Putting $x^0 = y_1 - S(t_u)x_1$, we obtain the required control.

Of course the existence of a universal time t_u in system (5) implies that

$$\bigcup_{0\leqslant t\leqslant t_u} C_t X = Y. \tag{24}$$

If C_t has only a countable number of values and (5) is controllable, then (25) holds. An assumption of type (18) (or about a countable number of values of C_t) cannot be replaced by continuity, as follows from the two examples given below, even in finite-dimensional spaces.

EXAMPLE 1. Let $X = Y = C$ be a complex plane considered as a two-dimensional real Banach space.

Let $C_t z = e^{\pi\left(\frac{t}{t+1}\right)i}\operatorname{Re} z$, where $\operatorname{Re} z$ denotes the real part of z. It is easy to verify that

$$Y = \bigcup_{0\leqslant t\leqslant +\infty} C_t X \tag{25}$$

and that for every $t_0 < +\infty$

$$Y \neq \bigcup_{0\leqslant t\leqslant t_0} C_t X. \tag{26}$$

On the other hand, C_t is continuous in the norm topology.

The next example shows that we can replace (26) by a stronger condition,

$$Y = \bigcap_{t\leqslant 0}\ \bigcup_{t\leqslant \tau<+\infty} C_\tau X, \tag{27}$$

and still inequality (26) holds.

EXAMPLE 2. Let X, Y be as before. Let $C_t z = e^{\pi\left(\frac{t}{t+1}\right)\sin^2 ti}\operatorname{Re} z$. It is easy to verify that C_t is norm-continuous. Of course, C_t satisfies (27) and (26).

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Saturation for Favard operators in weighted function spaces

Dedicated to Jean Favard on the occasion of the tenth anniversary of his death on January 21, 1965

by

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Abstract. This note continues the investigation of the operators

$$F_n^\gamma f(x) := \frac{1}{\sqrt{\pi\gamma n}}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right) \quad (\gamma>0,\ n\in\boldsymbol{N})$$

introduced by J. Favard in 1944 for $\gamma = 1$ as a discrete analog of the familiar Gauss–Weierstrass convolution integral. These Favard operators give approximation on the whole real axis $\boldsymbol{R}$ and are of special interest with regard to approximation in locally convex spaces. The saturation problem for $F_n^\gamma f$ on the Banach space

$$X_N := \{f\in C(\boldsymbol{R});\ (1+x^{2N})^{-1}f(x) = o(1),\ |x|\to\infty\} \quad (N\in\boldsymbol{N})$$

is solved by employing a theorem of H. F. Trotter (1958/59) on the convergence of semigroups of operators. Thus the family of noncommutative operators $\{F_n^\gamma;\ n\in\boldsymbol{N}\}$ is associated with a family of commutative operators having the same saturation class, in this case just the Gauss–Weierstrass integral. For this purpose asymptotic estimates are derived which are needed for verifying the hypotheses of the Trotter theorem. Finally, instead of the weight functions $(1+x^{2N})^{-1}$, also the functions $\exp(-\beta x^2)$, $\beta>0$, are considered.

1. Introduction.

In this note we would like to study the Favard operators

$$F_n^\gamma f(x) := \frac{1}{\sqrt{\gamma\pi n}}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right) \tag{1.1}$$

with $\gamma>0$, $n\in\boldsymbol{N}$, the set of positive integers. These operators were introduced by Favard [8], pp. 229, 239, in 1944 for $\gamma = 1$ as discrete analogs of the familiar Weierstrass operators

$$W_n^\gamma f(x) := \sqrt{\frac{n}{\gamma\pi}}\int_{-\infty}^{\infty} f(u)\exp\left(-\frac{n}{\gamma}(u-x)^2\right)du. \tag{1.2}$$

Note that (1.1) is also related to the $(e, 1)$ or Valiron method (cf. [18], p. 143) of summability of a sequence $\{s_k\}_{k=0}^{\infty}$ defined by

$$\frac{1}{\sqrt{\pi y}}\sum_{k=0}^{\infty}\exp\left(-y\left(\frac{k}{y}-1\right)^2\right)s_k \qquad (y\to\infty).$$

Favard was able to prove the following result.

THEOREM F. *Let f be continuous on the real axis $\boldsymbol{R}$ such that $|f(x)| \leqslant A\exp(Bx^2)$ for positive constants A, B, and $x\in\boldsymbol{R}$. Then $F_n^1 f(x)$ converges to $f(x)$ as $n\to\infty$ pointwise for every $x\in\boldsymbol{R}$, and even uniformly on any compact subinterval of $\boldsymbol{R}$.*

Although the Favard operators do not seem to have been studied by other authors, they deserve much more attention since they are an example of a process giving genuine approximation on the whole real axis. They will, moreover, turn out to be of particular interest with regard to approximation in locally convex spaces.

In this paper we therefore continue Favard's investigations, particularly discussing saturation for the family $\{F_n^\gamma\}_{n=1}^{\infty}$ on the Banach space $(N\in\boldsymbol{N})$

$$\begin{aligned} &X_N := \{f\in C(\boldsymbol{R});\ f(x) = o(1+x^{2N}),\ |x|\to\infty\},\\ &\|f\|_N := \|(1+x^{2N})^{-1}f(x)\| := \sup_{x\in\boldsymbol{R}}|(1+x^{2N})^{-1}f(x)|, \end{aligned} \tag{1.3}$$

$C(\boldsymbol{R})$ being the set of all continuous functions on $\boldsymbol{R}$. The results will be deduced as an application of a general theory in Banach spaces (and even in certain locally convex spaces) elaborated in [2] and based upon a theorem of Trotter [16] on the convergence of sequences of semigroups. Thus, not only does the Weierstrass operator (1.2) serve as a guide for what might be expected in connection with (1.1), but the results for (1.1) are actually obtained by reduction to the known ones for (1.2) via the theorem of Trotter.

Whereas this abstract theory will be formulated in Section 3, Section 2 provides some fundamental estimates needed for the application of the general theory to the operators (1.1) to be given in Section 4, at the same time improving certain asymptotic estimates of Favard. Finally, in Section 5 the operators (1.1) are considered on the more general spaces (5.1), already envisaged by Favard's result. However, from the point of view of an abstract theory the situation seems to be less satisfactory so that the result in this case should be proved by direct (and more elaborate) methods.

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2. Some fundamental estimates. Following the standard procedure in the discussion of the approximation-theoretical properties of interpolatory analogs (Bernstein polynomials, Szász operators, etc.) we have to derive precise asymptotic estimates for the quantities ($x\in\boldsymbol{R}$, $n\in\boldsymbol{N}$, $r\in\boldsymbol{P}$, the set of non-negative integers)

$$F_n^\gamma 1(x) := \frac{1}{\sqrt{\gamma\pi n}}\sum_{k=-\infty}^{\infty}\exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right), \tag{2.1}$$

$$T_{n,r}^\gamma(x) := \frac{1}{\sqrt{\gamma\pi n}}\sum_{k=-\infty}^{\infty}\left(\frac{k}{n}-x\right)^r\exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right). \tag{2.2}$$

Now by a classical relation for the theta-function θ_3 (cf. [3], p. 26) one has

$$F_n^\gamma 1(x) = 1+2\sum_{k=1}^{\infty}\exp(-n\gamma\pi^2k^2)\cos(2\pi nkx). \tag{2.3}$$

To estimate the right-hand side of (2.3) we need two lemmas.

LEMMA 2.1. *Let $f_a(t) := \big(t/(ae)\big)^t$ for $a, t>0$. Then for any y, $\alpha+\beta>0$*

$$y^\alpha e^{-ay} \leqslant f_a(\alpha+\beta)\cdot y^{-\beta}.$$

Proof. Setting $g(y) := y^{\alpha+\beta}e^{-ay}$ for $y>0$, it is easily verified that g attains its maximum at $(\alpha+\beta)/a$ so that for any $y>0$

$$y^{\alpha+\beta}e^{-ay} \leqslant \big((\alpha+\beta)/a\big)^{\alpha+\beta}e^{-(\alpha+\beta)}.$$

LEMMA 2.2. *For any $\gamma>0$, $\alpha,\beta\geqslant 0$, and $n\in\boldsymbol{N}$,*

$$\sum_{k=1}^{\infty}k^{2\alpha}\exp(-n\gamma\pi^2k^2) = O(n^{-(\alpha+\beta)}) \qquad (n\to\infty). \tag{2.4}$$

Proof. By choosing $y = nk^2$, $a = \gamma\pi^2$ in Lemma 2.1 and summing over k one obtains

$$n^\alpha\sum_{k=1}^{\infty}k^{2\alpha}\exp(-n\gamma\pi^2k^2) \leqslant f_{\gamma\pi^2}(\alpha+\beta)n^{-\beta}\sum_{k=1}^{\infty}k^{-2\beta}.$$

Thus, (2.4) follows for $\beta>1/2$, which in turn implies (2.4) for any $\beta\geqslant 0$.

THEOREM 2.3. *The following estimates hold uniformly on $\boldsymbol{R}$ for $n\to\infty$:*

$$F_n^\gamma 1(x) = T_{n,0}^\gamma(x) = 1+O(n^{-2}), \tag{2.5}$$

$$T_{n,1}^\gamma(x) = O(n^{-2}), \tag{2.6}$$

$$T_{n,2}^\gamma(x) = \gamma/2n+O(n^{-3}). \tag{2.7}$$

Proof. Obviously (2.5) follows from (2.3) and (2.4) for $\alpha = 0$, $\beta = 2$. Differentiation of (2.3) gives

$$(2.8)\qquad (F_n^\gamma 1)'(x) = (2n/\gamma)T_{n,1}^\gamma(x) = -4\pi n\sum_{k=1}^{\infty} k\exp(-n\gamma\pi^2k^2)\sin(2\pi nkx),$$

so that (2.6) is obtained from (2.4) for $\alpha = 1/2$, $\beta = 3/2$. A further differentiation yields

$$(2.9)\qquad (F_n^\gamma 1)''(x) = -(2n/\gamma)F_n^\gamma 1(x)+(4n^2/\gamma^2)T_{n,2}^\gamma(x) = -8\pi^2n^2\sum_{k=1}^{\infty}k^2\exp(-n\gamma\pi^2k^2)\cos(2\pi nkx).$$

Thus

$$T_{n,2}^\gamma(x) = (\gamma/2n)F_n^\gamma 1(x)-2\gamma^2\pi^2\sum_{k=1}^{\infty}k^2\exp(-n\gamma\pi^2k^2)\cos(2\pi nkx).$$

This proves (2.7) with the aid of (2.5) and (2.4) for $\alpha = 1$, $\beta = 2$.

Let $[x]$ denote the greatest integer less than or equal to x.

THEOREM 2.4. *For every $n, k\in \boldsymbol{N}$ there are constants $c_{j,k}$, independent of n such that for $x\in\boldsymbol{R}$*

$$(2.10)\qquad T_{n,k}^\gamma(x) = \left(\frac{\gamma}{2n}\right)^k (F_n^\gamma 1)^{(k)}(x)+\sum_{j=0}^{[k/2]-1} c_{j,k}\left(\frac{\gamma}{2n}\right)^{[k/2]-j} T_{n,2j+e_k}^\gamma(x),$$

where $e_k = 1$ for odd k and $e_k = 0$ for even k.

Proof. First of all, (2.10) is valid for $k = 1, 2$ by (2.8), (2.9). Supposing the equation to be valid for $k = 2m$, we show that it is valid for $k = 2m+2$ and so by induction for all even k. To this end assume

$$(2.11)\qquad T_{n,2m}^\gamma(x) = \left(\frac{\gamma}{2n}\right)^{2m}(F_n^\gamma 1)^{(2m)}(x)+\sum_{j=0}^{m-1}c_{j,2m}\left(\frac{\gamma}{2n}\right)^{m-j}T_{n,2j}^\gamma(x)$$

to be valid. Then by virtue of

$$(2.12)\qquad \frac{d}{dx}T_{n,r}^\gamma(x) = \frac{2n}{\gamma}T_{n,r+1}^\gamma(x)-rT_{n,r-1}^\gamma(x),$$

$$(2.13)\qquad \frac{d^2}{dx^2}T_{n,r}^\gamma(x) = \left(\frac{2n}{\gamma}\right)^2T_{n,r+2}^\gamma(x)-(2r+1)\frac{2n}{\gamma}T_{n,r}^\gamma(x)+r(r-1)T_{n,r-2}^\gamma(x),$$

we have

$$\begin{aligned}
T_{n,2m+2}^\gamma(x) &= \left(\frac{\gamma}{2n}\right)^2\frac{d^2}{dx^2}T_{n,2m}^\gamma(x)+(4m+1)\frac{\gamma}{2n}T_{n,2m}^\gamma(x)- \\
&\qquad -2m(2m-1)\left(\frac{\gamma}{2n}\right)^2T_{n,2m-2}^\gamma(x) \\
&= \left(\frac{\gamma}{2n}\right)^{2m+2}(F_n^\gamma 1)^{(2m+2)}(x)+\sum_{j=0}^{m-1}c_{j,2m}\left(\frac{\gamma}{2n}\right)^{m-j+2}\frac{d^2}{dx^2}T_{n,2j}^\gamma(x)+ \\
&\qquad +(4m+1)\frac{\gamma}{2n}T_{n,2m}^\gamma(x)-2m(2m-1)\left(\frac{\gamma}{2n}\right)^2T_{n,2m-2}^\gamma(x) \\
&= \left(\frac{\gamma}{2n}\right)^{2m+2}(F_n^\gamma 1)^{(2m+2)}(x)+\sum_{j=0}^{m-1}c_{j,2m}\left(\frac{\gamma}{2n}\right)^{m-j}T_{n,2j+2}^\gamma(x)- \\
&\quad -\sum_{j=0}^{m-1}(4j+1)c_{j,2m}\left(\frac{\gamma}{2n}\right)^{m-j+1}T_{n,2j}^\gamma(x)+\sum_{j=1}^{m-1}2j(2j-1)c_{j,2m}\left(\frac{\gamma}{2n}\right)^{m-j+2}\times \\
&\qquad\times T_{n,2j-2}^\gamma(x)+ \\
&\qquad +(4m+1)\frac{\gamma}{2n}T_{n,2m}^\gamma(x)-2m(2m-1)\left(\frac{\gamma}{2n}\right)^2T_{n,2m-2}^\gamma(x) \\
&= \left(\frac{\gamma}{2n}\right)^{2m+2}(F_n^\gamma 1)^{(2m+2)}(x)+\sum_{j=0}^{m}c_{j,2m+2}\left(\frac{\gamma}{2n}\right)^{m+1-j}T_{n,2j}^\gamma(x),
\end{aligned}$$

with appropriate constants $c_{j,2m+2}$. Differentiating (2.11) and using (2.12) delivers the desired result also for odd k.

THEOREM 2.5. *For every $n, k\in\boldsymbol{N}$ one has uniformly on $\boldsymbol{R}$ for $n\to\infty$*

$$T_{n,k}^\gamma(x) = O(n^{-[(k+1)/2]}).$$

In particular, $T_{n,k}^\gamma(x) = O(n^{-1})$.

Proof. By (2.3) and Lemma 2.2 we have

$$\left(\frac{\gamma}{2n}\right)^k(F_n^\gamma 1)^{(k)}(x) = O\Big(2(\pi\gamma)^k\sum_{j=1}^{\infty}j^k\exp(-n\gamma\pi^2j^2)\Big) = O(n^{-[(k+1)/2]})\qquad(n\to\infty).$$

As Theorem 2.3 treats the cases $k = 1, 2$ one may proceed via induction using Theorem 2.4, thus

$$T_{n,k}^\gamma(x) = O(n^{-[(k+1)/2]})+\sum_{j=0}^{[k/2]-1}c_{j,k}\left(\frac{\gamma}{2n}\right)^{[k/2]-j}O(n^{-j-e_k}) = O(n^{-[(k+1)/2]}),$$

since $[k/2]+e_k = [(k+1)/2]$.

3. General theory. The abstract background for our investigations is provided by a theorem of H.F. Trotter (cf. [16]; [17]) which asserts that an approximation process $\{S_n\}_1^\infty$ satisfying a Voronovskaja-type condition is closely related to a semigroup of operators which, as we shall see, has the same saturation class as $\{S_n\}_1^\infty$. These results are part of [2].

THEOREM T. *Let $\{S_n\}_1^\infty$ be a sequence of bounded linear operators mapping the Banach space X into itself, and let B be a closed operator mapping its domain $D(B) \subset X$ into X such that*

(3.1) (Voronovskaja-type condition) *there exists a sequence $\{h_n\}_1^\infty$ of positive numbers with $h_n \to 0$ such that for each $f \in D(B)$*

$$\lim_{n\to\infty}(1/h_n)[S_nf-f] = Bf,$$

(3.2) (stability condition) *there exist constants $M, K > 0$, independent of j, $n \in \mathbf{N}$, $f \in X$, such that*

$$\|S_n^j f\| \leqslant M e^{Kjh_n}\|f\|,$$

(3.3) *$D(B)$ is dense in X,*

(3.4) *there exists $\lambda > K$ such that the range of $\lambda I - B$ is dense in X.*

Then B generates a semigroup $\{T(t);\, t \geqslant 0\}$ of class (C_0) (cf. [5], p. 8) *such that for a sequence $\{k(n)\}_1^\infty \subset \mathbf{N}$*

(3.5) $$\lim_{n\to\infty} h_n\cdot k(n) = t \Rightarrow s-\lim_{n\to\infty} S_n^{k(n)} = T(t).$$

For a proof of this theorem we refer to [11]; [16].

In the following $\widetilde{D(B)}^X$ denotes the completion of $D(B)$ relative to X (cf. [1]; [4]).

THEOREM 3.1. *Let X, $\{S_n\}_1^\infty$, $\{T(t);\, t \geqslant 0\}$, and B be as in Theorem T. Then the following assertions are equivalent for $f \in X$:*

(3.6) $$\|S_nf-f\| = O(h_n) \quad (n\to\infty),$$

(3.7) $$\|T(t)f-f\| = O(t) \quad (t\to 0+),$$

(3.8) $$f \in \widetilde{D(B)}^X,$$

(3.9) $$f \in D(B) \quad \textit{if } X \textit{ is reflexive.}$$

For the convenience of the reader let us sketch certain steps of the proof (see also [2]; [9]; [14]).

(3.6) ⇒ (3.7). Let $t \geqslant 0$, $\{k(n)\}_1^\infty \subset \mathbf{N}$ be such that $\lim_{n\to\infty} h_n\cdot k(n) = t$. Then (3.2) yields

$$\|S_n^{k(n)}f-f\| \leqslant \sum_{j=1}^{k(n)} \|S_n^{j-1}(S_nf-f)\| \leqslant M\cdot e^{Kk(n)h_n}[h_n^{-1}\|S_nf-f\|]h_n\cdot k(n).$$

Thus letting $n\to\infty$ (3.7) follows in view of (3.5), (3.6). In particular, one has the following "o"-result: If $f \in X$ is such that $\|S_nf-f\| = o(h_n)$, then $\|T(t)f-f\| = o(t)$, and therefore $f \in D(B)$ and $Bf = 0$ by the classical semigroup-theory (cf. [5], p. 88).

(3.8) ⇒ (3.6). As $D(B)$ is complete with respect to the norm $\|g\|_{D(B)} := \|g\| + \|Bg\|$, in view of (3.1) and the uniform boundedness principle there exists a constant $C > 0$ such that for all $g \in D(B)$

$$\|(1/h_n)[S_ng-g]\| \leqslant C\|g\|_{D(B)}.$$

Given $f \in \widetilde{D(B)}^X$, by definition there exists a sequence $\{f_m\}_1^\infty \subset D(B)$ bounded in $D(B)$ and converging to f in X. Thus there exists $C_1 > 0$ such that

$$\|(1/h_n)[S_nf_m-f_m]\| \leqslant C_1$$

uniformly for $m, n \in \mathbf{N}$. Letting $m\to\infty$ delivers (3.6).

The equivalence of (3.7), (3.8), and (3.9) follows by Theorem 4.1 of [4] (see also [6], p. 505).

4. Saturation for the Favard operators. The foregoing preparations now enable one to study the approximation behaviour of the operators (1.1), in particular to determine their saturation class. In this section the underlying Banach space will be X_N as defined in (1.3), $N \in \mathbf{N}$ being arbitrarily fixed. Let us commence with the stability condition (3.2).

LEMMA 4.1. *For each $N \in \mathbf{N}$ there exists a constant M_N such that for $f \in X_N$*

(4.1) $$\|F_n^\gamma f\|_N \leqslant (1+M_Nn^{-1})\|f\|_N.$$

Thus $\{F_n^\gamma\}_1^\infty$ is a family of bounded linear operators of the Banach space X_N into itself satisfying the stability condition (3.2) *with $M = 1$, $K = M_N$, $h_n = n^{-1}$.*

Proof. We have for $x, u \in \mathbf{R}$

(4.2) $$\frac{1+(x+u)^{2N}}{1+x^{2N}} = 1+\sum_{j=0}^{2N-1}\binom{2N}{j}\frac{x^j}{1+x^{2N}}u^{2N-j}.$$

Since $\|x^j\|_N \leqslant 1$ for $0 \leqslant j \leqslant 2N$, letting $u = (k/n)-x$ it follows by Theorems 2.3, 2.5 that

$$\frac{|F_n^\gamma(1+x^{2N})(x)|}{1+x^{2N}} \leqslant |F_n^\gamma 1(x)| + \sum_{j=0}^{2N-1}\binom{2N}{j}\frac{|x|^j}{1+x^{2N}}|T_{n,2N-j}^\gamma(x)|$$
$$\leqslant 1+O(n^{-2})+\sum_{j=0}^{2N-1}\binom{2N}{j}O(n^{-1}) \leqslant 1+M_Nn^{-1}$$

with some constant M_N uniformly for $x \in \boldsymbol{R}$. Now let $f \in X_N$. Then by definition

$$|F_n^\gamma f(x)| \leqslant \|f\|_N |F_n^\gamma(1+x^{2N})(x)|,$$

so that there follows (4.1). Since $1+x \leqslant e^x$, this implies

$$\|(F_n^\gamma)^j f\| \leqslant e^{M_N j/n} \|f\|_N \tag{4.3}$$

for all $j \in \boldsymbol{N}$, completing the proof.

Next we note that $\{F_n^\gamma\}_1^\infty$ constitutes an approximation process on each X_N.

LEMMA 4.2. *There holds* $\lim\limits_{n\to\infty} \|F_n^\gamma f - f\|_N = 0$ *for each* $f \in X_N$.

Proof. Setting $\omega_1(X_N, f; h) := \sup\limits_{|t|<h} \|f(x+t) - f(x)\|_N$ for $f \in X_N$, it follows that $\lim\limits_{h\to 0+} \omega_1(X_N, f; h) = 0$. Therefore for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega_1(X_N, f; h) < \varepsilon$ for all $|h| < \delta$. Now splitting the subsequent sum into two parts according to $|(k/n)-x| < \delta$ and $|(k/n)-x| \geqslant \delta$, Theorems 2.3, 2.5 imply as in the proof of Lemma 4.1 that

$$\frac{|F_n^\gamma f(x) - f(x)|}{1+x^{2N}} \leqslant \|f\|_N |F_n^\gamma 1(x) - 1| + \frac{1}{\sqrt{\gamma\pi n}} \sum_{k=-\infty}^{\infty} \frac{\left|f\left(\frac{k}{n}\right) - f(x)\right|}{1+x^{2N}} \times$$
$$\times \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n} - x\right)^2\right)$$
$$\leqslant O(n^{-2}) + \varepsilon F_n^\gamma 1(x) + \frac{1}{\sqrt{\gamma\pi n}} \sum_{\left|\frac{k}{n}-x\right| \geqslant \delta} \left[\frac{\left|f\left(\frac{k}{n}\right)\right|}{1+x^{2N}} + \|f\|_N\right] \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n} - x\right)^2\right)$$
$$\leqslant O\left(\varepsilon + \frac{\varepsilon+1}{n^2}\right) + \frac{2\|f\|_N}{\sqrt{\gamma\pi n}} \sum_{\left|\frac{k}{n}-x\right| \geqslant \delta} \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n} - x\right)^2\right) + \|f\|_N \sum_{j=0}^{2N-1} \binom{2N}{j} \times$$
$$\times |T_{n,2N-j}^\gamma(x)|$$
$$\leqslant O\left(\varepsilon + \frac{\varepsilon+1}{n^2}\right) + 2\|f\|_N \delta^{-2} T_{n,2}^\gamma(x) + O\left(\frac{1}{n}\right) = O(\varepsilon + n^{-1})$$

uniformly for $x \in \boldsymbol{R}$.

The next step is the verification of the Voronovskaja-type condition (3.1), thus the determination of the operator B.

THEOREM 4.3. *With* $B := (\gamma/4)\ (d/dx)^2$ *and* $D_N(B) := \{f \in X_N;\ f',\ f'' \in X_N\}$ *one has for any* $f \in D_N(B)$

$$\lim_{n\to\infty} \|n[F_n^\gamma f - f] - Bf\|_N = 0. \tag{4.4}$$

Proof. The Taylor expansion

$$f(x+u) = f(x) + uf'(x) + \frac{u^2}{2} f''(x) + u^2 \int_0^1 (1-t)[f''(x+ut) - f''(x)]\,dt$$

and Theorem 2.3 deliver for $u = (k/n) - x$

$$F_n^\gamma f(x) = f(x) + f(x)\,O(n^{-2}) + f'(x)\,O(n^{-2}) + (\gamma/4n) f''(x) + f''(x)\,O(n^{-3}) +$$
$$+ \frac{1}{\sqrt{\gamma\pi n}} \sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x\right)^2 \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n} - x\right)^2\right) \times$$
$$\times \int_0^1 (1-t)\left[f''\left(x + \left(\frac{k}{n} - x\right)t\right) - f''(x)\right] dt,$$

so that for $f \in D_N(B)$

$$|(1+x^{2N})^{-1}\{n[F_n^\gamma f(x) - f(x)] - (\gamma/4) f''(x)\}|$$
$$\leqslant (\|f\|_N + \|f'\|_N + \|f''\|_N)\,O(n^{-1}) + S(x),$$

$$S(x) := \frac{n}{\sqrt{\gamma\pi n}} \sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x\right)^2 \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n} - x\right)^2\right) \times$$
$$\times \int_0^1 (1-t) \frac{\left|f''\left(x + \left(\frac{k}{n} - x\right)t\right) - f''(x)\right|}{1+x^{2N}}\,dt.$$

Using (4.2) with $u = \big((k/n) - x\big)t$ one may proceed as in the proofs of Lemmas 4.1, 4.2 to obtain

$$S(x) = O\big(\varepsilon + n\|f''\|_N \delta^{-2} T_{n,4}^\gamma(x)\big) = O(\varepsilon + n^{-1}).$$

Thus the operator B is just the infinitesimal generator of the Weierstrass semigroup (1.2), which therefore is the associated semigroup according to Theorem 3.1. Obviously $D_N(B)$ is dense in X_N. Hence it remains to show condition (3.4) to be valid. To this end it is a consequence of the theorem of Stone–Weierstrass that the functions $\{x^j \exp(-\alpha x^2);\ \alpha > 0,\ j \in \{0,1\}\}$ are dense in X_N for each $N \in \boldsymbol{N}$ (cf. [1a], p. 25). Thus (3.4) follows in view of

LEMMA 4.4. *For each* $\alpha > 0$, $\lambda > 0$, $N \in \boldsymbol{N}$ *there is a function* $F_{\alpha,\lambda} \in D_N(B)$ *such that for* $x \in \boldsymbol{R}$

$$(\lambda I - B) F_{\alpha,\lambda}(x) = \lambda F_{\alpha,\lambda}(x) - (\gamma/4) F''_{\alpha,\lambda}(x) = x^j \exp(-\alpha x^2) \quad (j \in \{0,1\})$$

Proof. Setting $\varphi(y) := (\alpha\gamma y/\lambda) + 1$ and

$$F_{\alpha,\lambda}(x) := \frac{1}{\lambda} \int_0^\infty \frac{1}{\sqrt{\varphi(y)}} \left\{\frac{x}{\varphi(y)}\right\}^j \exp\left(-y - \frac{\alpha x^2}{\varphi(y)}\right) dy,$$

one has $(j=0)$

$$F''_{a,\lambda}(x) = \frac{1}{\lambda}\int_0^\infty \frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{\varphi(y)}}\exp\left(-y-\frac{ax^2}{\varphi(y)}\right)dy$$
$$= \frac{1}{\lambda}\int_0^\infty \left\{\frac{4a^2x^2}{[\varphi(y)]^{5/2}}-\frac{2a}{[\varphi(y)]^{3/2}}\right\}\exp\left(-y-\frac{ax^2}{\varphi(y)}\right)dy,$$

so that

$$\lambda F_{a,\lambda}(x)-\frac{\gamma}{4}F''_{a,\lambda}(x) = \int_0^\infty \left\{\frac{1}{\sqrt{\varphi(y)}}+\frac{a\gamma/2\lambda}{[\varphi(y)]^{3/2}}-\frac{\gamma a^2x^2/\lambda}{[\varphi(y)]^{5/2}}\right\}\times$$
$$\times\exp\left(-y-\frac{ax^2}{\varphi(y)}\right)dy$$
$$= \int_0^\infty -\frac{\partial}{\partial y}\frac{1}{\sqrt{\varphi(y)}}\exp\left(-y-\frac{ax^2}{\varphi(y)}\right)dy = \exp(-ax^2).$$

Hence $F_{a,\lambda}$ is a solution of the differential equation. In fact, $F_{a,\lambda}$ belongs to $D_N(B)$ since

$$0 \leqslant F_{a,\lambda}(x) \leqslant \lambda^{-1}\int_0^\infty e^{-y}dy = \lambda^{-1}$$

and analogously, using the differential equation for $F''_{a,\lambda}$,

$$|F'_{a,\lambda}(x)| \leqslant 2a\lambda^{-1}|x|, \quad |F''_{a,\lambda}(x)| \leqslant (4/\gamma)\big(1+\exp(-ax^2)\big).$$

So far we have shown that the Favard operators satisfy the conditions of Theorem 3.1. So we can conclude

THEOREM 4.5. *Let $f \in X_N$ for some $N \in \mathbf{N}$. Then the following three assertions concerning the Favard operators are equivalent:*

(4.5) $$\|F_n^\gamma f - f\|_N = O(n^{-1}) \qquad (n\to\infty),$$

(4.6) $$f \in \widetilde{D_N(B)}^{X_N},$$

(4.7) $$\|f(x+h)-2f(x)+f(x-h)\|_N = O(h^2) \qquad (h\to 0).$$

Proof. (4.5)$\Leftrightarrow$(4.6). This is given by Theorem 3.1.

(4.6)$\Rightarrow$(4.7). Take $f \in \widetilde{D_N(B)}^{X_N}$, i.e., there exists a sequence $\{f_m\}_1^\infty \subset D_N(B)$ bounded in $D_N(B)$ and converging in X_N to f. Then for $x \in \mathbf{R}$ and $h>0$

$$|f_m(x+h)-2f_m(x)+f_m(x-h)| \leqslant \int_{-h/2}^{h/2}\int_{-h/2}^{h/2}|f''_m(x+s+t)|\,ds\,dt$$
$$\leqslant \|f''_m\|_N \int_{-h/2}^{h/2}\int_{-h/2}^{h/2}[1+(x+s+t)^{2N}]\,ds\,dt$$
$$= \|f''_m\|_N\left[h^2+\frac{(x+h)^{2N+2}-2x^{2N+2}+(x-h)^{2N+2}}{(2N+1)(2N+2)}\right]$$
$$= \|f''_m\|_N h^2\left[1+\frac{1}{(2N+1)(N+1)}\sum_{j=0}^{N}\binom{2N+2}{2j}x^{2j}h^{2N-2j}\right],$$

and hence

$$\|f_m(x+h)-2f_m(x)+f_m(x-h)\|_N = O(h^2) \qquad (h\to 0)$$

uniformly for $m \in \mathbf{N}$. Letting $m\to\infty$ delivers (4.7).

(4.7)$\Rightarrow$(4.6). Let $f \in X_N$ satisfy (4.7) and define

$$f_h(x) := \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} f(x+s+t)\,ds\,dt \qquad (h>0).$$

Then $f_h \to f$ in X_N since

$$\|f_h(x)-f(x)\|_N \leqslant \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\|f(x+s+t)-f(x)\|_N\,ds\,dt \leqslant \omega_1(X_N,f;h).$$

Furthermore by assumption

$$\|f''_h\|_N = h^{-2}\|f(x+h)-2f(x)+f(x-h)\|_N = O(1),$$

so that $\{f_h;\ h>0\}$ is also bounded in $D_N(B)$, thus $f \in \widetilde{D_N(B)}^{X_N}$.

5. Functions of exponential growth. So far we have treated the Favard operators on the Banach space X_N, for any fixed $N \in \mathbf{N}$, thus at most polynomial growth of f at infinity is allowed. Obviously the results, derived separately in each X_N, may be taken together and expressed equivalently in terms of the inductive limit $X := \bigcup_{N\in\mathbf{N}} X_N$. In this event one may use the generalizations of Theorems T and 3.1 to locally convex spaces as presented in [13]; [15] and [2], respectively. In this connection let us only point out that the relevant stability condition in the locally convex setting is satisfied by the example of Section 4 since the Favard operators are order-preserving in the sense of [10], i.e., F_n^γ maps X_N into itself (this was already used implicitly in [2] when treating the Szász operators on the corresponding polynomial weight spaces on $[0,\infty)$).

These observations together with Theorem F immediately lead to the problem of how to discuss the Favard operators on more general (locally convex) weight spaces. For example, the Weierstrass operators (1.2) have been considered on (cf. [1]; [7]; [12])

$$X_2 := \bigcap_{\beta>0} X_{2,\beta} := \bigcap_{\beta>0} \{f \in C(\mathbf{R});\ \|f\|_\beta := \|e^{-\beta x^2} f(x)\| < \infty\}. \tag{5.1}$$

This space deserves much more attention in view of the more general weights as well as of its more interesting locally convex space structure.

To consider the Favard operator (1.1) on X_2 let us observe that (compare with (4.2))

$$\|e^{-\beta x^2} f(x+u)\| \leqslant e^{\beta u^2} \|e^{-\beta x^2/2} f(x)\|. \tag{5.2}$$

Thus with $\omega_1(X_{2,\beta}, f; h) := \sup_{|t|<h} \|f(x+t) - f(x)\|_\beta$ it follows that for each $f \in X_2$ and $\beta > 0$ (cf. [12], p. 11)

$$\omega_1(X_{2,\beta}, f; h) \leqslant 2e^{\beta h^2} \|e^{-\beta x^2/2} f(x)\|, \tag{5.3}$$

$$\lim_{h\to 0+} \omega_1(X_{2,\beta}, f; h) = 0. \tag{5.4}$$

This implies that $\{F_n^\gamma\}_1^\infty$ forms an approximation process on X_2. Indeed,

THEOREM 5.1. *For any $f \in X_2$, $\beta > 0$ we have*

$$\lim_{n\to\infty} \|e^{-\beta x^2}[F_n^\gamma f(x) - f(x)]\| = 0.$$

Proof. Given $f \in X_2$ and $\beta > 0$ we may proceed as in the proof of Lemma 4.2. Thus (5.3), (5.4) and Theorem 2.3 deliver for $n > 2\gamma\beta$

$$|e^{-\beta x^2}[F_n^\gamma f(x) - f(x)]| \leqslant \|f\|_\beta |F_n^\gamma 1(x) - 1| +$$

$$+ \frac{1}{\sqrt{\gamma\pi n}} \sum_{k=-\infty}^{\infty} \omega_1\left(X_{2,\beta}, f; \left|\frac{k}{n} - x\right|\right) \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n} - x\right)^2\right)$$

$$\leqslant \|f\|_\beta O(n^{-2}) + O(\varepsilon) + \frac{2\|f\|_{\beta/2}}{\sqrt{\gamma\pi n}} \sum_{\left|\frac{k}{n}-x\right|\geqslant\delta} \exp\left(\left(\beta - \frac{n}{\gamma}\right)\left(\frac{k}{n} - x\right)^2\right)$$

$$= O\left(n^{-2} + \varepsilon + \delta^{-2} T_{n,2}^{2\gamma}(x)\right) = O(\varepsilon + n^{-1}) \qquad (n\to\infty)$$

uniformly for $x \in \mathbf{R}$. Note that as a consequence of the $\exp(\beta u^2)$-factor in (5.2) we have to use $T_{n,2}^{2\gamma}$ instead of $T_{n,2}^\gamma$.

As a counterpart to Theorem 4.3 we have the Voronovskaja-type condition.

THEOREM 5.2. *With $B := (\gamma/4)(d/dx)^2$ and $D(B) := \{f \in X_2; f', f'' \in X_2\}$ one has for any $f \in D(B)$ and $\beta > 0$*

$$\lim_{n\to\infty} \|e^{-\beta x^2}\{n[F_n^\gamma f(x) - f(x)] - Bf(x)\}\| = 0.$$

The proof is quite the same as for Theorem 4.3, using (5.3), (5.4) and replacing the weights $(1+x^{2N})^{-1}$ by $\exp(-\beta x^2)$ and $T_{n,4}^\gamma(x)$ by $T_{n,4}^{2\gamma}(x)$.

As the result of Lemma 4.4 also carries over to X_2 we have verified all conditions of the locally convex counterpart to Theorem T except the stability condition. In this respect, one may strengthen (5.2) to

$$\|e^{-\beta x^2} f(x+u)\| \leqslant e^{\beta\sigma u^2/(\beta-\sigma)} \|e^{-\sigma x^2} f(x)\|$$

for any $0 < \sigma < \beta$. Thus F_n^γ maps $X_{2,\beta}$ only into $X_{2,\sigma}$ with $\sigma < \beta$. Indeed, since for any $a \in \mathbf{R}$

$$\frac{1}{\sqrt{\gamma\pi n}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{n-a}{\gamma}\left(\frac{k}{n} - x\right)^2\right) = \sqrt{\frac{n}{n-a}}\, F_n^{\frac{n\gamma}{n-a}} 1(x),$$

it follows that ($f \in X_2$, $\sigma < \beta$)

$$\|F_n^\gamma f\|_\beta \leqslant \|f\|_\sigma \sqrt{A(\beta,\sigma)} \cdot \|F_n^{A(\beta,\sigma)\gamma} 1(x)\|,$$

where $A(\beta,\sigma) := n/\big(n - \beta\sigma\gamma/(\beta-\sigma)\big)$. To formulate a stability condition to be satisfied by $\{F_n^\gamma\}_1^\infty$ on X_2, in view of $h_n = n^{-1}$ one is led to choose $\sigma := \beta/(1+n^{-1})$ which would imply (cf. Theorem 2.3)

$$\|F_n^\gamma f\|_\beta \leqslant \|f\|_{\beta/(1+n^{-1})} \sqrt{\frac{1}{1-\beta\gamma}}\,[1 + O(n^{-2})].$$

Since $\lim_{n\to\infty}(1+n^{-1})^{k(n)} = e^t$, for this choice of σ a $k(n)$-fold iterative application of F_n^γ would keep the weight in $\|f\|_{\beta/(1+n^{-1})}$ to be positive as $n\to\infty$, whereas this would imply a multiplicative factor

$$\prod_{j=0}^{k(n)} \sqrt{\frac{(1+n^{-1})^j}{(1+n^{-1})^j - \beta\gamma}} = \prod_{j=0}^{k(n)} \sqrt{1 + \frac{\beta\gamma}{(1+n^{-1})^j - \beta\gamma}}$$

which does not converge as $n\to\infty$.

Thus it seems that the locally convex versions of Trotter's Theorem (cf. [13]; [15]) established so far essentially only work for contraction semigroups with a possible multiplicative factor $\exp(Kh_n)$, where K is independent of the seminorms (cf. (4.3) where M_N depends upon N).

So it seems that the abstract theory of Trotter's theorem in locally convex spaces is not yet satisfactory from the point of view of applications.

However, using the classical direct methods as employed in [12] it should be possible to obtain (using very long calculations)

THEOREM 5.3. *Let $f \in X_2$. For the Favard operators the following assertions are equivalent*:

(5.5) *for each* $\beta > 0$: $\|e^{-\beta x^2}[F_n^\gamma f(x) - f(x)]\| = O(n^{-1}) \quad (n \to \infty)$,

(5.6) *for each* $\beta > 0$: $\|e^{-\beta x^2}[f(x+h) - 2f(x) + f(x-h)]\| = O(h^2) \; (h \to 0)$.

Once this is shown characterizations via the relative completion (cf. Theorem 4.3) may be taken over from [1].

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