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A local characterization of Banach lattices with order continuous norm

by

S. J. BERNAU and H. ELTON LACEY (Austin, Tex.)

Abstract. A characterization, in terms of local unconditional structure, of Banach spaces linearly isomorphic to a Banach lattice with order continuous norm is obtained.

Banach spaces with a prescribed local (i.e., finite dimensional) structure are plentiful and, indeed, comprise the majority of Banach spaces studied for their nice properties. Let us make precise what we mean. Let X be a (real or complex) Banach space. Following Dubinsky, Pełczyński and Rosenthal [6] we say that X has *local unconditional structure* if there is a $\lambda > 1$ (depending only on X) such that for each finite dimensional subspace F of X there is a finite dimensional subspace G of X containing F and a finite dimensional Banach lattice H which is λ-isomorphic to G (i.e., there is a linear operator $T\colon G\to H$ such that $\|T\|\|T^{-1}\|\leqslant\lambda$). We may re-phrase this as follows: X has local unconditional structure if and only if there is an upwards directed family $\mathscr{B}$ of finite dimensional subspaces of X whose union is (dense in) X and such that each member of $\mathscr{B}$ is λ-isomorphic to a Banach lattice for some fixed $\lambda > 1$ depending only on X.

It might be thought that all Banach spaces have local unconditional structure since each finite dimensional subspace is certainly isomorphic to a Banach lattice. But, it is the uniform bound on the isomorphisms which restrict the class of such spaces. Indeed, Johnson has shown in [9] that if X has local unconditional structure, then either X is super-reflexive or it contains uniformly isomorphic copies of $l_1(n)$ or $l_\infty(n)$ for all n. Moreover, Gordon and Lewis [8] have exhibited examples of Banach spaces without local unconditional structure.

If $\mathscr{B}$ is the class $\{l_p(n)\}$ for a fixed p, $1\leqslant p\leqslant\infty$, and $n=1,2,\ldots$, then we obtain the spaces called $\mathscr{L}_{p,\lambda}$ spaces by Lindenstrauss and Pełczyński [13]. We note that $\mathscr{L}_{\infty,\lambda}$ spaces were first systematically studied by Lindenstrauss [12] and the $\mathscr{L}_{p,\lambda}$ spaces were studied by Lindenstrauss and Pełczyński in [13] and Lindenstrauss and Rosenthal in [14].

A general study of spaces with local unconditional structure has recently been undertaken by Figiel, Johnson, and Tzafriri [7].

If we assume that X has local unconditional structure for all $\lambda > 1$, then we obtain an isometric theory of such spaces. For example, X is an $\mathscr{L}_{p,\lambda}$ $(1 \leqslant p < \infty)$ for all $\lambda > 1$ if and only if $X = L_p(\mu)$ for some measure μ. This result traces back to Zippin [21] who proved it under the assumption that X is the union of an upwards directed family of finite dimensional subspaces each of which is linearly isometric to $l_p(n)$ for some n. Tzafriri [20] established it for the real case. The reader may see [4] or [10] for complete details in the complex case. Thus, we obtain, in particular, that X is a Banach lattice. On the other hand, X is an $\mathscr{L}_{\infty,\lambda}$ space for all $\lambda > 1$ if and only if X^* is an $L_1(\mu)$ space for some measure μ. The reader may check [10] for the details of this. We shall see later that there is such a space which is not isomorphic to a Banach lattice.

In this paper we investigate general isomorphic local unconditional structure conditions to obtain a characterization of Banach spaces which are linearly isomorphic to Banach lattices with order continuous norm. The characterization is given in terms of local lattice conditions, that is, lattice conditions on the finite dimensional lattices involved in the local unconditional structure of the space.

There are essentially trivial conditions one can impose to obtain such a characterization. For example, one can use conditions which yield that the natural embedding of a space X with local unconditional structure into an ultraproduct of finite dimensional lattices has the property that the image of X is a sublattice of the ultraproduct. These types of conditions seem to be uninteresting. A reasonable test of whether or not conditions are interesting is whether or not one can prove that an $\mathscr{L}_{p,\lambda}$ (for all $\lambda > 1$) space is an $L_p(\mu)$ space from them. We shall demonstrate that this is indeed the case with our approach.

We shall give the proofs in the complex case. In most instances, the real case is established by the same proof by simply dropping the complex notation.

§ 1. Notations and preliminaries. For convenience we now state what we mean by a complex Banach lattice.

DEFINITION 1.1. A *complex vector lattice* is a complex vector space X such that:

(1) there is a real linear subspace Y of X such that $X = Y \oplus iY$;

(2) Y is partially ordered so as to be a real vector lattice;

(3) for each $x \in X$ there is an element $|x| \in Y^+$ such that

$$|x| = \sup\{\mathrm{Re}(e^{i\theta}x)\colon\ 0 \leqslant \theta \leqslant 2\pi\}$$

where for $w \in X$, $\mathrm{Re}(w)$ is the unique component of w in the decomposition (1) above, i.e., $w - \mathrm{Re}(w) \in iY$.

A *normed complex vector lattice* X is a complex normed linear space which is also a complex vector lattice and where the norm satisfies the condition, if $|x| \leqslant |y|$, then $\|x\| \leqslant \|y\|$.

Thus a *complex Banach lattice* is a complete complex normed vector lattice.

The interested reader may consult [10] for other properties of complex Banach lattices.

If X is a finite dimensional Banach lattice of dimension n, then there are positive, normalized, disjoint elements $x_1, \ldots, x_n$ in X which form a basis for X. Moreover, all the lattice operations are "coordinatewise" e.g., $\left|\sum_{i=1}^{n} a_i x_i\right| = \sum_{i=1}^{n} |a_i| x_i$. The x_i's are unique up to permutation. We shall put

$$\mathrm{sgn}\Big(\sum_{i=1}^{n} a_i x_i\Big) = \sum_{i=1}^{n} \mathrm{sgn}\, a_i x_i$$

where $\mathrm{sgn}\, a$ is the usual signum of a. Also, the conjugate of $\sum_{i=1}^{n} a_i x_i$ is taken to be $\sum_{i=1}^{n} \bar{a}_i x_i$, where $\bar{a}$ is the complex conjugate of a.

Of course, if X is a complex vector lattice of complex valued functions on a set T under the usual pointwise operations, we have that for $f \in X$, $|f|$, $\mathrm{sgn} f$, $\bar{f}$, etc. have their usual meaning.

As mentioned in the introduction, we shall show in § 5 that there is a Banach space X with local unconditional structure for all $\lambda > 1$, but X is not even isomorphic to a Banach lattice. Thus, it follows that in order to obtain a lattice structure we must have some extra conditions on X. We have not been able to determine conditions which are necessary and sufficient for a general lattice structure. Furthermore, evidence indicates that such conditions may, indeed, be so complicated as to be of uncertain interest. A general Banach lattice does have local unconditional structure for all $\lambda > 1$. To see this, first observe that it is true for order complete Banach lattices. This can be easily seen by the approximation theorem in such lattices (see [10] or [11]). The general result then follows from the principle of local reflexivity [14] and the well-known fact that if X is a Banach lattice, then X^{**} is an order complete Banach lattice. (In fact, the principle of local reflexivity immediately shows that if X^{**} has local unconditional structure, then so does X. The converse is also true, see [7].)

The conditions with which we will be concerned in this paper give a local characterization of (complex) Banach lattices with order continuous norm. (Recall that X has *order continuous norm* means that if $Z \subset X^+$, Z is downwards directed and $\inf Z = 0$, then $\inf_{z \in Z} \|z\| = 0$. The reader may consult [19] for various equivalences of this notion.)

It is not surprising that this case is amenable to investigation since under order continuity, nets which converge in order also converge in norm. The conditions which we impose are conditions on the "bonding maps" between the finite dimensional subspaces of X and the Banach lattices given by local unconditional structure.

Let X be a complex Banach space and $\mathscr{P}$ a class of finite dimensional complex Banach lattices (we assume a fixed lattice structure in each member of $\mathscr{P}$).

We shall say that $\mathscr{P}$ is a *pre-lattice paving* of X if

(A) $X = \bigcup_\gamma G_\gamma$ where the family G_γ is an upwards directed family of finite dimensional subspaces such that there are invertible operators $T_\gamma\colon G_\gamma \to V_\gamma$ where $V_\gamma \in \mathscr{P}$ such that $\|T_\gamma\| = 1$ and $\sup_\gamma \|T_\gamma^{-1}\| < \infty$ (when we consider the isometric case the last condition becomes $\|T_\gamma^{-1}\| \to 1$).

(B) There is an $\eta(\gamma) > 0$ such that if $\gamma < \delta$ and $u, v \in V_\gamma$ with $|u| \wedge |v| = 0$, then

$$\big\| |T_\delta T_\gamma^{-1} u| \wedge |T_\delta T_\gamma^{-1} v| \big\| \leqslant \eta(\gamma)(\|u\| + \|v\|) \quad \text{and} \quad \lim_\gamma \eta(\gamma) = 0.$$

Condition (A) just says that X has local unconditional structure. Condition (B) is new to the literature. As we shall shortly see, however, it is implicitly found in the isometric theory of $\mathscr{L}_{p,\lambda}$ spaces. It was motivated by preceding work in [3] and [4].

It is clear that pre-lattice paving is an isomorphic invariant. In order to check that a space has a pre-lattice paving it is sometimes easier to check $\bigcup_\gamma G_\gamma$ is dense in X. We shall now show that this is indeed sufficient. That is condition (A) is modified to condition (A′) where the only change is that $\overline{\bigcup_\gamma G_\gamma} = X$.

Lemma 1.1. *If X satisfies conditions* (A′) *and* (B) *relative to $\mathscr{P}$, then it satisfies* (A) *and* (B) *relative to $\mathscr{P}$.*

Proof. We recall that in [4], Lemmas 5.2, 5.3, we showed that if $X = \overline{\bigcup_\gamma G_\gamma}$ where the G_γ's form an upwards directed family of finite dimensional spaces, then for each finite dimensional subspace F of X and each $\theta > 0$ there are a finite dimensional subspace H containing F and a G_γ such that there is an isomorphism T of H onto G_γ such that $\|x - Tx\| \leqslant \theta\|x\|$ for all $x \in H$.

Suppose we have the conditions (A′) and (B) satisfied. Let $\lambda = \sup_\gamma \|T_\gamma^{-1}\|$ and for each finite dimensional subspace F of X let H_F be a finite dimensional subspace of X such that $F \subset H_F$ and there is a G_{γ_F} and an isomorphism S_F of H_F onto G_{γ_F} such that

$$\|x - S_F x\| \leqslant \theta_F \|x\| \quad \text{where} \quad 0 < \theta_F \leqslant \min\left\{\frac{\lambda - 1}{\lambda + 1}, \frac{1}{\dim F}\right\}.$$

Then

$$(1 - \theta_F)\|x\| \leqslant \|S_F x\| \leqslant (1 + \theta_F)\|x\| \quad \text{and} \quad \|S_F\|\,\|S_F^{-1}\| \leqslant \frac{1+\theta}{1-\theta} \leqslant \lambda.$$

Moreover,

$$\|x - S_F^{-1}x\| \leqslant \|S_F^{-1}\|\,\|x - S_F x\| \leqslant \left(\frac{\theta_F}{1 - \theta_F}\right)\|x\|.$$

Now let $a_F = 1/\|T_{\gamma_F} S_F\|$. Then $\|x - a_F S_F x\| \leqslant K\theta_F\|x\|$ for a constant $K > 0$ not dependent on F. We put $T_F = a_F T_{\gamma_F} S_F$. Then the family $\{G_F, T_F\}$ as F ranges over all finite dimensional subspaces of X satisfies conditions (A) and (B). The family $\{G_F\}$ is clearly upwards directed, $\|T_F\| = 1$, and $\sup_F \|T_F\| < \infty$. Moreover, clearly $X = \bigcup_F G_F$. Thus we need only establish that condition (B) holds.

Suppose V_{F_1} $(= V_{\gamma_{F_1}})$ is contained in V_{F_2} and u, v are disjoint in V_{F_1}. Put $u_1 = T_{F_2} T_{F_1}^{-1} u$, $v_1 = T_{F_2} T_{F_1}^{-1} v$, $u_2 = T_{\gamma_{F_2}} T_{\gamma_{F_1}}^{-1} u$, and $v_2 = T_{\gamma_{F_2}} T_{\gamma_{F_1}}^{-1} v$. Then a simple computation shows that

$$\|u_1 - u_2\| \leqslant \lambda\left(\theta_{F_2} + \left(\frac{\theta_{F_1}}{1 - \theta_{F_1}}\right)\right)\|u\|$$

and similarly for $\|v_1 - v_2\|$. Now

$$|u_1| \wedge |v_1| \leqslant |u_2| \wedge |v_2| + |u_1 - u_2| + |v_1 - v_2|$$

and, hence, an $\eta(F)$ can be chosen to satisfy condition (B).

§ 2. Existence of pre-lattice pavings. In this section we show how pre-lattice pavings occur implicitly in $L_p(\mu)$ spaces and, indeed, in all Banach lattices with order continuous norm.

We first need the following extension of Clarkson's inequality.

Lemma 2.1. *Let $x, y \in L_p(\mu)$ and $g(x, y) = \|x+y\|^p + \|x-y\|^p - 2\|x\|^p - 2\|y\|^p$ where $1 \leqslant p < \infty$, $p \neq 2$. Then*

$$|g(x, y)| \geqslant 2\,|2^{p/2} - 2|\,\big\| |x| \wedge |y| \big\|^p.$$

Proof. Let z, w be complex numbers and put $h(z, w) = |z+w|^p + |z-w|^p - 2|z|^p - 2|w|^p$. Since $h(z, w) = 0$ if $zw = 0$, we may assume that $|z| = 1$ and $r = |w| \geqslant 1$. Put $t = \cos(\arg z\overline{w})$; then

$$|z+w|^p + |z-w|^p = (1 + 2rt + r^2)^{p/2} + (1 - 2rt + r^2)^{p/2}$$

and for fixed $r \geqslant 1$ this has a minimum (maximum) value $2(1+r^2)^{p/2}$ if $p \geqslant 2$ $(1 \leqslant p \leqslant 2)$. Since $2(1+r^2)^{p/2} - 2 - 2r^p$ is an increasing (decreasing) function of r on $[1, \infty)$ if $p \geqslant 2$ $(1 \leqslant p \leqslant 2)$ and has value $2(2^{p/2} - 2)$ at $r = 1$, Clarkson's inequality and our extension both follow.

Suppose now that $p > 2$ and let $T\colon L_p(\mu_1) \to L_p(\mu_2)$ be an isomorphism into such that $\|T\| \geqslant 1$, $\|T^{-1}\| \geqslant 1$. If $x, y \in L_p(\mu_1)$ and $|x| \wedge |y| = 0$, then

$$\begin{aligned} &2|2^{p/2}-2|\,\big\||Tx| \wedge |Ty|\big\|^p \\ &\leqslant \|Tx+Ty\|^p + \|Tx-Ty\|^p - 2\|Tx\|^p - 2\|Ty\|^p \\ &\leqslant \|T\|^p(\|x+y\|^p + \|x-y\|^p - 2\|T^{-1}\|^{-p}(\|x\|^p+\|y\|^p)) \\ &= \|T\|^p(2\|x+y\|^p) - 2\|T^{-1}\|^{-p}\|x+y\|^p = 2(\|T\|^p\| - \|T^{-1}\|^{-p})\|x+y\|^p. \end{aligned}$$

It is wellknown that $L_p(\mu)$ satisfies condition (A) in the isometric sense (see [10]). We shall show that it satisfies condition (B) with respect to any set of isomorphisms with suitable restrictions on their norms. This is not true for $p = 2$ and we know of no other examples where it is true.

Proposition 2.2. *Let $X = L_p(\mu)$ for $1 \leqslant p < \infty$, $p \neq 2$. Then X has isometric pre-lattice paving with respect to $\{l_p(n)\colon n = 1, 2, \ldots\}$.*

Proof. Let $X = \bigcup_\alpha G_\alpha$ where the G_α's form an upwards directed family of finite dimensional spaces such that there are isomorphisms $T_\alpha\colon G_\alpha \to l_p(n_\alpha)$ such that $\|T_\alpha\| = 1$ and $\lambda(\alpha) = \|T_\alpha^{-1}\|$ tends to 1.

Suppose $\alpha < \beta$ and $x, y \in l_p(n_\alpha)$ are disjoint. Then by the extended Clarkson inequality applied to $T = T_\beta T_\alpha^{-1}$,

$$\big\||Tx| \wedge |Ty|\big\|^p \leqslant |2^{p/2}-2|^{-1}(\|T\|^p - \|T^{-1}\|^{-p})\|x+y\|^p.$$

Since $\|T\| \leqslant \lambda(\alpha)$ and $\|T^{-1}\| \leqslant \lambda(\beta)$,

$$\eta(\alpha) = |2^{p/2}-2|^{-1/p}\big(\lambda(\alpha)^p - \lambda(\alpha)^{-p}\big)^{1/p}$$

is the required net for condition (B).

For general Banach lattices we have the following result whose proof is similar to that of Lemma 1.1.

Proposition 2.3. *Let X be a Banach lattice and $\mathscr{L}$ a set of finite dimensional sublattices of X which is upwards directed and $\bigcup \mathscr{L}$ is dense in X. Then X has isometric pre-lattice paving with respect to $\mathscr{L}$.*

§ 3. Existence of a pre-lattice structure on spaces with a pre-lattice paving. In the process of developing the theory of contractive projections in an $L_p(\mu)$ space we proved that if $M \subset L_p(\mu)$ is the range of a contractive projection on $L_p(\mu)$, then for all $x, y \in M$, $\big((\operatorname{Re} x \operatorname{sgn} \bar{y})^+ \wedge |y|\big)\operatorname{sgn} y \in M$ (see [3] or [4]). In [3] Bernau proved that the converse is true. (Actually in [3] it is stated in the form that $|x| \operatorname{sgn} y \in M$ and this is called the *exchange property* by Bernau. The existence of $|x|\operatorname{sgn} y$ in M from the above is a simple consequence of the monotone convergence theorem in $L_p(\mu)$, something which is difficult to formalize in a general non-lattice theoretic setting. We shall have more to say about this problem later.) This is interesting and motivational for our situation since ranges of contractive projections in $L_p(\mu)$ spaces are exactly the Banach spaces linearly isometric to $L_p(\nu)$ for some measure ν (see [4] or [10]).

We shall demonstrate in this section that *via* local approximations we can obtain in Banach spaces X with a pre-lattice paving $\mathscr{P}$ the beginning of a lattice structure in that for $x, y \in X$ we can produce in X an element which behaves like $\big((\operatorname{Re} Tx \operatorname{sgn} \overline{T}y)^+ \wedge |Ty|\big)\operatorname{sgn} Ty$ where the appropriate interpretation is given in V as mentioned in § 1. We show that if we consider the net of such elements, we can use it to construct a corresponding Cauchy net in X. The limit of this net plays the role of $\big((\operatorname{Re} x \operatorname{sgn} \bar{y})^+ \wedge |y|\big)\operatorname{sgn} y$ in the space.

A comment on our procedure is perhaps in order. There are two established methods of embedding Banach spaces with a local unconditional structure into a Banach lattice Y. One is an adaption of the embedding of $\mathscr{L}_{p,\lambda}$ spaces into $L_p(\mu)$ spaces as developed by Lindenstrauss and Pełczyński [13] and it can be found in [7]. The other is the elegant ultraproduct method developed by Dacunha-Castelle and Krivine in [5]. Using either of these embeddings one shows that an $\mathscr{L}_{p,\lambda}$ space for all $\lambda > 1$ $(1 \leqslant p < \infty)$ is an $L_p(\mu)$ space by showing that it is linearly isometric to a sublattice of an $L_p(\nu)$ space, the $L_p(\nu)$ space being obtained by the above methods. We have found it more convenient do not embed X into a Banach lattice, but to construct the lattice structure on X directly.

We now describe in detail the method of approach. Suppose X has a prelattice paving with respect to $\mathscr{P}$. Thus $X = \bigcup_\gamma G_\gamma$ where the G_γ's form an upwards directed family of finite dimensional subspaces and there are $V_\gamma \in \mathscr{P}$ and isomorphisms T_γ of G_γ onto V_γ satisfying conditions (A) and (B). For convenience we shall adopt the following notation. If $x, y \in V_\gamma$,

$$a(x, y, \gamma) = \big((\operatorname{Re} x \operatorname{sgn} \bar{y})^+ \wedge |y|\big)\operatorname{sgn} y$$

and, similarly, if α, β are complex numbers,

$$a(\alpha, \beta) = \big((\operatorname{Re} \alpha \operatorname{sgn} \bar{\beta})^+ \wedge |\beta|\big)\operatorname{sgn} \beta.$$

Now suppose that $\{e_1^\gamma, \ldots, e_{n(\gamma)}^\gamma\}$ is the disjoint, positive, normalized basis for V_γ (we shall assume that the basis is fixed throughout the paper). If $x, y \in G_\gamma$, then $T_\gamma x = \sum \alpha_j e_j^\gamma$ and $T_\gamma y = \sum \beta_j e_j^\gamma$ and we put

$$u_1(x, y, \gamma) = T_\gamma^{-1} a(T_\gamma x, T_\gamma y, \gamma) = T_\gamma^{-1}\Big(\sum a(\alpha_j, \beta_j) e_j^\gamma\Big),$$

if $\{x, y\} \not\subset G_\gamma$, we put $u_1(x, y, \gamma) = 0$. Similarly, we define $u_2(x, y, \gamma) = u_1(-x, y, \gamma)$, $u_3(x, y, \gamma) = u_1(-ix, y, \gamma)$ and $u_4(x, y, \gamma) = u_1(ix, y, \gamma)$.

What we shall show in this section is that the net $\{u_1(x, y, \gamma)\}$ converges in X (we call its limit, appropriately, $a(x, y)$). This shall be proved as a consequence of a series of lemmas. We are grateful to W.B. Johnson and L. Dor for the improvements in exposition of our original proofs in this section and for some substantial remarks and improvements of the lemmas involved.

PROPOSITION 3.1. *For each $x, y \in X$, $\{u_1(x, y, \gamma)\}$ is a convergent net in X.*

To prove this we are concerned with the following situation $x, y \in G_{\gamma_1}$ and $V_{\gamma_1} \subset V_{\gamma_2}$. What we need to estimate is $\|u_1(x, y, \gamma_1) - u_1(x, y, \gamma_2)\|$. Since

$$\|u_1(x, y, \gamma_1) - u_1(x, y, \gamma_2)\| \leqslant \|T_{\gamma_2}^{-1}\| \, \|T_{\gamma_2} u_1(x, y, \gamma_1) - T_{\gamma_2} u_1(x, y, \gamma_2)\|,$$

we consider the right hand expression. Let $S = T_{\gamma_2} T_{\gamma_1}^{-1}$, $e_j^* = S e_j^{\gamma_1}$, $x^* = T_{\gamma_2} x$, and $y^* = T_{\gamma_2} y$. Thus if $T_{\gamma_1} x = \sum a_j e_j^{\gamma_1}$ and $T_{\gamma_1} y = \sum \beta_j e_j^{\gamma_1}$, then $x^* = \sum a_j e_j^*$, and $y^* = \sum \beta_j e_j^*$, $T_{\gamma_2} u_1(x, y, \gamma_2) = a(x^*, y^*, \gamma_2)$, and $T_{\gamma_2} u_1(x, y, \gamma_1) = \sum a(a_j, \beta_j) e_j^*$. Thus we wish to estimate $\|a(x^*, y^*, \gamma_2) - \sum a(a_j, \beta_j) e_j^*\|$. Now if $|a_j'| \leqslant |\beta_j'|$, then

$$\Big\|\sum a_j' e_j^*\Big\| = \Big\|S\Big(\sum a_j' e_j^{\gamma_1}\Big)\Big\| \leqslant \|S\| \, \Big\|\sum a_j' e_j^{\gamma_1}\Big\| \leqslant \|S\| \, \Big\|\sum \beta_j' e_j^{\gamma_1}\Big\|$$
$$\leqslant \|S\| \, \Big\|S^{-1}\Big(\sum \beta_j' e_j^*\Big)\Big\| \leqslant \|S\| \, \|S^{-1}\| \, \Big\|\sum \beta_j' e_j^*\Big\| \leqslant \lambda^2 \Big\|\sum \beta_j' e_j^*\Big\|,$$

where $\lambda = \sup_\gamma \|T_\gamma^{-1}\|$. Furthermore, condition (B) tells us that if J_1, J_2 are disjoint subsets of $\{1, \ldots, n(\gamma_1)\}$ and $\mu_j \in C$, then

$$\Big\|\Big|\sum_{j \in J_1} \mu_j e_j^*\Big| \wedge \Big|\sum_{j \in J_2} \mu_j e_j^*\Big|\Big\| \leqslant 2\eta(\gamma_1) \Big\|\sum \mu_j e_j\Big\|.$$

Thus we have reduced the problem to the following abstract situation: V is a finite dimensional Banach lattice, $f_1, \ldots, f_n$ in V satisfy

(1) if $|a_i| \leqslant |\beta_j|$, then $\|\sum a_j f_j\| \leqslant K\|\sum \beta_j f_j\|$ for some fixed constant $K > 0$ independent of V, and

(2) if J_1, J_2 are disjoint subsets of $\{1, \ldots, n\}$ and $a_j \in C$, then

$$\Big\|\Big|\sum_{j \in J_1} a_j f_j\Big| \wedge \Big|\sum_{j \in J_2} a_j f_j\Big|\Big\| \leqslant \eta \Big\|\sum_{j=1}^{n} a_j f_j\Big\|$$

for some fixed $\eta > 0$. We wish to estimate $\|a(x, y) - \sum_{j=1}^{n} a(a_j, \beta_j) f_j\|$ where $x = \sum a_j f_j$ and $y = \sum \beta_j f_j$ and $a(x, y) = ((\operatorname{Re} x \operatorname{sgn} \bar{y})^+ \wedge |y|) \operatorname{sgn} y$. This is accomplished in the following series of lemmas which are also of independent interest.

LEMMA 3.2. *For each $\delta > 0$ there is a positive integer $N < (4/\delta^2 + 1)^2$ such that if V is a finite dimensional Banach lattice and $f_1, \ldots, f_n$ in V satisfy* (1) *above, then for $x = \sum a_j f_j$ and $y = \sum \beta_j f_j$ there is a partition $K_0, \ldots, K_N$ of $\{1, \ldots, n\}$ and $h_r = \sum_{j \in K_r} t_j f_j$, $A_r, B_r \in C$ such that $A_0 = 1$, $B_0 = 0$ and $B_r = 1$ for $r \neq 0$ such that*

$$\Big\|x - \sum_{r=0}^{N} A_r h_r\Big\| + \Big\|y - \sum_{r=0}^{N} B_r h_r\Big\| \leqslant K\delta(\|x\| + \|y\|).$$

Proof. By partitioning the closed $1/\delta$ ball in C with a rectangular grid we can obtain disjoint sets $H_0, \ldots, H_N$ such that the diameter of H_i is less than δ, $0 \in H_0$, $A_r \in H_r$, and $H_r \subset \{z\colon |z - A_r| < \delta\}$ and $N < (4/\delta^2 + 1)^2$. Let

$$K_r = \{j\colon a_j/\beta_j \in H_r\} \quad \text{and} \quad K_0 = \{j\colon |\beta_j| < |a_j|\,\delta\}.$$

Then we put

$$h_r = \sum_{j \in K_r} \beta_j f_j \quad \text{for } r \neq 0 \quad \text{and} \quad h_0 = \sum_{j \in K_0} a_j f_j.$$

Now

$$\Big\|y - \sum_{r=0}^{N} B_r h_r\Big\| = \Big\|\sum_{j \in K_0} \beta_j e_j\Big\| \leqslant K\delta\|x\|$$

and

$$\Big\|x - \sum_{r=0}^{N} A_r h_r\Big\| = \Big\|\sum_{r=1}^{N} \sum_{j \in K_r} \beta_j \Big(\frac{a_j}{\beta_j} - A_r\Big) f_j\Big\| \leqslant K\delta\|y\|.$$

COROLLARY 3.3. *In Lemma 3.2 the K_r, A_r, h_r also satisfy*

$$\Big\|\sum_{r=0}^{N} a(A_r h_r, B_r h_r) - \sum_{r=0}^{N} \sum_{j \in K_r} a(a_j, \beta_j) f_j\Big\| < K\delta(\|x\| + \|y\|).$$

Proof. Now we have $a(A_r h_r, B_r h_r) = B_r((\operatorname{Re} A_r h_r \operatorname{sgn} \bar{h}_r)^+ \wedge |h_r|) \operatorname{sgn} h_r = B_r((\operatorname{Re} A_r)^+ \wedge 1) h_r$. We choose the partition as in Lemma 1. If $r \neq 0$, then, for $j \in K_r$,

$$|((\operatorname{Re} A_r)^+ \wedge 1)\beta_j - ((\operatorname{Re} a_j \operatorname{sgn} \bar{\beta}_j)^+ \wedge |\beta_j|) \operatorname{sgn} \beta_j|$$
$$= |\beta_j| \Big|\Big[(\operatorname{Re} A_r)^+ \wedge 1 - \Big(\operatorname{Re} \frac{a_j}{\beta_j}\Big)^+ \wedge 1\Big]\Big| \leqslant |\beta_j| \Big|A_r - \frac{a_j}{\beta_j}\Big| \leqslant \delta|\beta_j|.$$

If $r = 0$, since $B_0 = 0$, $a(A_0 h_0, B_0 h_0) = 0$ and $|a(a_j, \beta_j)| < |\beta_j| < \delta|a_j|$ for $j \in K_0$. Thus the result follows from condition (1).

LEMMA 3.4. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that if V is a finite dimensional Banach lattice and u_1, u_2, v_1, v_2 are in V such that*

$$\|u_1 - u_2\| + \|v_1 - v_2\| < \delta(\|u_1\| + \|v_1\|),$$

then

$$\|a(u_1, v_1) - a(u_2, v_2)\| < \varepsilon(\|u_1\| + \|v_1\|).$$

Proof. Let $e_1, \ldots, e_k$ be a disjoint positive normalized basis for V and put

$$u_1 = \sum \alpha_i e_i, \quad v_1 = \sum \beta_i e_i, \quad u_2 = \sum \alpha_i' e_i, \quad \text{and} \quad v_2 = \sum \beta_i' e_i.$$

Then

$$\begin{aligned}
&|a(\alpha_i, \beta_i) - a(\alpha_i', \beta_i')| \\
&\leqslant (\operatorname{Re} \alpha_i \operatorname{sgn} \bar{\beta}_i)^+ \wedge |\beta_i|\, |\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i'| + \\
&\qquad\qquad + \big|(\operatorname{Re} \alpha_i \operatorname{sgn} \bar{\beta}_i)^+ \wedge |\beta_i| - (\operatorname{Re} \alpha_i' \operatorname{sgn} \bar{\beta}_i')^+ \wedge |\beta_i'|\big| \\
&\leqslant |\beta_i|\, |\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i'| + |\alpha_i \operatorname{sgn} \beta_i - \alpha_i' \operatorname{sgn} \beta_i'| + \big||\beta_i| - |\beta_i'|\big| \\
&\leqslant |\beta_i - \beta_i' + (|\beta_i'| - |\beta_i|) \operatorname{sgn} \beta_i'| + |\beta_i - \beta_i'| + |\alpha_i|\, |\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i'| + |\alpha_i - \alpha_i'| \\
&\leqslant 3\, |\beta_i - \beta_i'| + |\alpha_i|\, |\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i'| + |\alpha_i - \alpha_i'|.
\end{aligned}$$

If $|\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i'| < \varepsilon/2$, then

$$|a(\alpha_i, \beta_i) - a(\alpha_i', \beta_i')| < 3\, |\beta_i - \beta_i'| + |\alpha_i|\, \varepsilon/2 + |\alpha_i - \alpha_i'|.$$

If $|\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i'| \geqslant \varepsilon/2$, then

$$|\beta_i - \beta_i'| = \big||\beta_i| (\operatorname{sgn} \beta_i - \operatorname{sgn} \beta_i')\big| + \big|(|\beta_i| - |\beta_i'|) \operatorname{sgn} \beta_i'\big| \geqslant |\beta_i|\varepsilon/2 - |\beta_i - \beta_i'|.$$

Hence $|\beta_i'|$, $|\beta_i| \leqslant 4\, |\beta_i - \beta_i'|/\varepsilon$ so that

$$|a(\alpha_i, \beta_i) - a(\alpha_i', \beta_i')| \leqslant |\beta_i| + |\beta_i'| \leqslant 8\, |\beta_i - \beta_i'|/\varepsilon.$$

Thus

$$\begin{aligned}
\|a(u_1, v_1) - a(u_2, v_2)\| &\leqslant (3 + 8/\varepsilon)(\|u_1 - u_2\| + \|v_1 - v_2\|) + \varepsilon/2\, \|u_1\| \\
&\leqslant \big((3 + 8/\varepsilon)\, \delta + \varepsilon/2\big)(\|u_1\| + \|v_2\|).
\end{aligned}$$

Thus a δ which is independent of V can be so chosen.

LEMMA 3.5. *For each N and $\varepsilon > 0$ there is a $\theta > 0$ such that for each finite dimensional Banach lattice V if $h_0, \ldots, h_N$ in V satisfy condition* (1) *above and $\big\||h_n| \wedge |h_p|\big\| \leqslant \theta$ for $n \neq p$, then for $x = \sum A_n h_n$ and $y = \sum B_n h_n$,*

$$\Big\|a(x, y) - \sum a(A_n, B_n)\, h_n\Big\| \leqslant \varepsilon(\|x\| + \|y\|).$$

Proof. Put $h_n^* = h_n - \big(|h_n| \wedge \sum_{s \neq n} |h_s|\big) \operatorname{sgn} h_n$. Then the h_n^* are disjoint and

$$\|h_n - h_n^*\| \leqslant \sum_{s \neq n} \big\||h_n| \wedge |h_s|\big\| \leqslant N\theta$$

so that

$$\begin{aligned}
&\Big\|\sum A_n h_n - \sum A_n h_n^*\Big\| \\
&\qquad \leqslant \sum |A_n|\, \|h_n - h_n^*\| \leqslant N(N+1)\, \theta \max |A_n| \leqslant 2N^2 \theta K \|x\|,
\end{aligned}$$

where K is the constant given in condition (1) above. Similarly,

$$\Big\|\sum B_n h_n - \sum B_n h_n^*\Big\| \leqslant 2N^2 \theta K \|y\|.$$

Now

$$\begin{aligned}
&\Big\|a(x, y) - \sum a(A_n, B_n)\, h_n\Big\| \\
&\qquad \leqslant \Big\|a(x, y) - a\Big(\sum A_n h_n^*, \sum B_n h_n^*\Big)\Big\| + \\
&\qquad\qquad + \Big\|a\Big(\sum A_n h_n^*, \sum B_n h_n^*\Big) - \sum a(A_n, B_n)\, h_n\Big\| \\
&\qquad \leqslant \Big\|a(x, y) - a\Big(\sum A_n h_n^*, \sum B_n h_n^*\Big)\Big\| + \sum |B_n|\, \|h_n - h_n^*\| \\
&\qquad \leqslant \sum \Big\|a(x, y) - a\Big(\sum A_n h_n^*, \sum B_n h_n^*\Big)\Big\| + 2N^2 \theta K \|y\|.
\end{aligned}$$

Choose $\delta > 0$ to satisfy Lemma 3.4 for $\varepsilon/2$. If θ is chosen so that $2N^2 \theta K < \min(\delta, \varepsilon/2)$, then the conclusion follows.

Proof of Proposition 3.1. We adopt the notation following the statement of Proposition 3.1. That is, we assume that $f_1, \ldots, f_n$ in V satisfy conditions (1) and (2) and that $x = \sum \alpha_i f_i$ and $y = \sum \beta_i f_i$. Let $\varepsilon > 0$ be given and suppose $\delta > 0$ (to be chosen precisely later). From Lemma 3.2 we find an $N < (4/\delta^2 + 1)^2$ and a partition $K_0, \ldots, K_N$ of $\{1, \ldots, n\}$ and $h_0 = \sum_{j \in K_0} \alpha_j f_j$, $h_r = \sum_{j \in K_r} \beta_j f_j$ for $r \neq 0$, $A_0 = 1$, $B_0 = 0$, $B_r = 1$, and A_r chosen as in Lemma 3.2 such that

$$\Big\|x - \sum A_r h_r\Big\| + \Big\|y - \sum B_r h_r\Big\| \leqslant K\delta(\|x\| + \|y\|),$$

and

$$\Big\|\sum_{r=0}^{N} a(A_r h_r, B_r h_r) - \sum_{j=1}^{n} a(\alpha_j, \beta_j) f_j\Big\| \leqslant K\delta(\|x\| + \|y\|).$$

By Lemma 3.4 we choose $0 < \delta < \varepsilon/K$ so that

$$\Big\|a(x, y) - a\Big(\sum A_r h_r, \sum B_r h_r\Big)\Big\| < \varepsilon(\|x\| + \|y\|).$$

Now by conditions (1) and (2),

$$\big\||h_r| \wedge |h_s|\big\| \leqslant \eta \Big(\Big\|\sum_{j \in K_0} \alpha_j f_j\Big\| + \Big\|\sum_{j \notin K_0} \beta_j f_j\Big\|\Big) \leqslant \eta K(\|x\| + \|y\|).$$

Thus if $\eta K(\|x\| + \|y\|) < \theta$, where θ is the constant of Lemma 3.5, then

$$\begin{aligned}
\Big\|a\Big(\sum A_r h_r, \sum B_r h_r\Big) - \sum a(A_r h_r, B_r h_r)\Big\| &\leqslant \varepsilon\Big(\Big\|\sum A_r h_r\Big\| + \Big\|\sum B_r h_r\Big\|\Big) \\
&\leqslant (\varepsilon + K\delta)(\|x\| + \|y\|) < 2\varepsilon(\|x\| + \|y\|).
\end{aligned}$$

Thus

$$\Big\|a(x,y)-\sum_{j=1}^{n}a(\alpha_i,\beta_i)f_i\Big\|\leqslant 4\varepsilon(\|x\|+\|y\|).$$

Since in our situation $\eta(\gamma)\to 0$, we can, indeed, choose γ_1 so large that $\eta(\gamma_1)K(\|x\|+\|y\|)<\theta$ and obtain the conclusion of Proposition 3.1.

§ 4. Order continuous pre-lattice pavings. As we mentioned at the end of § 2, pre-lattice pavings do not quite yield a lattice structure on X. What we need is a condition which will assure that the sequence $v_n=a(x,ny)=\lim u_1(x,ny,\gamma)$ (whose existence was shown in § 3) converges in norm to an element which we shall label $P_y(x)$.

The element $P_y(x)$ will turn out to be a piece of the band projection of x onto $\{y\}^{\perp\perp}$ (this is the motivation for this limit). In order to do this it turns out that we must be in an order continuous situation. Hence we impose another axiom which we shall show is valid in any Banach lattice with order continuous norm. We are still assuming that X is a complex Banach space with a pre-lattice paving $\mathscr{P}$ and that $X=\bigcup G_\gamma$ where G_γ, V_γ, and T_γ have the same meaning as before.

DEFINITION 4.1. Let (v_n) be a sequence in X. We say that (v_n) is *$\mathscr{P}$-increasing* if there is a subnet $\{G_\lambda\}$ of $\{G_\gamma\}$ such that for each λ there is a sequence $(v_n(\lambda))$ such that

(i) $v_n(\lambda)\operatorname{sgn}\overline{v_m(\lambda)}\geqslant 0$ for all n,m;

(ii) $(|v_n(\lambda)|)$ is increasing (in V_λ);

(iii) $T_\lambda^{-1}v_n(\lambda)\to v_n$.

The sequence (v_n) is $\mathscr{P}$-increasing and *dominated* if there exists $x\in X$ such that the G_λ's which are chosen above also satisfy:

(iv) $x\in G_\lambda$ for each G_λ;

(v) $|v_n(\lambda)|\leqslant|T_\lambda x|$ for all n and all $\lambda>1$.

The paving $\mathscr{P}$ of X is called *order continuous* if every dominated $\mathscr{P}$-increasing sequence (v_n) in X is norm convergent in X.

To show that we are not working in a vaccuum we note at once the following two propositions.

PROPOSITION 4.2. *Suppose $1\leqslant p<\infty$, that $\mathscr{P}$ isometrically paves X and that the elements of $\mathscr{P}$ are finite dimensional $l_p(n)$ spaces, then $\mathscr{P}$ is an order continuous pre-lattice paving of X.*

Proof. Let (v_n) be a dominated $\mathscr{P}$-increasing sequence in X. By (ii) and (v) we have

$$\|v_n(\lambda)\|\leqslant\|v_{n+1}(\lambda)\|\leqslant\|T_\lambda x\|\quad\text{ for all } n \text{ and } \lambda.$$

Apply T_λ^{-1}, and take the limit over λ using (iii) to obtain $\|v_n\|\leqslant\|v_{n+1}\|\leqslant\|x\|$ for all n. Since each V_λ is an l_p-space, (i) and (ii) give, for $m>n$,

$$\|v_m(\lambda)-v_n(\lambda)\|^p=\big\||v_m(\lambda)|-|v_n(\lambda)|\big\|^p\leqslant\|v_m(\lambda)\|^p-\|v_n(\lambda)\|^p.$$

Use (iii) again to deduce

$$\|v_m-v_n\|^p\leqslant\|v_m\|^p-\|v_n\|^p\to 0\quad(m,n\to\infty).$$

Our result follows.

Next we consider Banach lattices with order continuous norm. If a Banach lattice has order continuous norm, then it is order complete [10] in the sense that its real part is an order complete vector lattice.

THEOREM 4.3. *Let X be a Banach lattice with order continuous norm, then X has an isometric order continuous pre-lattice paving.*

Proof. It is well known that under the hypothesis of the theorem there is an upwards directed set $\mathscr{P}$ of finite dimensional sublattices of X whose union is dense. One way to prove this is to start with a maximal, pairwise disjoint set D of positive elements of X and to consider the set $\mathscr{P}$ of finite dimensional sublattices of X generated by components of elements of D. Theorem 1.3 of [11] and the fact that an order bounded increasing sequence in X is norm-convergent to its supremum provides most of the necessary equipment.

By Proposition 2.3, $\mathscr{P}$ is a pre-lattice paving of X and the maps $T_\lambda\colon G_\lambda\to V_\lambda$ satisfy the extra condition

$$\|x-T_\lambda x\|\leqslant\{4(\lambda-1)/(\lambda+1)\}\|x\|.$$

Suppose then that (v_n) is a $\mathscr{P}$-increasing dominated (by x) sequence in X. Since

$$\|v_n(\lambda)-T_\lambda^{-1}v_n(\lambda)\|\leqslant\{4(\lambda-1)/(\lambda+1)\}\|T_\lambda^{-1}v_n(\lambda)\|,$$

condition (iii) shows that $v_n(\lambda)\to v_n$ for each n, and that $T_\lambda x\to x$. If $m\geqslant n$, (ii) and (v) give $|v_n(\lambda)|\leqslant|v_m(\lambda)|\leqslant|T_\lambda x|$. By (iii) we have $|v_n|\leqslant|v_n|\leqslant|x|$. In addition (i) and (ii) give $|v_m(\lambda)-v_n(\lambda)|=|v_m(\lambda)|-|v_n(\lambda)|$ so that the limit over λ gives $|v_m-v_n|=|v_m|-|v_n|$. Since X has order continuous norm, the order bounded increasing sequence $(|v_n|)$ is convergent and hence (v_n) converges as required.

Our condition that a pre-lattice paving be order continuous is highly local in character. It would be nice to have an equivalent simple global condition. Ultimately we will show that X has an order continuous pre-lattice paving if and only if X is linearly isometric to a Banach lattice with order continuous norm. By a result of Meyer–Nieberg [19], order-continuity of the norm is equivalent to the non-existence of sublattices which are linearly and lattice isomorphic to c_0 and have order bounded unit ball. With this in mind our next result is not too surprising.

THEOREM 4.4. *Let $\mathscr{P}$ be a pre-lattice paving of X and suppose X contains no subspace linearly isomorphic to c_0, then $\mathscr{P}$ is an order-continuous pre-lattice paving of X.*

Proof. Suppose (v_n) is a $\mathscr{P}$-increasing sequence in X which is dominated by x. Let (n_k) be any strictly increasing sequence of integers and let $w_k = v_{n_{k+1}} - v_{n_k}$. Let μ_j, μ be complex numbers such that $|\mu_j| \leqslant \mu$ $(j = r, r+1, \ldots, s)$, then

$$\Big|\sum_{j=r}^{s} \mu_j w_j(\lambda)\Big| \leqslant \sum_{r}^{s} |\mu_j|\,|w_j(\lambda)| = \sum_{r}^{s} |\mu_j|\big(|v_{n_{j+1}}(\lambda)| - |v_{n_j}(\lambda)|\big)$$
$$\leqslant \mu\big(|v_{n_{s+1}}(\lambda)| - |v_{n_r}(\lambda)|\big) \leqslant \mu\,|T_\lambda x|.$$

By taking the limit over λ we have $\|\sum_{j=r}^{s} \mu_j w_j\| \leqslant \mu\|x\|$. Hence if $(\xi_j) \in c_0$, the series $\sum \xi_j w_j$ is norm convergent in X and $\|\sum \xi_j w_j\| \leqslant \|x\|\,\|(\xi_j)\|$.

Suppose now that (v_n) is not convergent in X, and let $\varepsilon = \inf_n \sup_{m \geqslant n} \|v_m - v_n\|$. Since $\|v_m(\lambda) - v_n(\lambda)\|$ is an increasing function of m for each fixed n and each λ, we have $\varepsilon > 0$. Choose a subsequence (v_{n_k}) of (v_n) such that $\varepsilon \leqslant \|v_{n_{k+1}} - v_{n_k}\| \leqslant 3\varepsilon/2$ for each k, put $w_k = v_{n_{k+1}} - v_{n_k}$ $(k = 1, 2, \ldots)$; and define $T\colon c_0 \to X$ by $T(\xi_j) = \sum \xi_j w_j$ $((\xi_j) \in c_0)$. The argument above shows that $\|T\| \leqslant \|x\|$. Let $(\xi_j) \in c_0$ and suppose $\xi_p = \|(\xi_j)\|$. Since the $v_n(\lambda)$, and hence the $w_j(\lambda)$ are all parallel, we have, for $r \geqslant p$,

$$\Big\|\sum_{1}^{r} \xi_j w_j(\lambda)\Big\| \geqslant \Big\|\sum_{1}^{r} (\operatorname{Re}\xi_j) w_j(\lambda)\Big\|$$
$$= \Big\|\sum_{1}^{r} (\operatorname{Re}\xi_j) w_j(\lambda)\Big\| + \Big\|\sum_{1}^{r} |\operatorname{Re}\xi_j| w_j(\lambda)\Big\| - \Big\|\sum_{1}^{r} |\operatorname{Re}\xi_j| w_j(\lambda)\Big\|$$
$$\geqslant \Big\|\sum_{1}^{r} 2(\operatorname{Re}\xi_j)^+ w_j(\lambda)\Big\| - \Big\|\sum_{1}^{r} |\operatorname{Re}\xi_j| w_j(\lambda)\Big\|$$
$$\geqslant 2\xi_p\|w_p(\lambda)\| - \xi_p\|v_{n_{r+1}} - v_{n_1}\| \geqslant 2\xi_p\varepsilon - \xi_p \cdot 3\varepsilon/2 = \xi_p \cdot \varepsilon/2.$$

Let λ vary and then let $r \to \infty$ to see that $\|T(\xi_j)\| \geqslant \xi_p \cdot \varepsilon/2 = \|(\xi_j)\| \cdot \varepsilon/2$.

It follows that T is an isomorphism of c_0 into X, contrary to hypothesis. Thus $\mathscr{P}$ is an order continuous pre-lattice paving of X.

§ 5. The lattice structure on X. In this section we give in complete detail the proof that a Banach space with an isometric order continuous pre-lattice paving has a lattice structure under which it is a Banach lattice. Since the complex case presents a degree of difficulty over the real one, we write all of the proofs in the complex situation. At the end of the section we indicate how to obtain the corresponding isomorphic case from the isometric one.

We now suppose that $X = \bigcup G_\gamma$ where G_γ, V_γ, and T_γ have the same meaning as before and we assume that $\|T_\gamma^{-1}\| \to 1$, i.e., that we have an isometric order continuous pre-lattice paving on X. We recall the following definition formally since it will be used repeatedly.

DEFINITION 5.1. For $x, y \in X$,

$$a(x, y) = \lim u_1(x, y, \gamma).$$

THEOREM 5.2. *For $x, y \in X$ the sequence (v_n), with $v_n = a(x, ny)$, is convergent.*

Proof. We take the net of all of the G_λ's such that $x, y \in G_\lambda$ and put

$$v_n(\lambda) = a(T_\lambda x, nT_\lambda y, \lambda).$$

Then (v_n) is clearly $\mathscr{P}$-increasing and dominated by x. Hence (v_n) is convergent because $\mathscr{P}$ is an order continuous pre-lattice paving.

DEFINITION 5.3. For each $x, y \in X$,

$$P_y(x) = \lim_{n\to\infty} a(x, ny)$$

and

$$J_y(x) = P_y(x) - P_y(-x) + i\big(P_y(-ix) - P_y(ix)\big).$$

The motivation for $J_y(x)$ is that it is the component of x onto $\{y\}^{\perp\perp}$ under the band projection in a Banach lattice.

We shall now prove some computational rules for the operator P_y. These should be interpreted as properties of the positive part of the real part of an element of a complex vector lattice.

PROPOSITION 5.4. *For $x, y, x_1, x_2, x_3, x_4 \in X$ we have the following.*

(i) $\|P_y(x)\| \leqslant \|P_y(x) - P_y(-x)\| = \|P_y(x) + P_y(-x)\| \leqslant \|J_y(x)\|$
$= \|P_y(x) \pm P_y(-x) \pm i\big(P_y(-ix) \pm P_y(ix)\big)\| \leqslant \|x\|.$

(ii) $\|P_y(x_1 + x_2)\| \leqslant \|P_y(x_1)\| + \|P_y(x_2)\|$.

(iii) *If* $P_y(x_1) = 0$ *and* $P_y(x_2) = 0$, *then* $P_y(x_1 + x_2) = 0$.

(iv) $\|P_y(x_1) - P_y(x_2)\| \leqslant \|P_y(x_1 - x_2)\| \leqslant \|x_1 - x_2\|$, *so that* P_y *is continuous. Also* $\|a(x_1, y) - a(x_2, y)\| \leqslant \|x_1 - x_2\|$.

(v) $P_y\big(P_y(x_1) + P_y(x_2)\big) = P_y(x_1) + P_y(x_2)$.

(vi) $P_y\big(-P_y(x)\big) = 0 = P_y\big(\pm iP_y(x)\big)$.

(vii) $P_y\big[P_y(x_1) - P_y(x_2) + i\big(P_y(ix_3) - P_y(ix_4)\big)\big] = P_y\big(P_y(x_1) - P_y(x_2)\big)$.

(viii) $P_y\big(P_y(x) - P_y(-x)\big) = P_y(x)$.

(ix) $P_y\big(J_y(x)\big) = P_y(x) = J_y\big(P_y(x)\big)$.

Proof. For each integer n and each complex number α, since $u_1(\alpha x, ny, \gamma) \to a(\alpha x, ny)$ and $\|T_\gamma^{-1}\| \to 1$,

$$\|a(T_\gamma \alpha x, nT_\gamma y, \gamma)\| \to \|a(\alpha x, ny)\|.$$

Moreover, in V_γ we have that

$$\begin{aligned}\|a(T_\gamma x, nT_\gamma y, \gamma)\| &\leqslant \|a(T_\gamma x, nT_\gamma y, \gamma) - a(-T_\gamma x, nT_\gamma y, \gamma)\| \\ &= \|a(T_\gamma x, nT_\gamma y, \gamma) + a(-T_\gamma x, nT_\gamma y, \gamma)\| \\ &\leqslant \big\|a(T_\gamma x, nT_\gamma y, \gamma) \pm a(T_\gamma x, nT_\gamma y, \gamma) \pm \\ &\qquad \pm i\big(a(-iT_\gamma x, nT_\gamma y, \gamma) \pm a(iT_\gamma x, nT_\gamma y, \gamma)\big)\big\|,\end{aligned}$$

with equality of norms for all eight possible choices of sign in the last expression. Let γ vary to get corresponding inequalities involving $a(x, ny)$ and similar terms, then let $n \to \infty$ to obtain (i).

For (ii) we have in V_γ,

$$\begin{aligned}0 &\leqslant a\big(T_\gamma(x_1+x_2), nT_\gamma y, \gamma\big)\operatorname{sgn}\overline{T_\gamma y} \\ &\leqslant a(T_\gamma x_1, nT_\gamma y, \gamma)\operatorname{sgn}\overline{T_\gamma y} + a(T_\gamma x_2, nT_\gamma y)\operatorname{sgn}\overline{T_\gamma y}.\end{aligned}$$

Hence

$$\big\|a\big(T_\gamma(x_1+x_2), nT_\gamma y, \gamma\big)\big\| \leqslant \|a(T_\gamma x_1, nT_\gamma y, \gamma)\| + \|a(T_\gamma x_2, nT_\gamma y, \gamma)\|$$

and (ii) follows.

Part (iii) is immediate from (ii).

To obtain (iv) recall that if $b \geqslant 0$

$$|a^+ \wedge b - c^+ \wedge b| \leqslant |(a-c)^+ \wedge b| \leqslant |a-c|$$

in any lattice group. Hence,

$$\begin{aligned}\|a(T_\gamma x_1, nT_\gamma y, \gamma) - a(T_\gamma x_2, nT_\gamma y, \gamma)\| \\ \leqslant \big\|a\big(T_\gamma(x_1-x_2), nT_\gamma y, \gamma\big)\big\| \leqslant \|T_\gamma(x_1-x_2)\|\end{aligned}$$

and (iv) follows.

Let m, n be integers with $m > 2n$, and let $w_j(n) = a(x_j, ny)$ for $j = 1, 2$. In V_γ,

$$\begin{aligned}&\Big|a\big(T_\gamma\big(w_1(n)+w_2(n)\big), mT_\gamma y, \gamma\big) - \big(a(T_\gamma x_1, nT_\gamma y, \gamma) + a(T_\gamma x_2, nT_\gamma y, \gamma)\big)\Big| \\ &\quad = \Big|a\big(T_\gamma\big(w_1(n)+w_2(n)\big), mT_\gamma y, \gamma\big) - \\ &\qquad\qquad - a\big(a(T_\gamma x, nT_\gamma y, \gamma) + a(T_\gamma x_2, nT_\gamma y, \gamma), mT_\gamma y, \gamma\big)\Big| \\ &\quad \leqslant |T_\gamma w_1(n) - a(T_\gamma x_1, nT_\gamma y, \gamma)| + |T_\gamma w_2(n) - a(T_\gamma x_1, nT_\gamma y, \gamma)|\end{aligned}$$

where we have assumed throughout that γ is large enough so that G_γ contains all the elements above.

Take norms, let γ vary and then $m \to \infty$ to conclude

$$\Big\|P_y\big(w_1(n)+w_2(n)\big) - \big(w_1(n)+w_2(n)\big)\Big\| = 0.$$

Now let $n \to \infty$ and use (iv) to obtain (v).

For (vi), if $w_1(n)$ is as defined above

$$a\big(-T_\gamma w_1(n), mT_\gamma y, \gamma\big) = 0 = a\big(\pm iT_\gamma w_1(n), mT_\gamma y, \gamma\big) \quad \text{for all } m.$$

Letting γ vary, $m \to \infty$ and then $n \to \infty$ we have (vi).

The proof of (vii) is similar to that of (v) and (vi).

For (viii), put $x_1 = x$, $x_2 = -x$. As in the proof of (v) we have, for $m > n$,

$$\begin{aligned}&\Big\|a\big(T_\gamma\big(w_1(n)-w_2(n)\big), mT_\gamma y, \gamma\big) - \\ &\qquad - a\big(a(T_\gamma x_1, nT_\gamma y, \gamma) - a(T_\gamma x_2, nT_\gamma y, \gamma), mT_\gamma y, \gamma\big)\Big\| \to 0.\end{aligned}$$

If $m > n$,

$$\begin{aligned}a\big(a(T_\gamma x_1, nT_\gamma y, \gamma) - a(T_\gamma x_2, nT_\gamma y, \gamma), mT_\gamma y, \gamma\big) \\ = a(T_\gamma x_1, nT_\gamma y, \gamma).\end{aligned}$$

Hence letting γ vary and then $m \to \infty$, we have

$$P_y\big(w_1(n) - w_2(n)\big) = w_1(n).$$

Now let $n \to \infty$ and use (iv) to obtain (viii).

Part (ix) now follows from (vi), (vii) and (viii).

Next we define disjointness so that it means what it should mean after we put a complex Banach lattice structure on X.

DEFINITION 5.5. If $x, y \in X$, x is *disjoint* from y, in symbols $x \perp y$ if

$$a(i^k x, y) = 0 \quad \text{for} \quad k = 0, 1, 2, 3.$$

The main properties of disjointness are now established.

LEMMA 5.6. *For $x, y \in X$, and any positive integer n,*

(i) $\|a(x, ny)\| \leqslant n\|a(x, y)\| \leqslant n\|P_y(x)\|$.

(ii) $\|a(y, x)\| \leqslant \|4a(x, y)\|$.

Proof. Use the corresponding facts in V_γ and take the limit over γ.

PROPOSITION 5.7. *For $x, y \in X$ the following are equivalent.*

(i) $x \perp y$.

(ii) $P_y(i^k x) = 0$ $(k = 0, 1, 2, 3)$.

(iii) $J_y(x) = 0$.

(iv) $y \perp x$.

Proof. The equivalence of (i) and (iv) follows from Lemma 5.6 (ii). Equivalence of (i) and (ii) follows from Lemma 5.6 (i). Clearly, (ii) implies (iii) and if $J_y(x) = 0$, $P_y(i^k x) = 0$ $(k = 0, 1, 2, 3)$ by Proposition 5.4 (i).

PROPOSITION 5.8. *If $a \perp b$,*

(i) $\|a+b\| = \|a + e^{i\theta}b\| \geqslant \|a\|$ *for all real* θ;

(ii) $P_y(a+b) = P_y(a) + P_y(b)$ $(y \in X)$.

Proof. In V_γ we have

$$|T_\gamma a| \wedge |T_\gamma b| \leqslant \sum_{k=0}^{3} |a(i^k T_\gamma a, T_\gamma b, \gamma)|$$

so $\big\| |T_\gamma a| \wedge |T_\gamma b| \big\| \to 0$. Since

$$\big| |T_\gamma a + e^{i\theta} T_\gamma b| - |T_\gamma a + T_\gamma b| \big| \leqslant 2|T_\gamma a| \wedge |T_\gamma b|,$$

(i) follows.

Since

$$(\operatorname{Re} T_\gamma a \operatorname{sgn} \overline{T_\gamma y})^+ \wedge (\operatorname{Re} T_\gamma b \operatorname{sgn} \overline{T_\gamma y})^+ \leqslant |T_\gamma a| \wedge |T_\gamma b|,$$

we conclude similarly that

$$a(a+b, ny) = a(a, ny) + a(b, ny)$$

and (ii) follows.

PROPOSITION 5.9. *We have*

$$P_y\big(P_y(x) - P_y(z)\big) = P_y\big(P_y(x) - P_y(-x) - P_y(z)\big) \perp P_y(-x).$$

Proof. Let

$$a_{m,n} = a\big(a(x, ny) - a(z, ny), my\big) \quad \text{and} \quad b_r = a(-x, ry).$$

Since the elements of V_γ obtained by substituting $T_\gamma x$, $T_\gamma z$ and $T_\gamma y$ in the expressions for $a_{m,n}$, b_r are disjoint it follows by the usual limiting arguments that $a_{m,n} \perp b_r$.

Let $r \to \infty$ to obtain $P_y(-x) \perp a_{m,n}$ for all m, n and then let $m, n \to \infty$ to conclude $P_y(-x) \perp P_y\big(P_y(x) - P_y(z)\big)$. By Propositions 5.8 (ii) and 5.4 (vi)

$$P_y(a_{m,n}) = P_y(a_{m,n}) + P_y\big(-P_y(-x)\big) = P_y\big(a_{m,n} - P_y(-x)\big).$$

Now let $m \to \infty$, $n \to \infty$ and use Proposition 5.4 (iv). This proves our result. (In fact, we can show $P_y\big(J_y(x) - P_y(z)\big) = P_y\big(x - P_y(z)\big)$ but this will not be needed.)

PROPOSITION 5.10. *The set $\{x \in X\colon x \perp y\}$ is a closed subspace of X.*

Proof. Use Propositions 5.7 (ii) and 5.4 (iii) and (iv).

PROPOSITION 5.11. *If $y_1 \perp y_2$, then $P_{y_1}(x_1) \perp P_{y_2}(x_2)$ $(x_1, x_2 \in X)$ and hence*

$$J_{y_1}(x_1) \perp J_{y_2}(x_2) \quad (x_1, x_2 \in X).$$

Proof. Clearly,

$$\big\| a\big(a(x_1, ny_1), y_2\big) \big\| \leqslant \| J_{y_2}(ny_1) \| = 0.$$

Letting $n \to \infty$ we conclude $P_{y_2}\big(P_{y_1}(x_1)\big) = 0$ and our result follows easily.

PROPOSITION 5.12. *Suppose $w_m \perp w_n$ $(m \neq n)$ and $v_n = \sum_{m=1}^{n} w_m$, then v_n is $\mathscr{P}$-increasing. If in addition $y_m \perp y_n$ $(m \neq n)$ and $w_n = P_{y_m}(x)$ for some fixed $x \in X$, then (v_n) is also dominated by x.*

Proof. Let N be a positive integer and write $w_n(\lambda) = T_\lambda w_n$ if $w_n \in G_\lambda$ and 0 otherwise. As in the proof of Proposition 5.9 we have

$$\big\| |w_n(\lambda)| \wedge |w_m(\lambda)| \big\| \to 0 \quad \text{for} \quad n \neq m.$$

Thus we can determine λ_N such that

$$\big\| |w_n(\lambda)| \wedge |w_m(\lambda)| \big\| < 1/N^2 \quad \text{for} \quad \lambda \geqslant \lambda_N,\ 1 \leqslant m < n \leqslant N.$$

In V_{λ_N} define

$$w_n^*(\lambda_N) = w_n(\lambda_N) - \sum_{j=1}^{N}{}' \big(|w_j(\lambda_N)| \wedge |w_n(\lambda_N)|\big) \operatorname{sgn} w_n(\lambda_N),$$

where the prime denotes omission of the terms with $j = n$, for $n = 1, \ldots, N$, and $w_n^*(\lambda_N) = 0$ $(n > N)$. Note that the $w_n^*(\lambda_N)$ are pairwise disjoint in V_{λ_N} and

$$\|w_n(\lambda_N) - w_n^*(\lambda_N)\| \leqslant (N-1)\cdot 1/N^2 < 1/N.$$

If $\|T_{\lambda_N}^{-1}\|$ is chosen to be strictly decreasing to 1 and we take $v_n(\lambda) = \sum_{m=1}^{n} w_m^*(\lambda_N)$ for $\lambda_N \geqslant \lambda \geqslant \lambda_{N-1}$ we have exhibited (v_n) as a $\mathscr{P}$-increasing sequence.

In the special case considered, we can for fixed N choose M such that $\|P_{y_m}(x) - a(x, My_m)\|$ is as small as we choose for $1 \leqslant m \leqslant N$. Then for $w_m(\lambda)$ we choose $T_\lambda u_1(x, My_m, \lambda)$ in place of $T_\lambda w_m = T_\lambda\big(P_{y_m}(x)\big)$ for $1 \leqslant m \leqslant N$ and zero otherwise. We still have

$$\big\| |w_m(\lambda)| \wedge |w_{m'}(\lambda)| \big\| \to 0 \quad \text{if} \quad m \neq m'.$$

Now if we mimic the earlier definition of the $w_m^*(\lambda)$ and $v_n(\lambda)$, the disjointness of the $w_m^*(\lambda)$ gives us $|v_n(\lambda)| \leqslant |T_\lambda x|$ for all n, λ as required.

THEOREM 5.13. *For each $y \in X$, J_y is a contractive linear projection on X.*

Proof. That $J_y^2 = J_y$ and $\|J_y(x)\| \leqslant \|x\|$ is a consequence of Proposition 5.4. It remains to prove linearity of J_y and for this it suffices to show that $x \mapsto P_y(x) - P_y(-x)$ is additive. Let a, b, c be elements of a vector lattice with $c \geqslant 0$; for any positive integer n,

$$\begin{aligned}
(a+b)^+ \wedge c + a^- \wedge c + b^- \wedge c &\leqslant \big((a+b)^+ + a^- + b^-\big) \wedge 3c \\
&= \big((a+b)^- + a^+ + b^+\big) \wedge 3c \\
&\leqslant (a+b)^- \wedge 3c + a^+ \wedge 3c + b^+ \wedge 3c \\
&\leqslant \big((a+b)^- + a^+ + b^+\big) \wedge 9c \\
&\leqslant (a+b)^+ \wedge 9c + a^- \wedge 9c + b^- \wedge 9c.
\end{aligned}$$

Now

$$\|a(T_\lambda(x_1+x_2),9nT_\lambda y,\lambda)-a(T_\lambda(x_1+x_2),nT_\lambda y,\lambda)+a(-T_\lambda x_1,9nT_\lambda y,\lambda)-$$
$$-a(-T_\lambda x_1,nT_\lambda y,\lambda)+a(-T_\lambda x_2,9nT_\lambda y,\lambda)-a(-T_\lambda x_2,nT_\lambda y,\lambda)\|$$
$$\geqslant \|a(-T_\lambda(x_1+x_2),3nT_\lambda y,\lambda)+a(T_\lambda x_1,3nT_\lambda y,\lambda)+a(T_\lambda x_2,3nT_\lambda y,\lambda)-$$
$$-\big(a(T_\lambda(x_1+x_2),nT_\lambda y,\lambda)+a(-T_\lambda x_1,nT_\lambda y,\lambda)+a(-T_\lambda x_2,nT_\lambda y,\lambda)\big)\|.$$

The usual double limit argument produces

$$\|P_y(-x_1-x_2)+P_y(x_1)+P_y(x_2)-[P_y(x_1+x_2)+P_y(-x_1)+P_y(-x_2)]\|$$
$$\leqslant \|P_y(x_1+x_2)+P_y(-x_1)+P_y(-x_2)-\big(P_y(x_1+x_2)+P_y(-x_1)+P_y(-x_2)\big)\|$$
$$=0.$$

Hence $x\mapsto P_y(x)-P_y(-x)$ is additive as required.

We now have the machinery and can make X into a complex vector lattice quite quickly. Choose a maximal pairwise disjoint subset D of x.

THEOREM 5.14. *For fixed $x\in X$,*

$$\{y\in D\colon J_y(x)\neq 0\}\supset\{y\in D\colon P_y(x)\neq 0\}$$

and the larger set is at most countable. Further, the series $\sum_{y\in D}P_y(x)$ is convergent so that $\sum_{y\in D}J_y(x)$ is also convergent. In addition $x=\sum_{y\in D}J_y(x)$.

Proof. Let $\{y_n\}$ be a countable subset of D. By Proposition 5.12, if $w_n=P_{y_n}(x)$ and $v_n=\sum_{m=1}^{n}P_{y_m}(x)$, then (v_n) is $\mathscr{P}$-increasing and dominated by x. Since $\mathscr{P}$ is an order continuous paving, (v_n) is a convergent sequence in X. In particular, $\|P_{y_n}(x)\|\to 0$ $(n\to\infty)$. The countability assertions and the convergence of $\sum P_y(x)$ and $\sum J_y(x)$ are immediate.

Let $z=\sum_{y\in D}J_y(x)$. Since J_y is linear and continuous, $J_{y_0}(x-z)$ $=J_{y_0}x-J_{y_0}x=0$ $(y_0\in D)$. By maximality of D, $x-z=0$ and we are done.

DEFINITION 5.15. The *real part*, Y, of X is defined by

$$Y=\{x\in X\colon P_y(\pm ix)=0\ \ (y\in D)\}.$$

The *positive cone*, K, of X is defined by

$$K=\{x\in X\colon J_y(x)=P_y(x)\quad (y\in D)\}.$$

Observe that $K\subset Y$ by Proposition 5.4(vi) and that Y is a real subspace of X by Proposition 5.4(iii). If $x\in Y\cap iY$, we have $P_y(i^kx)=0$ $(k=0,1,2,3)$ so that $J_y(x)=0$ $(y\in D)$ and $x=0$ by maximality. Also K is a cone in Y by Proposition 5.4 (v).

THEOREM 5.16. *The cone K induces a lattice order of Y under which Y has order continuous norm.*

Proof. If $x\in K\cap(-K)$, we have $0=J_y(x)+J_y(-x)=P_y(x)+P_y(-x)$. By Proposition 5.4 (i), $J_y(x)=P_y(x)-P_y(-x)=0$ $(y\in D)$ and by maximality $x=0$. Thus K partially orders Y.

Suppose $x\in Y$. By Theorem 5.14 and the definition of Y,

$$x=\sum_{y\in D}P_y(x)-\sum_{y\in D}P_y(-x).$$

Define $x^+=\sum_{y\in D}P_y(x)$, then $x^+\in K$ (Propositions 5.11, 5.10, 5.4 (iv)). Similarly, $x^+-x=(-x)^+\in K$. Suppose $z\in K$ and $z-x\in K$. Consider $P_y(x^+-z)$. By Propositions 5.4 (ix) and 5.9

$$\begin{aligned}P_y(x^+-z)&=P_y\big(J_y(x^+-z)\big)=P_y\big(J_y(x^+)-J_y(z)\big)\\&=P_y\big(P_y(x)-P_y(z)\big)=P_y\big(J_y(x)-P_y(z)\big)=P_y\big(J_y(x)-J_y(z)\big)\\&=P_y\big(J_y(x-z)\big)=P_y(x-z)=P_y(z-x)-J_y(z-x)=0.\end{aligned}$$

Thus $J_y(z-x^+)=P_y(z-x^+)$ $(y\in D)$ and $z-x^+\in K$.

This shows that K lattice orders Y, and that x^+ as defined above is $x\vee 0$ in the lattice order induced by K. If $x\in Y$, $|x|=\sum_{y\in D}\big(P_y(x)+P_y(-x)\big)$. Since $P_y(x)\perp P_y(-x)$ $(y\in D)$ and $P_y(\pm x)\perp y'$ $(y'\in D,\ y'\neq y)$, Proposition 5.8 (i) gives

$$\||x|\|=\Big\|\sum_{y\in D}\big(P_y(x)+P_y(-x)\big)\Big\|=\Big\|\sum_{y\in D}P_y(x)-P_y(-x)\Big\|=\|x\|.$$

An obvious, and easy, limiting argument shows that if $x,\ y-x\in K$, $\|y\|\geqslant\|x\|$. Thus Y is a Banach lattice under the order induced by K.

Suppose that the norm on Y (induced from X) is not order continuous. By a result of Meyer–Nieberg [19] there is a linear lattice isomorphism $\theta\colon c_0\to Y$ such that the unit ball of θc_0 has an upper bound in Y. Let $x\in K$ be this upper bound and let $\theta e_n=y_n$, where $\{e_n\}$ is the standard basis in c_0. Since θ is an order isomorphism, $\{y_n\}$ is a pairwise disjoint sequence in Y. Since $x\geqslant y_n\geqslant 0$ for all n, $J_{y_n}(x)=P_{y_n}(x)\geqslant y_n$. By Proposition 5.11, $\sum_{n=1}^{\infty}P_{y_n}(x)$ is convergent. Hence $\|y_n\|\leqslant\|P_{y_n}(x)\|\to 0$. Thus θ is not an isomorphism. This proves our theorem.

PROPOSITION 5.17. *With Y as its real part, X is a complex vector lattice.*

Proof. By Theorem 5.13, if $x\in X$,

$$x=\sum_{y\in D}J_y(x)=\sum_{y\in D}\big(P_y(x)-P_y(-x)\big)+i\Big(\sum_{y\in D}P_y(-ix)-P_y(ix)\Big).$$

By Propositions 5.4, 5.8, 5.11,

$$P_{y_0}\Big(\pm i\sum_{y\in D}\big(P_y(x)-P_y(-x)\big)\Big)=P_{y_0}\big(\pm iP_{y_0}(x)\mp iP_{y_0}(-x)\big)=0.$$

Thus $X=Y+iY$. If $x\in Y\cap iY$, we have $P_y(\pm ix)=0=P_y(\pm x)$ $(y\in D)$. Hence $x\perp y$ $(y\in D)$ and by maximality, $x=0$. Since Y has order continuous norm, Y is order complete. For any real θ,

$$P_y(e^{i\theta}x)=P_y\big(e^{i\theta}J_y(x)\big)\leqslant P_y(x)+P_y(-x)+P_y(-ix)+P_y(ix).$$

Hence

$$|x|=\bigvee\{\operatorname{Re}e^{i\theta}x\colon 0\leqslant\theta\leqslant 2\pi\}=\bigvee\Big\{\sum_{y\in D}P_y(e^{i\theta}x)\colon 0\leqslant\theta\leqslant 2\pi\Big\}.$$

At this point things get hard again. We have to show that X is a complex Banach lattice.

LEMMA 5.18. *If $y_1\perp y_2$, then $P_{y_1+y_2}=P_{y_1}+P_{y_2}$.*

Proof. Suppose u, v, w are complex numbers and $0<\theta<1$; we have

$$\Big|\big((\operatorname{Re}w\operatorname{sgn}\overline{u+v})^+\wedge|u+v|\big)\operatorname{sgn}(u+v)-$$
$$-\big((\operatorname{Re}w\operatorname{sgn}\bar u)^+\wedge|u|\big)\operatorname{sgn}u-\big((\operatorname{Re}w\operatorname{sgn}\bar v)^+\wedge|v|\big)\operatorname{sgn}v\Big|$$
$$\leqslant 2\theta|w|/(1-\theta^2)+(2+4\theta^{-1})|u|\wedge|v|.$$

If $\theta|u|\leqslant|v|\leqslant\theta^{-1}|u|$, then $|u|,|v|\leqslant\theta^{-1}(|u|\wedge|v|)$ and $4\theta^{-1}|u|\wedge|v|$ dominates the left-hand side. If $|v|<\theta|u|$, we have

$$|\operatorname{sgn}(u+v)-\operatorname{sgn}u|=|\operatorname{sgn}(1+v/u)-1|\leqslant\tan(\sin^{-1}\theta)$$
$$=\theta/(1-\theta^2)^{1/2}\leqslant\theta/(1-\theta^2).$$

Hence,

$$\Big|\big((\operatorname{Re}w\operatorname{sgn}\overline{u+v})^+\wedge|u+v|\big)\operatorname{sgn}(u+v)-\big((\operatorname{Re}w\operatorname{sgn}\bar u)^+\wedge|u|\big)\operatorname{sgn}u\Big|$$
$$\leqslant 2|w|\,|\operatorname{sgn}(u+v)-\operatorname{sgn}u|+\big||u+v|-|u|\big|\leqslant 2\theta|w|/(1-\theta^2)+|u|\wedge|v|.$$

Our inequality follows for $|v|<\theta|u|$ and similarly for $|u|<\theta|v|$.

Now apply the inequality in V_λ with $w=T_\lambda x$, $u=nT_\lambda y_1$, $v=nT_\lambda y_2$. Since $y_1\perp y_2$ and $|u|\wedge|v|\leqslant\sum_{k=0}^{3}|a(i^k u, v)|\to 0$, we have

$$\big\|a\big(x, n(y_1+y_2)\big)-a(x, ny_1)-a(x, ny_2)\big\|\leqslant\theta\|x\|/(1-\theta^2).$$

Let $\theta\to 0$ and then $n\to\infty$ to obtain $P_{y_1+y_2}=P_{y_1}+P_{y_2}$ as required.

LEMMA 5.19. *If $y_1,\ldots,y_n$ are distinct elements of D, the real part Y of X and the positive cone K of X are unchanged if $y_1,\ldots,y_n$ are deleted from D and replaced by $y_0=y_1+\ldots+y_n$.*

Proof. Immediate.

LEMMA 5.20. *For each $x\in X$, $y\in D$, we have $\|J_yx\|=\big\||J_yx|\big\|$.*

Proof. Assume that $x=J_yx$. We will compute in V_λ with the following notation: $v=T_\lambda y$; $u=T_\lambda x\operatorname{sgn}\bar v$; n, N, $k_1,\ldots,k_n$ are positive integers; $a_1,\ldots,a_n$ are complex numbers of absolute value 1;

$$w_N=(\operatorname{Re}u)^+\wedge N|v|-(\operatorname{Re}u)^-\wedge N|v|+i\big((\operatorname{Im}u)^+\wedge N|v|-(\operatorname{Im}u)^-\wedge N|v|\big);$$

$$z(k_1,\ldots,k_n)=((\operatorname{Re}((\operatorname{Re}\ldots((\operatorname{Re}((\operatorname{Re}(a_1u-a_2u))^+\wedge k_1|v|)+$$
$$+a_2u-a_3u)^+\wedge k_2|v|)+\ldots+a_{n-1}u-a_nu))^++a_nu))^+\wedge k_n|v|;$$

and $z_N^*(k_1,\ldots,k_n)$ is the element obtained by replacing u by w_N in the formula for $z(k_1,\ldots,k_n)$. Now, observe that

$$\|z(k_1,\ldots,k_n)\|=\|z(k_1,\ldots,k_n)\operatorname{sgn}v\|$$

and that

$$\lim_{k_1,\ldots,k_n\to\infty}\ \lim_\lambda T_\lambda^{-1}z(k_1,\ldots,k_n)\operatorname{sgn}v$$
$$=P_y(P_y\ldots(P_y(P_y(a_1x-a_2x)+a_2x-a_3x)+\ldots+a_{n-1}x-a_nx)+a_nx)$$
$$=((\operatorname{Re}((\operatorname{Re}\ldots(\operatorname{Re}((\operatorname{Re}(a_1x-a_2x))^++a_2x-a_3x))^++\ldots+a_{n-1}x-a_nx))^++$$
$$+a_nx))^+$$
$$=\bigvee_{j=1}^{n}(\operatorname{Re}a_jx)^+.$$

Since,

$$z(k_1,\ldots,k_n)\leqslant((\operatorname{Re}\ldots((\operatorname{Re}(a_1u-a_2u))^++\ldots+a_nu))^+$$
$$=\bigvee_{j=1}^{n}(\operatorname{Re}a_ju)^+\leqslant|u|,$$

we have

$$\|z(k_1,\ldots,k_n)\operatorname{sgn}v\|\leqslant\big\||u|\operatorname{sgn}v\big\|=\|T_\lambda x\|.$$

This gives

$$\Big\|\bigvee_{j=1}^{n}(\operatorname{Re}a_jx)\Big\|=\lim_{k_1,\ldots,k_n\to\infty}\ \lim_\lambda\|z(k_1,\ldots,k_n)\|\leqslant\|x\|.$$

Because the norm in Y is order continuous we can conclude that $\big\||x|\big\|\leqslant\|x\|$.

For the reverse inequality we work a little harder. Since $|w_N|\leqslant 2N|v|$, we see that if $2nN\leqslant\min\{k_1,\ldots,k_n\}$,

$$z_N^*(k_1,\ldots,k_n)=((\operatorname{Re}\ldots((\operatorname{Re}(a_1w_N-a_2w_N))^++\ldots+a_nw_N))^+$$
$$=\bigvee_{j=1}^{n}(\operatorname{Re}a_jw_N)^+.$$

Let ω be a complex nth root of unity and observe that if $a_j=\omega^j$ $(j=1,\ldots,n)$,

$$\bigvee_{j=1}^{n}(\operatorname{Re}a_jw_N)^+\geqslant\cos(\pi/n)|w_N|=\cos(\pi/n)|w_N\operatorname{sgn}v|.$$

Hence, for this choice of a_j we have

$$\|z(k_1, \ldots, k_n)\| \geqslant \cos(\pi/n)\|w_N \operatorname{sgn} v\| - \|z(k_1, \ldots, k_n) - z_N^*(k_1, \ldots, k_n)\|.$$

Now

$$\|z(k_1, \ldots, k_n) - z_N^*(k_1, \ldots, k_n)\| \leqslant 2n\|u - w_N\| = 2n\|u \operatorname{sgn} v - w_N \operatorname{sgn} v\|.$$

Hence

$$\|z(k_1, \ldots, k_n)\| \geqslant \cos(\pi/n)\|w_N \operatorname{sgn} v\| - 2n\|u \operatorname{sgn} v - w_N \operatorname{sgn} v\|.$$

Let λ vary, $k_1, \ldots, k_n \to \infty$ and then $N \to \infty$. The left-hand side becomes

$$\Big\|\bigvee_{j=1}^{n} (\operatorname{Re} a_j x)^+\Big\| \leqslant \big\||x|\big\|;$$

the right-hand side becomes

$$\cos(\pi/n)\|J_y x\| - 2n\|x - J_y x\| = \cos(\pi/n)\|x\|,$$

because $x = J_y x$. Hence $\big\||x|\big\| \geqslant \cos(\pi/n)\|x\|$ and we can let $n \to \infty$ and obtain $\big\||x|\big\| = \|x\|$ as required.

THEOREM 5.21. *The complex vector lattice structure of X is that of a Banach lattice, with order continuous norm.*

Proof. Let $y_1, \ldots, y_n$ be distinct elements of D. By Lemma 5.18, the band projection $J_{y_1+\ldots+y_n} = J_{y_1} + \ldots + J_{y_n}$. If $x \in X$, we have $|x| = \sum |J_y x|$. Since

$$\Big|\sum_{j=1}^{n} J_{y_j} x\Big| = \sum_{j=1}^{n} |J_{y_j} x|,$$

we have only to show that

$$\Big\|\Big|\sum_{j=1}^{n} J_{y_j} x\Big|\Big\| = \Big\|\sum_{j=1}^{n} J_{y_j} x\Big\|.$$

By Lemma 5.18, $\sum_{j=1}^{n} J_{y_j} x = J_{y_0} x$ where $y_0 = y_1 + \ldots + y_n$. By Lemma 5.19, we can compute $|J_{y_0} x|$ in terms of P_{y_0} so by Lemma 5.20 $\big\||J_{y_0} x|\big\| = \|J_{y_0} x\|$. We are done.

If we now combine Theorem 5.21 with Theorem 4.2 we have the following.

THEOREM 5.22. *A Banach space X can be given the structure of a Banach lattice with order continuous norm if and only if X admits an order continuous pre-lattice paving.*

Let us note that we obtain as a consequence the $\mathscr{L}_{p,\lambda}$ characterization of $L_p(\mu)$ spaces.

COROLLARY 5.23. *Let X be a complex Banach space. Then X is an $\mathscr{L}_{p,\lambda}$ space for all $\lambda > 1$ if and only if $X = L_p(\mu)$ for some measure μ.*

For, we have noted that if X is an $\mathscr{L}_{p,\lambda}$ space ($1 \leqslant p < \infty$, $p \neq 2$) for all $\lambda > 1$, then $\mathscr{P} = \{l_p(n)\}$ is an order-continuous pre-lattice paving of X. Thus X has a Banach lattice structure as above. It is easy to see that this structure satisfies the condition that $|x| \wedge |y| = 0$ implies that $\|x+y\|^p = \|x\|^p + \|y\|^p$ (for $p = 2$ it is easily seen that X satisfies the parallelogram law). Hence $X = L_p(\mu)$ for some measure μ (see [10]). We note that this proof does not use conditional expectation and contractive projection theory as the proofs in [4] and [20] do.

We can also obtain as a corollary a well-known result of Grothendieck without using the theory of $\mathscr{P}_1$ spaces and normal measures.

COROLLARY 5.24. *Let X be a Banach space and suppose that $X^* = C(T)$ for some compact Hausdorff space T. Then $X = L_1(\mu)$ for some measure μ.*

Clearly, we need only show that X is an $\mathscr{L}_{1,\lambda}$ space for all $\lambda > 1$. But this follows from the principle of local reflexivity and the fact that $X^{**} = C(T)^*$ is an $\mathscr{L}_{1,\lambda}$ space for all $\lambda > 1$.

We now indicate how to obtain the isomorphic case from the isometric case by using a re-norming technique and lemma suggested to us by W. B. Johnson. The lemma is similar to Lemma 3.2 and we only state it.

LEMMA 5.25. *Let $\varepsilon > 0$ and k be given. Then there is an N such that for each finite dimensional Banach lattice V, if $x_1, \ldots, x_k$ are in V and $\|x_i\| = 1$, then there are disjoint $h_1, \ldots, h_N$ in V and A_{ij} ($i = 1, \ldots, k$; $j = 1, \ldots, N$) such that*

$$\sum_{i=1}^{k} \Big\|x_i - \sum_{j=1}^{N} A_{ij} h_i\Big\| < \varepsilon.$$

In particular, there is a linear map S of $\operatorname{sp}\{x_1, \ldots, x_k\} = F$ into $\operatorname{sp}\{h_1, \ldots, h_N\}$ such that

$$\|x - Sx\| < \varepsilon\|x\| \quad \textit{for all } x \in F.$$

The proof is similar to that of Lemma 3.2 and is obtained by choosing an appropriate partition of the closed unit ball in C^{k-1} (under supremum norm).

THEOREM 5.26. *Let X be a Banach space. Then X is linearly isomorphic to a Banach lattice with order continuous norm if and only if it has an order continuous pre-lattice paving with respect to some class $\mathscr{P}$ of finite dimensional Banach lattices.*

The necessity is immediate. We shall outline how the sufficiency is verified. Thus we assume that $X = \bigcup_\gamma G_\gamma$ where G_γ is an upwards directed family of finite dimensional subspaces and there are finite dimensional Banach lattices V_γ and isomorphisms T_γ of G_γ onto V_γ satisfying conditions (A) and (B) in Section 2 and the conditions of order continuity in Section 4. Without loss of generality we may assume that $\big\||x|\big\| = \lim \|T_\gamma x\|$.

exists for all $x \in X$ and, clearly, $|||\ |||$ is an equivalent norm on X. Thus it only remains to show that X has an isometric order continuous pre-lattice paving with respect to this norm.

Let $\varepsilon_\gamma \to 0$ and for a given γ choose N to satisfy Lemma 5.25 for $\varepsilon_\gamma = \varepsilon$ and $k = \dim G_\gamma$. Clearly, one can find a $\gamma_0 > \gamma$ such that $\eta(\gamma')$ is small for all $\gamma' > \gamma_0$. By Lemma 5.25, there is a block basis $h_1, \ldots, h_N$ in V_{γ_0}, relative to the positive disjoint normalized basis of V_{γ_0}, and an operator S of $Y = T_{\gamma_0}(G_\gamma)$ to $\mathrm{sp}\{h_1, \ldots, h_N\}$ such that $\|y - Sy\| < \varepsilon\|y\|$ for all $y \in Y$. From the definition of the norm it is possible to choose $\gamma_1 > \gamma_0$ such that

$$(1-\varepsilon)|||x||| \leqslant \|T_{\gamma_1}x\| \leqslant (1+\varepsilon)|||x|||$$

for all $x \in F = \mathrm{sp}\{x_1, \ldots, x_N\}$, where $x_i = T_{\gamma_0}^{-1}(h_i)$. Let

$$h_i^* = T_{\gamma_1}x_i - \Big(|T_{\gamma_1}x_i| \wedge \sum_{j \neq i} |T_{\gamma_1}x_i|\Big)\operatorname{sgn} T_{\gamma_1}x_i.$$

Then

$$|h_i^*| \wedge |h_j^*| = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad \|T_{\gamma_1}x_i - h_i^*\| \leqslant \eta(\gamma_0)N\Big\|\sum T_{\gamma_0}x_i\Big\|$$

and, hence,

$$\Big\|\sum a_i T_{\gamma_1}x_i - \sum a_i h_i^*\Big\| \leqslant \eta(\gamma_0)NK\Big\|\sum T_{\gamma_1}x_i\Big\|$$

where K is a constant depending only on G_γ (see the proof of Lemma 3.5). We can choose $\eta(\gamma_0)$ small and so we put $T'_\gamma = T_{\gamma_1}|F$, where in the range W_γ of T'_γ the norm is given by $\|\sum a_i T_{\gamma_1}x_i\| = \|\sum a_i h_i^*\|$ and positivity by $a_i \geqslant 0$ for $i = 1, \ldots, N$. Then W_γ is a Banach lattice and it can be shown that there is an isometric pre-lattice paving using the above and techniques in the proof of Lemma 1.1. A routine calculation also shows that it is order continuous since the original one is. Thus the conclusion follows from Theorem 5.22.

Finally, we give an example of a pre-dual L_1 space which is not isomorphic to a Banach lattice. We shall need the following proposition mentioned in [15] without proof. The proof indicated here was provided to us by Lior Tzafriri.

PROPOSITION 5.27. *The Banach space l_1 has a unique Banach lattice structure.*

Proof. Suppose X is a Banach lattice which is linearly isomorphic to l_1. Then since X does not contain c_0, the norm in X is order continuous [19]. Hence by Theorem 10 of [17] X is order isomorphic to $L_1(\mu)$ for some measure space μ. Clearly, μ is purely atomic since otherwise $L_1(\mu)$ contains a subspace isomorphic to separable Hilbert space l_2 (see [10]).

Now, Benyamini and Lindenstrauss [2] have given an example of a pre-dual L_1 space X such that $X^* = l_1$ and X is not isomorphic to a space $C(T)$. Suppose X is isomorphic to a Banach lattice Y. Then by the above Y^* is order isomorphic to l_1 and Y^{**} is order isomorphic to l_∞. Thus Y is order isomorphic to a separable M space. But, Benyamini [1] has shown that a separable M space is isomorphic to $C(T)$ for some compact metrizable space T. Thus X is not isomorphic to any Banach lattice, but X has local unconditional structure for all $\lambda > 1$ with respect to $\{l_\infty(n)\}$.

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Extension of real-valued α-additive set functions

by
R. B. KIRK and J. A. CRENSHAW (Carbondale, Ill.)

Abstract. The extension of real-valued α-additive finite finitely additive regula. real-valued set functions from an algebra of sets to larger σ-algebras of sets is givenr The extensions are then used to obtain results on $\sigma(A^*, A)$ convergence of τ-additive functionals on an algebra A of real-valued functions on a set X.

Introduction. Let A be a uniformly closed algebra of bounded real-valued functions on a set X which separates the points of X and contains the constants. Let X be equipped with the τ_A topology which is the weakest topology on X which makes each $f \in A$ continuous. In [4] the concept of α-additive set functions on a paving $\mathscr{W}$ of subsets of X was introduced to represent the α-additive functionals in A^*, and it was indicated that the α-additive set functions could be extended to α-additive elements on a larger paving (this includes the fact that τ-additive Baire measures in $C^b(X)$ can be extended to Borel measures on X). We shall establish this extension process which depends on which definition of outer measure is chosen. We then employ the extension to questions about weak, $\sigma(A^*, A)$, convergence of elements in A^*. We anticipate that working with a paving and that working with subalgebras of $C^b(X)$ will prove useful in probability theory, and in this direction we obtain a weakened form of Prochorovs' theorem. Also for subalgebras $A_2 \subset A_1$ we give sufficient conditions for weak convergence of τ-additive φ in A_1^* to be determined by the elements of A_2.

The authors wish to thank the referee for pointing out that our results in Section 1 should extend to exaustive functions with range a suitably endowed topological group. He also noted some of the rich literature on the subject such as done by Drewnowski [2], Sion [6] and Traynor [7]. The referee is of course correct and the authors intend to show this and that the weak additivity condition does yield the usual additivity condition in a different paper.

§ 1. Extension. We refer the reader to [4] for many of the basic definitions and results; however, we shall indicate here some of the essential definitions.

A *paving* on X is a family $\mathscr{W}$ of subsets which contains $\emptyset$, is closed under finite unions and intersections, and has $X = \bigcup \mathscr{W}$. The paving is *full* if $X \in \mathscr{W}$ and in this paper all pavings will be assumed to be full.