebra?

In this paper we give a positive answer to this question. The answer is also positive in a more general case, namely if we replace here a B_0-algebra by a barrelled algebra, but it fails in the general case. Moreover, even in the general case, the answer is "almost positive", namely we prove that all maximal ideals are of codimension one if and only if there exists on A a stronger (not necessarily strictly stronger) complete m-convex Q-algebra topology. This is contained in our main result. As a corollary we see (Corollary 1) that A possesses a dense maximal ideal of infinite codimension if and only if it possesses an element with unbounded spectrum. We discuss also the problem of possessing a dense maximal ideal (not necessarily of infinite codimension). Such an ideal always exists if the space $\mathfrak{M}(A)$ is non-compact (Proposition 1), but it may exist also otherwise. So in Proposition 2 we give two conditions equivalent to the closedness of all maximal ideals, under the assumption that the space $\mathfrak{M}(A)$ is compact. One of them is the equality of the algebraic and topological joint spectrum for any n-tuple of elements $x_1, \ldots, x_n \in A$.

I would like to express my gratitude to Dr Z. Słodkowski, who communicated me a part of the proof of the main theorem (implication (iii) ⇒ (iv)) and suggested a fruitful Example 3 which disproved a previous conjecture stating that A possesses a dense ideal if and only if the space $\mathfrak{M}(A)$ is non-compact.

We shall prove now our main result.

THEOREM. *Let A be a complete, complex, unital, m-convex algebra. Then the following conditions are equivalent:*

(i) *Every maximal ideal of A is of codimension 1.*

(ii) *For each element $x \in A$ the spectrum $\sigma(x)$ is bounded.*

(iii) *For each element $x \in A$ the spectrum $\sigma(x)$ is compact.*

(iv) *A is a complete m-convex Q-algebra under a topology stronger than the original one.*

(v) *A is a complete m-convex Q-algebra under some topology.*

(vi) *The space $\mathfrak{M}^{\#}(A)$ of all non-zero multiplicative-linear functionals on A is compact in the weak star topology.*

Proof. (i) ⇒ (ii). We have to show that if for some $x_0 \in A$ the spectrum $\sigma(x_0)$ is unbounded, then A possesses a maximal ideal of infinite codimension. So suppose that for some $x_0 \in A$ there exists a sequence $z_n \in \sigma(x_0)$ of complex numbers tending to an infinity. By the classical theorem of Weierstrass, there exists a sequence $\varphi_n(z)$ of entire functions such that

$$\varphi_n^{-1}(0) = \{z_n, z_{n+1}, z_{n+2}, \ldots\}, \quad n = 1, 2, \ldots \tag{3}$$

We put

$$x_n = \varphi_n(x_0), \quad n = 1, 2, 3, \ldots$$

Let I be the smallest ideal of A containing all the elements $x_1, x_2, \ldots$ It consists of all finite sums of the form $\sum_{i=1}^n x_i y_i$, $y_i \in A$. We shall show that I is a proper ideal in A. If not, then $e \in I$, $e = \sum_1^n x_i y_i$ for some elements $y_1, \ldots, y_n \in A$. Take any functional $f \in \mathfrak{M}(A)$ such that $f(x_0) = z_n$. For $1 \leqslant i \leqslant n$ we have

$$0 = \varphi_i(z_n) = \varphi_i\big(f(x_0)\big) = f\big(\varphi_i(x_0)\big) = f(x_i)$$

and it gives contradiction since

$$1 = f(e) = f\Big(\sum_{i=1}^n x_i y_i\Big) = \sum_{i=1}^n f(x_i) f(y_i) = 0.$$

Thus I is a proper ideal of A.

Let now M be any maximal ideal of A containing the ideal I. We shall show that its codimension is infinite. If not, then by the classical Frobenius theorem its codimension is 1, and so M is the kernel of a functional $F \in \mathfrak{M}^{\#}(A)$. Since $x_i \in I \subset M$, we have $F(x_i) = 0$ for $i = 1, 2, \ldots$

Let E be the algebra of all entire functions of one complex variable. It is known (cf. e.g. [3]) that every multiplicative linear functional on E is the point evaluation, i.e., if Φ is such a functional, then there exists a complex number z_0 such that

$$\Phi(\varphi) = \varphi(z_0)$$

for each $\varphi \in E$.

We put now

$$\Phi(\varphi) = F\big(\varphi(x_0)\big)$$

for each $\varphi \in E$. It is clearly a multiplicative linear functional on E, so there is a complex number z_0 such that

$$F\big(\varphi(x_0)\big) = \varphi(z_0)$$

for each $\varphi \in E$. Taking as φ the functions φ_i we see that

$$\varphi_i(z_0) = F\big(\varphi_i(x_0)\big) = F(x_i) = 0.$$

So all functions φ_i have a common zero z_0, which contradicts relation (3).

Thus the condimension of M is infinite.

(ii) $\Rightarrow$ (iii). We have to show that if for some $x \in A$ the spectrum $\sigma(x)$ is non-closed in C, then there is an element in A with an unbounded spectrum. So let $\lambda \in \overline{\sigma(x)} \setminus \sigma(x)$. Thus $x - \lambda e$ is an invertible element in A and, by formulas (1) and (2) the spectrum of the element $\bar{y} = (x - \lambda e)^{-1}$ is unbounded.

(iii) $\Rightarrow$ (iv). Let $(\|x\|_a)$ be a system of submultiplicative seminorms in A giving its topology, denoted by τ. We put

$$|x|_0 = \sup\{|\lambda| : \lambda \in \sigma(x)\} = \sup_{f \in \mathfrak{M}(A)} |x\hat{}\,(f)|$$

for each $x \in A$. It is clearly a submultiplicative seminorm on A, possibly discontinuous (cf. Example 1). We define on A a new system of submultiplicative seminorms

$$\|x\|_a^* = \max(|x|_0, \|x\|_a).$$

The system $(\|x\|_a^*)$ defines on A a new m-convex topology, denoted by τ^*. The topology τ^* is certainly stronger than the topology τ. We shall show that the algebra A complete in the topology τ^*. So let (x_ι) be a Cauchy net in the τ^* topology. Thus (x_ι) is also a Cauchy net in the topology τ, i.e., it τ-converges to an element $x_0 \in A$, and, moreover, the net $x_\iota\hat{}$ of Gelfand's transforms is uniformly convergent to a continuous function $u(f)$ defined on $\mathfrak{M}(A)$. In order to prove that A is τ^*-complete it is sufficient to show that $u(f) = x_0\hat{}\,(f)$. But for each $f \in \mathfrak{M}(A)$ it is $x_\iota\hat{}\,(f) = f(x_\iota) \to f(x_0) = x_0\hat{}\,(f)$, and at the same time $x_\iota\hat{}\,(f) \to u(f)$. So we are done. Clearly, (A, τ^*) is a Q-algebra since the neighbourhood of the unit element, given by $\{x \in A : |x - e|_0 < 1\}$, consists entirely of invertible elements.

Implication (iv) $\Rightarrow$ (v) is trivial and (v) $\Rightarrow$ (vi) follows from the fact that for a Q-algebra $\mathfrak{M}(A) = \mathfrak{M}^{\#}(A)$ is a compact space (the topology of $\mathfrak{M}^{\#}(A)$ depends only upon the linear space structure of A and not upon its topology).

To conclude the proof we observe that implication (vi) $\Rightarrow$ (ii) follow- immediately from formula (1) and implication (v) $\Rightarrow$ (i) follows from a suitable property of Q-algebras (all maximal ideals closed, therefore of codimension 1).

COROLLARY 1. *The algebra A possesses a dense maximal ideal of infinite codimension if and only if it possesses an element with unbounded spectrum.*

COROLLARY 2. *If the space $\mathfrak{M}(A)$ is compact, then the space $\mathfrak{M}^{\#}(A)$ is compact, too.*

We shall show now that the topology making of A a Q-algebra (as in condition (iv) of the theorem) can be essentially stronger than the original one.

EXAMPLE 1 ([2], Example 3.8). Let A be the algebra of all complex continuous functions on the closed unit interval $[0, 1]$. For any countable compact subset $a \subset [0, 1]$, and $x \in A$ we put

$$\|x\|_a = \sup_{t \in a} |(xt)|.$$

So A is a complete m-convex algebra under the system $(\|x\|_a)$. It is not a Q-algebra, but it satisfies conditions (i)–(vi) of the theorem. The stronger Q-algebra topology considered in the proof of (iii) $\Rightarrow$ (iv) is the usual Banach algebra topology on $C[0, 1]$.

The next example shows that condition (vi) cannot be replaced by the compactness of the space $\mathfrak{M}(A)$. It shows also that the converse of Corollary 2 fails to be true.

EXAMPLE 2 ([2], Example 3.7). Let T be the space of all ordinals less than Ω, the first uncountable ordinal, with the order topology, and let $A = C(T)$ with the compact open topology. Since every continuous function on T must be constant beginning from some ordinal on; then $\mathfrak{M}^{\#}(A)$ is the one-point compactification of the non-compact space $\mathfrak{M}(A) = T$, and the algebra A satisfies conditions (i)–(vi).

Both algebras in these examples are non-barrelled algebras. For a barrelled algebra we have a stronger version of our theorem:

COROLLARY 3. *Let A be a commutative, complex, unital complete m-convex algebra, which is a barrelled space. Then the following conditions are equivalent:*

(i) *Every maximal ideal of A is of codimension one.*

(ii) *Every maximal ideal of A is closed.*

(iii) *The spectrum $\sigma(x)$ of each element $x \in A$ is bounded.*

(iv) *The spectrum $\sigma(x)$ of each element $x \in A$ is compact.*

(v) *A is a Q-algebra.*

(vi) *The space $\mathfrak{M}(A)$ is compact.*

The proof follows from the theorem and from the fact that if A is a barrelled algebra, then (v) is equivalent to (vi) and to (iii) (cf. [2], Theorem 13.6).

As it was seen in Examples 1 and 2, conditions (ii), (v) and (vi) of the corollary fail in general to be equivalent with condition (i).

We shall discuss now the problem when an m-convex algebra possesses a dense ideal.

PROPOSITION 1. *Let A be a commutative, complex, unital, complete m-convex algebra. If the maximal ideal space $\mathfrak{M}(A)$ is non-compact, then A possesses a dense maximal ideal.*

Proof. If A possesses an element x having unbounded spectrum, then the result follows by Corollary 1. If all spectra of elements of A are bounded, then the space $\mathfrak{M}^{\#}(A)$ is compact (condition (vi) of the theorem). So $\mathfrak{M}^{\#}(A) \neq \mathfrak{M}(A)$ and there is an element in $\mathfrak{M}^{\#}(A) \setminus \mathfrak{M}(A)$ whose kernel is a dense maximal ideal.

The following example, due to Dr Z. Słodkowski, shows that the converse result fails to be true. It is a modification of Example 1.

EXAMPLE 3. Put

$$D = \{(z_1, z_2) \in C^2 \colon \tfrac{1}{2} \leqslant |z_1|^2 + |z_2|^2 \leqslant 1\}$$

and

$$D_0 = \left\{(z_1, z_2) \in C^2 \colon \tfrac{1}{2} \leqslant |z_1|^2 + |z_2|^2 < 1\right\}.$$

Let A be the algebra of all functions continuous in D and holomorphic in D_0. For any convergent sequence $\alpha = \{(z_1^{(n)}, z_2^{(n)})\} \subset D$, $\lim (z_1^{(n)}, z_2^{(n)}) = (z_1, z_2)$, $z_1^{(n)} \neq z_i$, $i = 1, 2$, we put

$$\|x\|_\alpha = \begin{cases} \sup\limits_n |x(z_1^{(n)}, z_2^{(n)})| & \text{if} \quad (z_1, z_2) \in D \setminus D_0, \\ \max\left(\sup\limits_n |x(z_1^{(n)}, z_2^{(n)})|,\ \sup\limits_n \left| \dfrac{x(z_1, z_2^{(n)}) - x(z_1, z_2)}{z_2^{(n)} - z_2} \right|,\right. & \\ \qquad \left. \sup\limits_n \left| \dfrac{x(z_1^{(n)}, z_2) - x(z_1, z_2)}{z_1^{(n)} - z_1} \right| \right), & \text{otherwise}. \end{cases}$$

Then A is a complete m-convex algebra in the topology introduced by the system $(\|x\|_\alpha)$, for every function in the completion of A has continuous first partial derivatives in D_0 and so it is holomorphic in D_0. Since every function holomorphic in D_0 extends uniquely to a function holomorphic in

$$D_1 = \{(z_1, z_2) \in C^2 \colon |z_1|^2 + |z_2|^2 < 1\},$$

it follows that $\mathfrak{M}^{\#}(A) = \bar{D}_1$, while $\mathfrak{M}(A) = D$. Thus $\mathfrak{M}(A)$ is a compact space different from $\mathfrak{M}^{\#}(A)$.

This example exhibits also a situation in which the space $\mathfrak{M}(A)$ is not dense in $\mathfrak{M}^{\#}(A)$.

In order to decide whether an algebra A possesses a dense maximal ideal we have, in view of Proposition 1, to consider the case where the maximal ideal space $\mathfrak{M}(A)$ is compact. Here we can prove the following:

PROPOSITION 2. *Let A be a commutative, complex, complete m-convex unital algebra, and suppose that the maximal ideal space $\mathfrak{M}(A)$ is compact. Then the following conditions are equivalent:*

(i) *Any proper finitely generated ideal of A is non-dense.*

(ii) *If $x_1, \ldots, x_n \in A$ and*

$$\sum_{i=1}^{n} |f(x_i)| > 0 \tag{4}$$

for each $f \in \mathfrak{M}(A)$, then there exist elements $y_1, \ldots, y_n \in A$ such that

$$\sum_{i=1}^{n} x_i y_i = e. \tag{5}$$

(iii) *Every maximal ideal of A is closed.*

Proof. (i) $\Rightarrow$ (ii). Suppose that for $x_1, \ldots, x_n \in A$ relation (4) holds true. It means that the elements $x_1, \ldots, x_n$ are not contained in any closed maximal ideal. So the ideal generated by those elements is a dense ideal. Thus it is an improper ideal and relation (5) holds true for a certain n-tuple $(y_1, \ldots, y_n) \subset A$.

(ii) $\Rightarrow$ (iii). Implication (4) $\Rightarrow$ (5) means that if the elements $x_1, \ldots, x_n$ belong to a proper ideal I, then there exists a functional $f \in \mathfrak{M}(A)$ such that $f(x_i) = 0$, $i = 1, 2, \ldots, n$. Thus the family of closed sets

$$Z(x) = \{f \in \mathfrak{M}(A): f(x) = 0\}, \quad x \in I,$$

has the finite intersection property. By the compactness of $\mathfrak{M}(A)$, there exists a functional $f \in \bigcap_{x \in I} Z(x)$, what means that $I \subset M = f^{-1}(0)$. In particular, every maximal ideal is closed.

(iii) $\Rightarrow$ (i). Obvious, since every proper ideal of the algebra A is contained in a maximal ideal.

Condition (ii) resembles a condition in the corona problem. It says, in fact, that the algebraic joint spectrum

$$\sigma_{\mathrm{alg}}(x_1, \ldots, x_n) = \Big\{(\lambda_1, \ldots, \lambda_n) \in C^n: \sum_{i=1}^{n} (x_i - \lambda_i e) y_i \neq e, \text{ for each } n\text{-tuple } (y_1, \ldots, y_n) \subset A\Big\}$$

is equal to the topological joint spectrum

$$\sigma(x_1, \ldots, x_n) = \big\{\big(f(x_1), \ldots, f(x_n)\big) \in C^n: f \in \mathfrak{M}(A)\big\}.$$

Example 3 shows that the equality of these two spectra may fail for an algebra with a compact maximal ideal space. Moreover, in this case it follows from our theorem that the algebraic joint spectrum is equal to

$$\sigma^{\#}(x_1, \ldots, x_n) = \big\{\big(f(x_1), \ldots, f(x_n)\big) \in C^n: f \in \mathfrak{M}^{\#}(A)\big\}.$$

It would be interesting to know whether the algebraic joint spectrum equals to the topological one in the case where A is a barrelled algebra. In the case where the maximal ideal space $\mathfrak{M}(A)$ is compact the affirmative answer to this question follows from Corollary 3 and Proposition 2. In general, it is only known that both spectra coincide in the case of a B_0-algebra ([1], cf. also [3]).

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241_3	$\mathfrak{F}(\boldsymbol{N}$	$\mathfrak{F}(\boldsymbol{N})$
242^{13}	$\mathrm{conv}\, x\, \big((\mathfrak{A})\big)$	$\mathrm{conv}\, \big(x(\mathfrak{A})\big)$
243_{17}	$\forall_{\Lambda \in \Gamma}$	$\forall_{\Lambda \subset \Gamma}$
246_4	(i) $\Rightarrow$ (ii)	(ii) $\Rightarrow$ (i)
246_1	$c_\gamma(\Gamma)$	$c_{\mathfrak{n}}(\Gamma)$
247^7	(ii) $\Rightarrow$ (i)	(i) $\Rightarrow$ (ii)
247_7	Edinburgh Math. Soc.	Proc. Edinburgh Math. Soc.

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