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On a local ergodic theorem

by

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Abstract. The purpose of this paper is to prove the following ergodic theorem: Let $(X, \mathscr{M}, m)$ be a σ-finite measure space with positive measure m and let $\Gamma = \{T_t;\ t>0\}$ be a strongly continuous semigroup of linear contractions on $L_1(X, \mathscr{M}, m)$ satisfying $\|T_t f\|_\infty \leqslant \|f\|_\infty$ for all $t>0$ and $f \in L_1(X, \mathscr{M}, m) \cap L_\infty(X, \mathscr{M}, m)$. Then the following local ergodic limit

$$\lim_{b\to 0} \frac{1}{b} \int_0^b T_t f\, dt$$

exists and is finite a.e. for any $f \in L_1(X, \mathscr{M}, m)$, from which it follows that if we let $T_0 f = \lim_{b\to 0} \frac{1}{b}\int_0^b T_t f\, dt$ a.e. for all $f \in L_1(X, \mathscr{M}, m)$, then $\{T_t;\ t \geqslant 0\}$ is a strongly continuous semigroup on $[0, \infty)$.

1. Introduction. Let $(X, \mathscr{M}, m)$ be a σ-finite measure space with positive measure m and let $L_p(X) = L_p(X, \mathscr{M}, m)$, $1 \leqslant p \leqslant \infty$, be the usual (complex) Banach spaces. $L_p^+(X)$ will denote the positive cone of $L_p(X)$ consisting of the non-negative $L_p(X)$-functions. A linear operator T on $L_p(X)$ is called a *contraction* if $\|T\|_p \leqslant 1$, and *positive* if $TL_p^+(X) \subset L_p^+(X)$. A *Dunford-Schwartz operator* T is a contraction on $L_1(X)$ satisfying $\|Tf\|_\infty \leqslant \|f\|_\infty$ for all $f \in L_1(X) \cap L_\infty(X)$. It follows from the Riesz convexity theorem (cf. [3], Theorem VI.10.11) that a Dunford–Schwartz operator T may be considered to be a contraction on each $L_p(X)$, $1 \leqslant p < \infty$.

Let $\Gamma = \{T_t;\ t>0\}$ be a strongly continuous semigroup of linear contractions on $L_1(X)$, i.e., each T_t is a linear contraction on $L_1(X)$, $T_t T_s = T_{t+s}$ for all $t, s>0$, and $\lim_{t\to s} \|T_t f - T_s f\|_1 = 0$ for all $s>0$ and $f \in L_1(X)$. It then follows (cf. [8], Section 4) that for any $f \in L_1(x)$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathscr{M}$, such that $T_t f(x)$ belongs to the equivalence class of $T_t f$ for each $t>0$. Moreover, there exists a set $N(f)$ with $m(N(f)) = 0$, dependent on f but independent of t, such that if $x \notin N(f)$, then $T_t f(x)$ is integrable on every finite interval (a, b) and the integral $\int_a^b T_t f(x)\, dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f\, dt$.

In [1] Akcoglu and Chacon proved that if all the T_t are positive, then the following local ergodic limit

$$\lim_{b\to 0}\frac{1}{b}\int_0^b T_t f(x)\,dt \tag{1}$$

exists and is finite a.e. for any $f\in L_1(X)$. On the other hand, Kubokawa [7] proved that if T_t converges strongly to the identity operator as $t\to 0$, then the limit (1) exists and is finite a.e. for any $f\in L_1(X)$. See also [2], [4], [5], [6], [8] and [9].

Our aim in this paper is to prove that if all the T_t are Dunford–Schwartz operators, then the limit (1) exists and is finite a.e. for any $f\in L_1(X)$, and T_t converges strongly as $t\to 0$.

Here I would like to express my hearty thanks to the referee by whom the proof of Lemma 1 was simplified.

2. The theorem. Throughout this section it will be assumed that $\Gamma=\{T_t;\, t>0\}$ is a strongly continuous semigroup of Dunford–Schwartz operators on $L_1(X)$. (We note that Γ may be considered to be a strongly continuous semigroup of linear contractions on each $L_p(X)$, $1\leqslant p<\infty$.) Then we have the following

THEOREM. (i) *For any $f\in L_1(X)$ the local ergodic limit* (1) *exists and is finite a.e.* (ii) *If $(T_0 f)(x)=\lim_{b\to 0}\frac{1}{b}\int_0^b T_t f(x)\,dt$ a.e. for all $f\in L_1(X)$, then $\{T_t;\, t\geqslant 0\}$ is a strongly continuous semigroup on $[0,\infty)$.*

For the proof of the theorem we need the following two lemmas.

LEMMA 1. *Let H be a Hilbert space and let $\{V_t;\, t>0\}$ be a strongly continuous semigroup of linear contractions on H. Then V_t converges strongly as $t\to 0$, and hence if we let $V_0=\text{strong-}\lim_{t\to 0} V_t$, then $\{V_t;\, t\geqslant 0\}$ is a strongly continuous semigroup on $[0,\infty)$.*

Proof. Let $\tilde H=\overline{\bigcup_{t>0} V_t H}$. If $f\in V_t H$, then $f=V_t g$ for some $g\in H$ and then $f=V_t g=\text{strong-}\lim_{s\to 0} V_{t+s} g=\text{strong-}\lim_{s\to 0} V_s f$. Therefore $\text{strong-}\lim_{t\to 0} V_t f=f$ for every $f\in\tilde H$.

Let now $f\in H$. Since $\|V_t f\|\leqslant\|f\|$ for all $t>0$, the net $(V_t f)_{t>0}$ has some weak cluster point $g\in H$ as $t\to 0$. $\tilde H$ is closed, so it is weakly closed, therefore $g\in\tilde H$. For any $t>0$, V_t is continuous, so it is weakly continuous, therefore $V_t g$ is a weak cluster point of $(V_{t+s}f)_{s>0}$ as $s\to 0$. But $V_t f=\text{strong-}\lim_{s\to 0} V_{t+s} f$, thus $V_t f=V_t g$. Hence

$$g=\text{strong-}\lim_{t\to 0} V_t g=\text{strong-}\lim_{t\to 0} V_t f,$$

and the proof is complete. (This argument is due to the referee.)

LEMMA 2. *If $1\leqslant p<\infty$ and $f\in L_p(X)$, then*

$$\sup_{b>0}\left|\frac{1}{b}\int_0^b T_t f(x)\,dt\right|<\infty \quad \text{a.e.}$$

Proof. This is immediate from Lemma VIII. 7.6 of [3].

Proof of the theorem. To prove that limit (1) exists and is finite a.e. for any $f\in L_1(X)$, we first consider the semigroup $\Gamma=\{T_t;\, t>0\}$ to be a strongly continuous semigroup of linear contractions on $L_2(X)$. It then follows from Lemma 1 that if we let $T_0=\text{strong-}\lim_{t\to 0} T_t$, then $\{T_t;\, t\geqslant 0\}$ is a strongly continuous semigroup on $[0,\infty)$. If $f\in L_2(X)$, write $f=g+h$, where $g=T_0 f$ and $h=f-g$. It follows that $T_t h=0$ for all $t>0$, and

$$\lim_{b\to 0}\left\|\frac{1}{b}\int_0^b T_t g\,dt-g\right\|_2=0.$$

Hence the set M of functions $f\in L_2(X)$ of the form

$$f=\frac{1}{b}\int_0^b T_t g\,dt+h,\quad \text{where}\quad T_t h=0\quad \text{for all}\quad t>0,$$

is dense in $L_2(X)$. Let $g'=\int_0^b T_t g\,dt$, and define a scalar function $\xi(t,x)$ on $(0,\infty)\times X$ by the relation

$$\xi(t,x)=\int_t^{t+b} T_s g(x)\,ds.$$

It is easy to see that $\xi(t,x)$ is measurable with respect to the product of the Lebesgue measurable subsets of $(0,\infty)$ and $\mathscr{M}$, and that $\xi(t,x)$ belongs, as a function of x, to the equivalence class of $T_t g'$ for each $t>0$. Moreover, we see that

$$\lim_{t\to 0}\xi(t,x)=\int_0^b T_s g(x)\,ds \quad \text{a.e.}$$

Therefore

$$\lim_{a\to 0}\frac{1}{a}\int_0^a T_t g'(x)\,dt=\lim_{a\to 0}\frac{1}{a}\int_0^a \xi(t,x)\,dt=\lim_{t\to 0}\xi(t,x)=g'(x)\quad \text{a.e.}$$

Thus the limit (1) exists and is finite a.e. for any $f\in M$. Hence we may apply Lemma 2 and Banach's convergence theorem (cf. [3], Theorem IV.11.3) to infer that the limit (1) exists and is finite a.e. for any $f\in L_2(X)$. Since $L_2(X)\cap L_1(X)$ is dense in $L_1(X)$, we may apply Lemma 2 and

Banach's convergence theorem again to infer that the limit (1) exists and is finite a.e. for any $f \in L_1(X)$.

Let now $(T_0 f)(x) = \lim_{b\to 0} \frac{1}{b} \int_0^b T_t f(x)\,dt$ a.e. for all $f \in L_1(X)$. It follows that T_0 is a Dunford–Schwartz operator on $L_1(X)$. Since Lemma 1 implies that $T_t T_s = T_{t+s}$ on $L_2(X)$ for all $t, s \geqslant 0$, we observe, by an approximation argument, that $T_t T_s = T_{t+s}$ on $L_1(X)$ for all t, $s \geqslant 0$. If $f \in L_1(X)$ satisfies $T_0 f = f$, let $A = \operatorname{supp} f$ and $e_f = f/|f|$ on A. Then, since $\bar{e}_f T_0(e_f|f|) = \bar{e}_f T_0 f = |f|$, we observe by an approximation argument that $g \to \bar{e}_f T_0(e_f g)$ is a positive linear contraction on $L_1(A) = \{g \in L_1(X) | \operatorname{supp} g \subset A\}$, and that $\|\bar{e}_f T_0(e_f g)\|_1 = \|g\|_1$ for all $g \in L_1^+(A)$. Since T_0 is a contraction, this implies that $T_0 L_1(A) \subset L_1(A)$. Using this, we can choose a function $h \in L_1(X)$ with $T_0 h = h$ such that if $f \in L_1(X)$ satisfies $T_0 f = f$, then $\operatorname{supp} f \subset \operatorname{supp} h$. Let us denote $Q = \operatorname{supp} h$. Since $T_0 T_t = T_t$ for all $t \geqslant 0$, $T_t L_1(Q) \subset L_1(Q)$. Therefore, in order to prove (ii) of the theorem, we may assume without loss of generality that $Q = X$. With this assumption, let $e = h/|h|$ and $S_t f = \bar{e} T_t(ef)$ for all $t \geqslant 0$ and $f \in L_1(X)$. It follows that S_0 is a positive linear contraction on $L_1(X)$, and that $\|S_0 g\|_1 = \|g\|_1$ for all $g \in L_1^+(X)$. Moreover, we have

$$(S_0 g)(x) = \lim_{b\to 0} \frac{1}{b} \int_0^b S_t g(x)\,dt \qquad \text{a.e.}$$

Hence it follows that

$$\lim_{b\to 0} \left\| \frac{1}{b} \int_0^b S_t g\,dt - S_0 g \right\|_1 = 0.$$

Therefore we conclude by the Hahn–Banach theorem that $\lim_{t\to 0} \|S_t g - S_0 g\|_1 = 0$. This completes the proof of (ii).

Remark. Applying Kubokawa's local ergodic theorem ([7]), the following can be proved: If $\Gamma = \{T_t;\ t \geqslant 0\}$ is a semigroup of linear contractions (not necessarily Dunford–Schwartz operators) on $L_1(X)$ and strongly continuous on $[0, \infty)$, then the local ergodic limit (1) exists and is finite a.e. for any $f \in L_1(X)$ (cf. [9]).

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