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An extension of Duhamel's integral to differential equations involving generalized functions; the steady-state solution

by

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Abstract. The two-sided algebraic operational calculus is applied to differential equations of the form

$$a_0y+a_1y'+\ldots+a_my^{(m)} = b_0F+b_1F'+\ldots+b_nF^{(n)},$$

where F is a Schwartz distribution of unbounded support, arbitrary growth, and arbitrary order; the coefficients are also arbitrary. Duhamel's integral has lower limit $=-\infty$.

Let Ω be a fixed open sub-interval (such that $0\in\Omega$) of the real line. Let $\mathfrak{P}$ be the space of functions which are piecewise continuous in Ω. We shall define a linear injection $f\mapsto\{f\}$ of $\mathfrak{P}$ into a space $\mathscr{K}$ of generalized functions. If $B\in\mathscr{K}$, then B has an *initial value* $[B]_0$ and a *derivative* ∂B; if $f\in\mathfrak{P}$, then $[\{f\}]_0=f(0-)$; if also $f'\in\mathfrak{P}$, then $\partial\{f\}=\{f'\}$ when the function f is continuous in the open interval Ω. If f and g are continuous (in Ω), then $\partial\{f\}=\{g\}$ if (and only if) $f'=g$.

Often, the components of a physical system are governed by an input-output differential equation of the form

$$a_0y+a_1\partial y+\ldots+a_m\partial^my = b_0F+b_1\partial F+\ldots+b_n\partial^nF, \tag{1}$$

where $F\in\mathscr{K}$ and $\partial^kF\in\mathscr{K}$ for $0\leqslant k\leqslant n$; both $(a_0, a_1, \ldots, a_m)$ and $(b_0, b_1, \ldots, b_n)$ are sequences of complex numbers; the input F could be a distribution with locally finite support and (possibly) infinite order, F could be an arbitrary series of impulses (Dirac deltas) going from $-\infty$ to $+\infty$. The steady-state solution of such equations is the main concern of this paper.

Given any sequence c_k $(0\leqslant k<m)$ of complex numbers, we shall prove the existence of a unique solution y of (1) such that $[\partial^ky]_0=c_k$ for $0\leqslant k<m$; this solution is easily obtainable by our operational calculus and is given explicitly by the simple equation in 6.1.

Let Ω be an interval of the form $(-\infty, \beta)$ with $0<\beta\leqslant\infty$; further, let μ and P be the polynomials

$$\mu(z) = a_0+a_1z+\ldots+a_mz^m$$

and

$$P(z) = b_0 + b_1 z + \ldots + b_n z^n;$$

Equation (1) can therefore be written $\mu(\partial)y = P(\partial)F$. In 9.12 we define the *steady-state* solution $[P(\partial)/\mu(\partial)]F$ of equation (1); the generalized function F can be a series of impulses; in case $n < m$ and $F = \{f\}$ (with f piecewise continuous) then

$$\frac{P(\partial)}{\mu(\partial)}\{f\} = \left\{\int_{-\infty}^{t} G(t-x)f(x)\,dx\right\} \quad (\textit{Duhamel's integral}),$$

where G is the inverse Laplace transform of the rational function P/μ. In contrast to the classical situation, no difficulty arises when $n \geqslant m$. Until further notice, suppose that $\Omega = (-\infty, \infty)$ and let all the roots of the polynomial μ have negative real parts. If a is a complex number with real part $= 0$, then $[P(\partial)/\mu(\partial)]\{e^{at}\}$ equals the *frequency response* $[P(a)/\mu(a)]\{e^{at}\}$. If the Fourier-transform procedure can be applied to obtain a solution of (1), then this solution y equals $[P(\partial)/\mu(\partial)]F$ (see 8.6); however, the inverse Fourier transform can fail to exist in the simplest cases: see Example 10.5, where $m = 1 = n$ and $F = \{f\}$ with $f(t) = e^t$ for $t < 0$ (but $f(t) = e^{2t}$ for $t > 0$).

1.1. Organization. Motivation and background are found in Sections 6 and 8 (entitled "The principal objective of this paper"). Section 9 is entitled "The main theorem"; Section 10 is devoted to applications; Section 11 is entitled "A generalization of Duhamel's integral". Some of the less easy proofs are found in Sections 12–15.

2. THE OPERATIONAL CALCULUS

Henceforth, Ω is an open sub-interval (α_1, β) of the real line: we suppose that $-\infty \leqslant \alpha_1 < 0 < \beta \leqslant \infty$. We shall denote by $L^{\mathrm{loc}}(\Omega)$ the family of all the functions which are integrable in every compact sub-interval of the open interval Ω.

2.1. The space of test-functions. Let W be the linear space of all the infinitely differentiable functions w such that $w^{(k)}(0) = 0$ for every integer $k \geqslant 0$. As usual, $w^{(k)}$ is the kth derivative of the function w.

2.2. Definitions. An *operator* is a linear mapping of W into W. If A is an operator, we denote by $A \cdot w$ the function that the mapping A assigns to w. As usual, the product $A_1 A_2$ of two operators is defined by

(2.3) $$A_1 A_2 \cdot w = A_1 \cdot (A_2 \cdot w) \quad (\text{for all } w \text{ in } W).$$

We denote by D the operator which assigns to each test-function its derivative:

(2.4) $$D \cdot w = w' \quad (\text{for all } w \text{ in } W).$$

If $f \in L^{\mathrm{loc}}(\Omega)$, we denote by $\{f\}$ the mapping that assigns to each test-function w the function

$$\int_0^t f(t-x)w'(x)\,dx.$$

The mapping $\{f\}$ is an operator, called the *operator* of the function f (see [10], 1.28).

We denote by $\mathscr{A}$ the family of all the operators A such that

$$A\{w\} = \{A \cdot w\} \quad \text{for all } w \text{ in } W.$$

2.5. The space $\mathscr{A}$ is a commutative algebra (see [10], 1.38).

2.6. The transformation $f \mapsto \{f\}$ is a linear injection of $L^{\mathrm{loc}}(\Omega)$ into the linear space $\mathscr{A}$:

(2.7) $$\{f_1\} = \{f_2\} \quad \text{implies} \quad f_1 = f_2;$$

of course, the equation $f_1 = f_2$ means that the functions f_1 and f_2 are equal almost-everywhere in the interval Ω (see [10], 1.34).

2.8. Notation. Given an operator B, let $\Omega(B)$ be the set of all the points τ in Ω such that the equations

(2) $$f(\tau-) = f(\tau) = f(\tau+) \quad \text{and} \quad \{f\} = B$$

hold for some function f in $L^{\mathrm{loc}}(\Omega)$. If $\tau \in \Omega(B)$ there exists a function f in $L^{\mathrm{loc}}(\Omega)$ such that (2) holds; the number $f(\tau)$ depends only on B and τ (this follows directly from 2.6): we can therefore set $B(\tau) = f(\tau)$. We shall denote by $B(t)$ the function $\tau \mapsto B(\tau)$ which assigns the number $B(\tau)$ to any point τ in $\Omega(B)$.

2.9. Definition. An operator B will be called *functionable* if the function $B(t)$ is defined somewhere (that is, if the set $\Omega(B)$ is not void).

2.10. If B is functionable, then $B(t) \in L^{\mathrm{loc}}(\Omega)$ and $B = \{B(t)\}$.

2.11. If $B = \{f\}$ and $f \in L^{\mathrm{loc}}(\Omega)$, then $B(t) = f$; moreover, if $f(\tau-) = f(\tau) = f(\tau+)$ for $\tau \in \Omega$, then $B(\tau) = f(\tau)$.

2.12. If $B = \{f\}$ and if f is continuous in Ω, then $B(\tau) = f(\tau)$ for $\tau \in \Omega$.

2.13. The operators $\wedge$ and T_α. We shall denote by $\wedge$ the operator of the unit ramp function; from 2.11 it therefore follows that

$$\wedge(\tau) = \tau \quad \text{for} \quad \tau \in \Omega.$$

If $\alpha \geqslant 0$ we denote by T_α the operator of the characteristic function of the interval (α, ∞); from 2.11 it therefore follows that

$$\mathsf{T}_\alpha(\tau) = \begin{cases} 0 & \text{when} \quad \tau < \alpha, \\ 1 & \text{when} \quad \tau > \alpha. \end{cases} \tag{2.14}$$

If $\alpha < 0$ we set $\mathsf{T}_\alpha = \{-\mathsf{T}_0(\alpha - t)\}$.

2.15. The unit impulse $D\mathsf{T}_x$. If $x \in \Omega$, then the operator $D\mathsf{T}_x$ is not functionable (it is the unit impulse applied at the point x: see [10], 4.5). The operator D was defined in (2.4).

2.16. The unit constant $\mathbf{1}$. We denote by $\mathbf{1}$ the unit constant (defined by $\mathbf{1}(\tau) = 1$ for $\tau \in \Omega$); if c is a complex number, then $c\mathbf{1}$ is the constant function c; if A is an operator, then cA is the product $\{c\mathbf{1}\}A$; moreover, $\{\mathbf{1}\}$ is the multiplicative unit of the algebra $\mathscr{A}$:

$$A\{\mathbf{1}\} = A = \{\mathbf{1}\}A. \tag{2.16.1}$$

2.17. An operator is *invertible* if the equation $AB = \{\mathbf{1}\}$ holds for some element B in $\mathscr{A}$. If A is an invertible element of $\mathscr{A}$, there exists a unique element A^{-1} of $\mathscr{A}$ such that $AA^{-1} = \{\mathbf{1}\} = A^{-1}A$.

2.18. The operator D is an invertible element of $\mathscr{A}$; in fact,

$$D\wedge = \{\mathbf{1}\} \quad \text{and} \quad D^{-1} = \wedge = \{t\}. \tag{2.19}$$

Therefore, $DB\wedge = \{\mathbf{1}\}B = B$ for any B in $\mathscr{A}$. Equations (2.19) are proved in [10], 2.6 (also, in 12.20).

2.20. Theorem. *If B_1 and B_2 are functionable, then $B_1 \wedge B_2$ is functionable and*

$$B_1 \wedge B_2 = \Big\{\int_0^t B_1(t-x) B_2(x)\, dx\Big\}. \tag{2.21}$$

See 12.2. From 2.11 it follows that

$$[B_1 \wedge B_2](t) = \int_0^t B_1(t-x) B_2(x)\, dx. \tag{2.22}$$

In particular,

$$[B_1 \wedge B[_2(\tau) = -\int_\tau^0 B_1(\tau - x) B_2(x)\, dx$$

for τ almost-everywhere in the interval $\Omega \cap (-\infty, 0)$; consequently, the function $[B_1 \wedge B_2](t)$ is not the usual convolution.

3. THE DERIVATIVE

3.1. Definitions. An operator A is said to *agree with a function f on an interval J* if

$$A \cdot w(\tau) = \int_0^\tau f(\tau - x) w'(x)\, dx \quad \text{for } \tau \in J \text{ and any } w \text{ in } W.$$

3.2. If $f \in L^{\mathrm{loc}}(\Omega)$, this means that

$$[A \cdot w](\tau) = [\{f\} \cdot w](\tau) \quad \text{for} \quad \begin{cases} \tau \in J, \\ w \in W. \end{cases}$$

3.3. We shall write $A \supset f$ (or $f \subset A$) to indicate the existence of a number $a < 0$ such that A agrees with the function f on the open interval $(a, 0)$.

3.4. Let $[\mathscr{K}]$ be the family of all the functions which are piecewise continuous in some open interval of the form $(x, 0)$. Thus, if $h \in [\mathscr{K}]$, then the limit

$$h(0-) \overset{\text{def}}{=} \lim h(\tau) \quad \text{as} \quad \tau \to 0 \quad \text{and} \quad \tau < 0$$

exists, and there is a number $a < 0$ such that h is continuous in the open interval $(a, 0)$. Piecewise continuity is called *sectional* continuity in [4], p. 5.

We shall denote by $\mathscr{K}$ the family of operators A in $\mathscr{A}$ such that the relation $A \supset h$ holds for some function h in $[\mathscr{K}]$.

3.5. The derivative. If $A \in \mathscr{K}$ there exists a unique number $[A]_0$ with the following property: the equation $[A]_0 = h(0-)$ holds for some function h in $[\mathscr{K}]$ with $h \subset A$ (see [10], 5.0); we set

$$\partial A \overset{\text{def}}{=} DA - [A]_0 D. \tag{3.6}$$

3.7. Thus, if $A \in \mathscr{A}$ and $A \supset h$ for some element h of $[\mathscr{K}]$, then $[A]_0 = h(0-)$.

3.8. Suppose that $f \in L^{\mathrm{loc}}(\Omega)$ and $f \in [\mathscr{K}]$; since $\{f\} \supset f$, it follows from 3.5–3.7 that $[\{f\}]_0 = f(0-)$ and

$$\partial\{f\} = D\{f\} - f(0-)D. \tag{3.9}$$

Suppose that $B \in \mathscr{K}$; if ∂B is functionable, then it is the operator of the derivative of the function $B(t)$; thus, $B'(t) = \partial B(t)$ (see 13.21).

3.10. Clearly, $\partial\mathsf{T}_x = D\mathsf{T}_x$ for all real values of x (see (3.9) and 2.13).

3.11. Suppose that $f \in L^{\mathrm{loc}}(\Omega)$ and let f' be continuous in Ω except possibly on a locally finite subset of Ω; further, suppose that $f' \in L^{\mathrm{loc}}(\Omega)$.

Therefore, f is continuous in some open sub-interval of the form $(a, 0)\cap\Omega$; since $f'\in L^{\mathrm{loc}}(\Omega)$, it also follows the existence of a limit $f(0-)$; consequently, $f\in[\mathcal{K}]$. If this function f is continuous in Ω, then $\partial\{f\} = \{f'\}$ (see [10], 5.5).

3.12. Let g be a function such that both g and g' are continuous in Ω except possibly on a locally finite subset of Ω; further, suppose that $g'\in L^{\mathrm{loc}}(\Omega)$. If so, then $g\in L^{\mathrm{loc}}(\Omega)$ and

$$\partial\{g\} = \{g'\}+\sum_{a=-\infty}^{\infty}[g(a+)-g(a-)]D\mathsf{T}_a;$$

this equation can be proved by observing that the equation

$$f = g-\sum_{a=-\infty}^{\infty}[g(a+)-g(a-)]\chi(a, \infty)$$

(where $\chi(a, \infty)$ is the characteristic function of the interval (a, ∞)) defines a continuous function f satisfying 3.11. The proof requires the definition of convergence given in [10].

3.13. Let $\mathcal{F}$ be the family of operators of functions satisfying 3.12. If $A\in\mathcal{F}$ and if $A(t)$ is continuous in Ω it follows from 3.12 that

$$\partial A = \{A'(t)\} \overset{\text{def}}{=} \left\{\frac{d}{dt}A(t)\right\}. \tag{3.14}$$

The correspondence $A\mapsto\partial A$ (of $\mathcal{F}$ into $\mathcal{A}$) is the only linear mapping such that (3.14) holds (when $A(t)$ is continuous) and $\partial\mathsf{T}_x = D\mathsf{T}_x$ (for all x in Ω).

3.15. There is a linear injection $R\mapsto\langle R\rangle$ of a certain space of Schwartz distributions into $\mathcal{K}$; under this injection, distributional derivation corresponds to the operation ∂; further, a distribution R is regular if (and only if) the operator $\langle R\rangle$ is functionable (see Shultz [18], pp. 177 and 180); finally, the Dirac distribution corresponds to the operator $D\mathsf{T}_a$ (see 2.15).

3.16. If A_1 and A_2 belong to $\mathcal{K}$, then $A_1\wedge A_2$ also belongs to $\mathcal{K}$ and $\partial[A_1\wedge A_2] = A_1A_2$: see 12.12.

4. HIGHER DERIVATIVES

As usual, $\partial^0 A = A$ and $\partial^n A$ is defined (recursively, for $n\geqslant 1$) by the equation

$$\partial^n A = \partial[\partial^{n-1}A] \quad \text{when} \quad \partial^k A\in\mathcal{K} \quad \text{for} \quad 0\leqslant k\leqslant n-1. \tag{4.1}$$

4.1.1. We say that $\partial^n A$ is *definable* if $\partial^k A\in\mathcal{K}$ for $0\leqslant k\leqslant n-1$. If $n\geqslant 0$ we denote by $[\mathcal{K}_n]$ the space of functions h in $[\mathcal{K}]$ such that $h^{(k)}\in[\mathcal{K}]$ for $0\leqslant k\leqslant n$. The space $\mathcal{K}_n$ will consist of all the operators A in $\mathcal{A}$ such that $A\supset h$ for some function h in $[\mathcal{K}_n]$. Note that

$$\mathcal{K}_{n+1}\subset\mathcal{K}_n\subset\ldots\subset\mathcal{K}_0 = \mathcal{K}.$$

4.2. If $B\in\mathcal{K}$ and $\partial B\in\mathcal{K}$, then $B\in\mathcal{K}_1$; more generally,

(4.3) *if $\partial^n B$ is definable and belongs to $\mathcal{K}$, then $B\in\mathcal{K}_n$;*

see (13.19) and (5.4).

4.4. Conversely, *if $B\in\mathcal{K}_n$, then $\partial^n B$ is definable and belongs to $\mathcal{K}$.* Indeed, if the relation $B\supset h$ holds for some function h in $[\mathcal{K}_n]$, then $\partial^n B\supset h^{(k)}$: see 12.1.

4.5. If k and r are non-negative integers, then

$$\partial^k[\partial^r B] = \partial^{k+r}B \quad \text{when} \quad \partial^{k+r}B \text{ is definable}.$$

4.6. If $\partial^n y$ is definable it follows easily from (3.6) and (4.1) that

$$\partial^n y = D^n y-\sum_{k=0}^{n-1}[\partial^k y]_0 D^{n-k}. \tag{4.7}$$

4.8. If $y\in\mathcal{K}_n$ and $\partial^n y = 0$ it is easily verified (using 4.6 and 4.4) that $y(t)$ is a polynomial of degree $= n-1$.

4.9. Example. Let c_k $(-\infty<k<\infty)$ be *any* sequence of complex numbers. If $a>0$ the series of impulses

$$y = \sum_{k=-\infty}^{\infty}c_k D\mathsf{T}_{ka}$$

is such that $\partial^k y\in\mathcal{K}$ and $[\partial^k y]_0 = 0$ for any integer $k\geqslant 0$. Of course, y is not functionable.

5. THE SPACE $\mathcal{C}^m$

5.1. Definitions. An operator g is *$\mathcal{C}^0$-functionable* if it is the operator of a function which is continuous in the interval Ω. When $m\geqslant 0$ we denote by $\mathcal{C}^m$ the space of all the $\mathcal{C}^0$-functionable operators g such that the kth derivative $g^{(k)}(t)$ (of the function $g(t)$) is continuous in Ω for $0\leqslant k\leqslant m$.

5.2. If $g\in\mathcal{C}^m$, then g is the operator of the function $g(t)$ which is m times continuously differentiable; its mth derivative $g^{(m)}(t)$ is continuous in Ω. Recall that $g(t)$ is the only continuous function f such that $g = \{f\}$

and $g(\tau) = f(\tau)$ for all τ in Ω (see 2.12); moreover, $[g]_0 = f(0)$ and $g \in \mathscr{K}$. If $g \in \mathscr{C}^1$ the equation $\partial g = \{g'(t)\}$ is immediate from 3.11.

5.3. If $g \in \mathscr{C}^m$, then $\partial^k g = \{g^{(k)}(t)\}$ for $0 \leqslant k \leqslant m$.

5.4. DEFINITIONS. Given a polynomial

$$p(z) = b_0 + b_1 z + \ldots + b_m z^m \quad (\text{with } b_m \neq 0 \text{ and } m \geqslant 1),$$

we write, if $\partial^m A$ is definable,

$$p(\partial)A \stackrel{\text{def}}{=} b_0 A + b_1 \partial A + \ldots + b_m \partial^m A;$$

we shall say that $p(\partial)A$ is *definable* if $\partial^m A$ is definable.

5.5. If $p(\partial)A$ is definable and belongs to $\mathscr{K}$, then $A \in \mathscr{K}_m$ (this wil be proved in 13.15); from 4.4 it therefore follows that

$A \in \mathscr{K}_m$ if (and only if) $p(\partial)A$ is definable and belongs to $\mathscr{K}$.

5.6. Suppose that $g \in \mathscr{C}^m$; it results directly from 5.3 and 2.12 that

$$[p(\partial)g](\tau) = b_0 g(\tau) + b_1 g'(\tau) + \ldots + b_m g^{(m)}(\tau)$$

for all τ in Ω.

5.7. Consequently, $p(\partial)g \in \mathscr{C}^0$ whenever $g \in \mathscr{C}^m$. If $p(\partial)g$ is definable and belongs to $\mathscr{C}^0$ it can be proved that $g \in \mathscr{C}^m$: see 13.21. Thus, if $p(\partial)y$ is definable, then the relation $p(\partial)y = R \in \mathscr{C}^0$ implies that the equation

$$b_0 y(\tau) + b_1 y'(\tau) + \ldots + b_m y^{(m)}(\tau) = R(\tau)$$

holds for all τ in Ω.

5.8. If p_1 and p_2 are two polynomials, then

$$p_1(\partial)[p_2(\partial)F] = p_1 p_2(\partial)F \quad (\text{if } p_1 p_2(\partial)F \text{ is definable}), \tag{5.9}$$

see (4.5), and

$$p_2(\partial)F + p_1(\partial)F = [p_2 + p_1](\partial)F \quad (\text{if } [p_2+p_1](\partial)F \text{ is definable}). \tag{5.10}$$

5.11. As in 5.4,

$$p(z) = \sum_{n=0}^{m} b_n z^n. \tag{5.12}$$

Let c_k $(0 \leqslant k < m)$ be a given sequence of complex numbers; observe that

$$p(D) \sum_{k=0}^{m-1} c_k D^{-k} = \sum_{n=0}^{m} \sum_{k=0}^{n-1} b_n c_k D^{n-k}. \tag{5.13}$$

5.14. Notation. We set

$$\left\| p(D) \sum_{k=0}^{m-1} c_k D^{-k} \right\| \stackrel{\text{def}}{=} \sum_{n=0}^{m} \sum_{k=0}^{n-1} b_n c_k D^{n-k}: \tag{5.15}$$

this is the result of deleting (from the right-hand side of (5.13)) *all the terms which do not contain positive powers of D.*

5.16. From (5.15) it follows easily the existence of a polynomial λ whose degree is less than the degree of p and such that

$$\left\| p(D) \sum_{k=0}^{m-1} c_k D^{-k} \right\| = D\lambda(D).$$

5.17. THEOREM. *Let p be a polynomial of positive degree m. If $p(\partial)y$ is definable, then*

$$p(\partial)y = p(D)y - \left\| p(D) \sum_{k=0}^{m-1} [\partial^k y]_0 D^{-k} \right\|. \tag{5.18}$$

Proof. Immediate from 4.6.

6. INITIAL-VALUE PROBLEMS

We shall now deal with equations of the form $\mu(\partial)y = P(\partial)F$, where μ is a polynomial of degree $m \geqslant 1$, and where P is a non-zero polynomial. Most equations in system engineering have such a form; this has been emphasized in Wunsch [19] (see also [13], p. 25, Berg [1], p. 76, and Example 10.1). The subject of steady-state solution will be brought up in 6.12.

6.1. Suppose that $P(\partial)F$ is definable and belongs to $\mathscr{K}$ (in view of 5.5, this means that $F \in \mathscr{K}_n$, where n is the degree of P). Let c_k $(0 \leqslant k < m)$ be a sequence of complex numbers. The initial-value problem

$$\mu(\partial)y = P(\partial)F \quad \text{with} \quad [\partial^k y]_0 = c_k \quad \text{for} \quad 0 \leqslant k < m$$

has a solution y in $\mathscr{K}_m$ (see 13.20): to solve it, we apply 5.17 twice:

$$y = \frac{P(D)}{\mu(D} F + \frac{1}{\mu(D)} \left\| P(D) \sum_{k=0}^{n-1} [\partial^k F]_0 D^{-k} \right\| + \frac{1}{\mu(D)} \left\| \mu(D) \sum_{k=0}^{m-1} c_k D^{-k} \right\|;$$

6.2. Consequently, this last equation determines the unique operator y such that $\mu(\partial)y = P(\partial)F$ and $[\partial^k y] = c_k$ (for $0 \leqslant k < m$).

6.3. Of course, the degree of P could exceed the degree of μ.

6.4. Example. The input F could be the series of impulses in 4.9; it could also be the operator

$$(6.5)\qquad \sum_{k=0}^{\infty} \partial^k (D\mathsf{T}_k)$$

which is such that $[\partial^n F]_0 = 0$ and $\partial^n F \in \mathscr{K}$ for any integer $n \geqslant 0$; it corresponds to a distribution of infinite order; a simple differential equation involving (6.5) is worked out in Zemanian [20], p. 164 with the initial condition $y(-\varepsilon) = 3$, where ε is *infinitesimally small* (and positive).

6.6. Zero initial values at time zero. Let λ be a non-zero polynomial of degree $n < m$ (as before, m is the degree of the polynomial μ). The operator

$$(6.7)\qquad G = D\frac{\lambda(D)}{\mu(D)}$$

belongs to $\mathscr{C}^1$ (see 13.6). If F is the unit impulse $D\mathsf{T}_0$, then $[\partial^k F]_0 = 0$; we can set $P = \lambda$ in 6.1 to conclude that the equation

$$(6.7.1)\qquad y = D\frac{\lambda(D)}{\mu(D)}\mathsf{T}_0 = \mathsf{T}_0 G$$

determines the solution of the initial-value problem $\mu(\partial)y = \lambda(\partial)[D\mathsf{T}_0]$ with $[\partial^k y]_0 = 0$ (for $0 \leqslant k < m$). The function (6.7.1) is the "response to the unit impulse"; we have $y(\tau) = G(\tau)$ for $\tau > 0$ (but $y(\tau) = 0$ for $\tau < 0$: see [10], 3.7).

If F is functionable it follows from (6.7) and (2.18) that

$$(6.8)\qquad \frac{\lambda(D)}{\mu(D)}F = G\wedge F;$$

setting $c_k = 0$ in 6.1, we see that the equation

$$(6.9)\qquad y = G\wedge F + \frac{1}{\mu(D)}\left\| \lambda(D)\sum_{k=0}^{n-1} [\partial^k F]_0 D^{-k}\right\|$$

determines the unique operator y which satisfies the intitial-value problem

$$(6.10)\qquad \mu(\partial)y = \lambda(\partial)F \quad \text{with} \quad [\partial^k y]_0 = 0 \quad \text{for} \quad 0 \leqslant k < m.$$

If $[\partial^k F]_0 = 0$ for $0 \leqslant k < n$ (= degree of λ), then (6.9) becomes $y = G\wedge F$; if F is functionable it results from (2.21) that

$$(6.11)\qquad y(t) = \int_0^t G(t-x)F(x)\,dx.$$

This solution of (6.10) corresponds therefore to zero initial values (equilibrium) at time zero; a similar solution is obtained by Wunsch [19] by adjoining a different type of conditions ("natürliche Anfangsbedingungen"). See also Berg [1], pp. 76–77; in both [19] and [1] the interval Ω is $[0, \infty)$.

6.12. The road ahead. We shall consider steady-state solutions; to that effect, integral (6.11) will be replaced by the integral

$$(6.13)\qquad \int_{-\infty}^{t} G(t-x)F(x)\,dx;$$

the resulting solution (of $\mu(\partial)y = \lambda(\partial)F$), although considered physically more appropriate by many engineering textbooks (e.g., [14], p. 330, [17], pp. 221–222, [16], p. 21, and Van Der Pol [15], p. 156) does not always correspond to zero initial values at time $-\infty$ (see 9.22). We shall require that F agrees on $(-\infty, 0)$ with a suitably integrable function.

7. THE SPACE $\mathfrak{B}$

Henceforth, Ω is an interval of the form $(-\infty, \beta)$ with $0 < \beta \leqslant \infty$. Let $\mathfrak{l}$ be the operator

$$(7.1)\qquad \mathfrak{l} \overset{\text{def}}{=} \{1\} - \mathsf{T}_0;$$

from (2.12) it therefore follows that

$$(7.2)\qquad \mathfrak{l}(\tau) = \mathbf{1}(\tau) - \mathsf{T}_0(\tau) = 1 - \mathsf{T}_0(\tau) = \begin{cases} 1 & \text{for} \quad \tau < 0, \\ 0 & \text{for} \quad \tau > 0. \end{cases}$$

7.3. Definition. $\mathfrak{B}$ will denote the space of all the operators A in $\mathscr{A}$ such that $\mathfrak{l}A$ is *functionable* and $\mathfrak{l}A(t) \in [\mathscr{K}]$.

7.4. If $A \in \mathscr{A}$, then $A = \mathfrak{l}A + \mathsf{T}_0 A$ (from (7.1) and (2.16.1)). Let B be a functionable operator:

$$(7.5)\qquad B = \{B(t)\} \quad \text{(from 2.10)};$$

it is easily verified that

$$(7.6)\qquad \mathfrak{l}\{B(t)\} = \mathfrak{l}B = \{\mathfrak{l}(t)B(t)\} \quad \text{(see Section 15)};$$

consequently, 2.11 gives

$$(7.7)\qquad [\mathfrak{l}B](t) = \mathfrak{l}(t)B(t),$$

whence

$$(7.8)\qquad [\mathfrak{l}B](\tau) = \begin{cases} 0 & \text{for} \quad \tau > 0, \\ B(\tau) & \text{for} \quad \tau < 0; \end{cases}$$

that is, $\mathfrak{l}B(\tau) = B(\tau)$ for τ almost-everywhere in the interval $(-\infty, 0)$. From (7.6) it follows directly that $\mathfrak{l}\mathsf{T}_0 = 0$ and

$$\mathfrak{l}\mathfrak{l} = \mathfrak{l}. \tag{7.9}$$

7.10. THEOREM. *Suppose that $A \in \mathscr{A}$ and $f \in L^{\mathrm{loc}}(\Omega)$. The equation $\mathfrak{l}A = \mathfrak{l}\{f\}$ holds if (and only if) the operator A agrees with the function f on the interval $(-\infty, 0)$.*

Proof. See 15.9 or [10], 3.24.

7.11. Suppose that A agrees with a function f on $(-\infty, 0)$: if $f \in [\mathscr{K}] \cap L^{\mathrm{loc}}(\Omega)$ it follows from 7.10 that $\mathfrak{l}A = \mathfrak{l}\{f\}$; from (7.7) therefore,

$$[\mathfrak{l}A](t) = \mathfrak{l}(t)f(t);$$

consequently, $\mathfrak{l}A(t) \in [\mathscr{K}]$, whence $A \in \mathfrak{B}$. Conversely, suppose that $A \in \mathfrak{B}$: this means that the equation

$$\mathfrak{l}A = \{f\} \tag{1}$$

holds for some function f in $[\mathscr{K}] \cap L^{\mathrm{loc}}(\Omega)$; from (7.9) and (1) therefore: $\mathfrak{l}A = \mathfrak{l}\mathfrak{l}A = \mathfrak{l}\{f\}$; consequently, we may conclude from 7.10 that the operator A agrees with f on $(-\infty, 0)$; in particular,

$$A \supset f; \tag{2}$$

since $f \in [\mathscr{K}]$, it follows from Definition 3.4 that $A \in \mathscr{K}$ and

$$[A]_0 = f(0-). \tag{3}$$

From (1) and 2.12 we see that $\mathfrak{l}A(t) = f$; since this gives $\mathfrak{l}A(0-) = f(0-)$, we can use (1)–(3) to write

$$A \supset [\mathfrak{l}A](t) \quad \text{and} \quad [A]_0 = [\mathfrak{l}A](0-). \tag{7.12}$$

7.13. Suppose that $A \in \mathfrak{B}$ and $\mathfrak{l}A(t) \in [\mathscr{K}_n]$; from (7.12) and 4.1.1 it follows that $\partial^n A$ is definable and $A \in \mathscr{K}_n$; in fact,

$$[\mathfrak{l}(\partial^n A)](t) = (d^n/dt^n)[\mathfrak{l}A(t)].$$

7.13.1. As we saw in 7.11, if $A \in \mathfrak{B}$, then $A \in \mathscr{K}$; therefore, $\mathfrak{B} \subset \mathscr{K}$.

7.14. Suppose that B is functionable. The operator $\mathfrak{l}B$ is the operator of the pointwise product $\mathfrak{l}(t)B(t)$ (see (7.6)); therefore, $B \in \mathfrak{B}$ whenever the function $B(t)$ belongs to $[\mathscr{K}]$.

7.15. Let $\mathfrak{B}_+$ be the space of all the operators A in $\mathscr{A}$ such that $\mathfrak{l}A = 0$. Note that

$$\mathfrak{B}_+ \subset \mathfrak{B} \subset \mathscr{K} \subset \mathscr{A}.$$

The space $\mathfrak{B}_+$ contains operators of infinite order such as (6.5); therefore, $\mathfrak{B}_+$ is larger than Berg's space of "distributions" [1], p. 104 and p. 108. The space $\mathfrak{B}_+$ corresponds to the space $\mathscr{D}'_+$ of distributions with support in the interval $[0, \infty)$.

7.16. As we shall see in 9.18.1, if $F \in \mathfrak{B}$, then the general solution y of the equation $\mu(\partial)y = \lambda(\partial)F$ belongs to $\mathfrak{B}$; if $F \in \mathfrak{B}_+$ this general solution y need not belong to $\mathfrak{B}_+$. For example, the equation $\partial y = D\mathsf{T}_0$ implies that $y(t) = c + \mathsf{T}_0(t)$ for some number c: since $\mathfrak{l}y(t) = c$, it follows that $y \in \mathfrak{B}$, but $y \notin \mathfrak{B}_+$ when $c \neq 0$ (as we shall see, $c = 0$ gives the steady-state solution).

8. THE PRINCIPAL OBJECTIVE OF THIS PAPER

Let G be a functionable operator. If $F \in \mathfrak{B}$ we write

$$G \otimes F \overset{\text{def}}{=} \Big\{ \int_{-\infty}^{0} g(t-x)\,[\mathfrak{l}F](x)\,dx \Big\}. \tag{8.1}$$

8.2. If F and $G \otimes F$ are functionable, then

$$G \wedge F + G \otimes F = \Big\{ \int_{-\infty}^{t} G(t-x)F(x)\,dx \Big\} = \Big\{ \int_{0}^{\infty} G(u)F(t-u)\,du \Big\}; \tag{8.3}$$

the first equation will be proved in 11.1. Of course, Ω is an interval of the form $(-\infty, \beta)$ with $0 < \beta \leqslant \infty$. A device which determines a correspondence $F \mapsto [G \wedge F + G \otimes F]$ is sometimes called a "filter of type II" [2], p. 132 or a "one-port" [20], 10.2.

8.4. The main theorem. Let μ be a polynomial of positive degree, let λ be a polynomial of degree less than the degree of μ. The operator

$$G = D\frac{\lambda(D)}{\mu(D)} \tag{8.5}$$

is functionable, and the function $G(t)$ is the inverse Laplace transform of the rational function λ/μ. By expanding λ/μ into partial fractions, the function $G(t)$ is obtained as a linear combination of a family $e^{at}t^n$ $(a \in Z,\ 0 \leqslant n \leqslant \pi(a))$ of exponential-monomials (see 9.2). Suppose that $F \in \mathfrak{B}$; if

$$\sum_{a \in Z} \sum_{n=0}^{\pi(a)} \Big| \int_{-\infty}^{0} e^{-ax}x^n\,[\mathfrak{l}F](x)\,dx \Big| < \infty, \tag{1}$$

then $G \otimes F$ is functionable and

$$\mu(\partial)[G \wedge F + G \otimes F] = \lambda(\partial)F \tag{2}$$

— provided that $\lambda(\partial)F \in \mathscr{K}$ (see 9.4–9.5 and 9.8).

8.6. Particular cases. In this section we suppose that $\Omega = (-\infty, \infty)$ and that F is functionable; further, we suppose that the polynomial μ is *stable* (that is, suppose that all the zeros of μ lie on the left-hand side of the imaginary axis). The equation (8.5) implies that

$$\int_0^\infty e^{-au} G(u)\, du = \frac{\lambda(a)}{\mu(a)} \quad \text{for} \quad \mathrm{Re}(a) \geqslant 0: \tag{8.7}$$

by hypothesis, $\mu(a) = 0$ implies $\mathrm{Re}(a) < 0$. Thus, if F satisfies (1) it follows from 8.2 that the equation

$$y = \Big\{ \int_{-\infty}^{t} G(t-x) F(x)\, dx \Big\} \tag{3}$$

implies $\mu(\partial) y = \lambda(\partial) F$ (provided that $\lambda(\partial) F \in \mathscr{K}$).

Condition (1) is satisfied when

$$0 = \lim_{\tau \downarrow -\infty} e^{\varepsilon\tau} [\mathfrak{l}F](\tau) \quad \text{for any } \varepsilon > 0; \tag{4}$$

this condition is therefore sufficient to ensure that (3) determines a solution y of the equation

$$\mu(\partial) y = \lambda(\partial) F: \tag{5}$$

this is proved in Kaplan [8], pp. 284–287, for the case $\lambda = 1$. The particular solution (3) of (5) is often calculated by the Fourier transform technique: the transform $f \mapsto \hat{f}$ is applied to both sides of (5), the resulting algebraic equation is solved for $\hat{y}$: the *inversion step* consists in finding the inverse Fourier transform of $\hat{y}$; if $F(t)$ and $[\lambda(\partial)F](t)$ belong to $L^1(-\infty, \infty)$, then

$$y = G \wedge F + G \otimes F = \Big\{ \lim_{N \uparrow \infty} \int_{-N}^{N} e^{i\omega t} \frac{\lambda(i\omega) \hat{F}(i\omega)}{2\pi\mu(i\omega)}\, d\omega \Big\}:$$

see [2], pp. 126–128, and [8], pp. 284–287.

However, when $\lambda(\partial) F \notin L^1(-\infty, \infty)$ the inversion step is not applicable to some equations governing very simple systems; for example, the inversion step is not applicable to the equation $\partial y + y = \partial F$ with $F(t) = \mathfrak{l}(t) e^t + \mathsf{T}_0(t) e^{2t}$ (see 10.5); note that Condition (4) (and therefore Condition (1)) is satisfied.

9. THE MAIN THEOREM

As in Sections 7–8, the interval Ω is of the form $(-\infty, \beta)$. Throughout, μ is a polynomial of positive degree and λ is a non-zero polynomial such that the degree of λ is less than the degree of μ. Also,

$$G = D \frac{\lambda(D)}{\mu(D)}. \tag{9.1}$$

9.2. Let Z be the set of roots of the polynomial μ. There exists a unique family $\pi(a)$ $(a \in Z)$ of non-negative integers such that

$$G = \sum_{a \in Z} \sum_{m=0}^{\pi(a)} c_a^m \{e^{at} t^m\}, \tag{9.3}$$

where the c_a^m are complex numbers (they are uniquely determined by (9.3)).

9.4. The numbers c_a^m can be found by expanding $\lambda(D)/\mu(D)$ into partial fractions; recall that $\{e^{at} t^m\} = D/(D-a)^{m+1}$. The multiplicity of the root a equals the integer $1 + \pi(a)$.

9.5. Suppose that $F \in \mathfrak{B}$ and $g \in \mathscr{C}^0$; from (8.1) we see that

$$[g \otimes F](0) = \int_{-\infty}^{0} g(-x) [\mathfrak{l}F](x)\, dx = \int_0^\infty g(u) [\mathfrak{l}F](-u)\, du. \tag{9.6}$$

On the other hand, it follows from (9.3) that

$$[G \otimes F](0) = \sum_{a \in Z} \sum_{m=0}^{\pi(a)} c_a^m [\{e^{at} t^m\} \otimes F](0):$$

we shall denote by $\mathfrak{S}[G]$ the space of all the operators F in $\mathfrak{B}$ such that

$$\sum_{a \in Z} \sum_{m=0}^{\pi(a)} |[\{e^{at} t^m\} \otimes F](0)| < \infty.$$

9.7. Therefore, $F \in \mathfrak{S}[G]$ if (and only if) $F \in \mathfrak{B}$ and if the integral

$$\int_{-\infty}^{0} e^{-ax} x^m [\mathfrak{l}F](x)\, dx \quad \textit{exists whenever} \quad \begin{cases} a \in Z, \\ 0 \leqslant m \leqslant \pi(a). \end{cases}$$

9.8. Let P be a non-zero polynomial. The division algorithm determines uniquely two polynomials Q and λ such that

$$\frac{P}{\mu} = Q + \frac{\lambda}{\mu} \quad \text{with} \quad \deg(\lambda) < \deg(\mu): \tag{9.9}$$

here deg = degree of the polynomial. Suppose that $F \in \mathfrak{S}[G]$, where $G = D\lambda(D)/\mu(D)$; we set

$$\frac{P(\partial)}{\mu(\partial)} F \overset{\text{def}}{=} Q(\partial) F + \frac{\lambda(D) F}{\mu(D)} + \frac{D\lambda(D)}{\mu(D)} \otimes F. \tag{9.10}$$

If $P(\partial) F$ is definable and belongs to $\mathscr{K}$, then

$$\mu(\partial) \Big[\frac{P(\partial)}{\mu(\partial)} F \Big] = P(\partial) F \quad \text{(see 14.6)}. \tag{9.11}$$

9.12. Definition. We shall call (9.10) the *steady-state solution* of the equation $\mu(\partial)y = P(\partial)F$.

9.13. Recall that $[D\lambda(D)/\mu(D)](t)$ is the inverse Laplace transform $\mathfrak{L}^{-1}(\lambda/\mu)$ of the rational function λ/μ (see (8.6)):

$$\frac{P(\partial)}{\mu(\partial)}F = Q(D)F + \frac{\lambda(D)F}{\mu(D)} + \left[\mathfrak{L}^{-1}\left(\frac{\lambda}{\mu}\right)\right]\otimes F. \tag{9.14}$$

9.15. As in (9.1), set $G = D\lambda(D)/\mu(D)$:

$$G = \left\{\mathfrak{L}^{-1}\left(\frac{\lambda}{\mu}\right)\right\}; \tag{9.16}$$

therefore, Definition (9.10) can be written

$$\frac{P(\partial)}{\mu(\partial)}F = Q(\partial)F + [G\wedge F + G\otimes F]. \tag{9.17}$$

9.17.1. Suppose that $F\in\mathfrak{S}[G]$; consequently, $G\otimes F\in\mathscr{C}^0$ (by 14.1); since $G\in\mathscr{C}^0$ (by (9.3)), it follows from (11.4) that

$$[G\wedge F + G\otimes F] \quad \text{belongs to } \mathfrak{B};$$

also, we may let $\tau\to 0$ in (11.5):

$$[G\wedge F + G\otimes F](0-) = \int_{-\infty}^{0} G(-x)[tF](x)\,dx;$$

from (9.17) therefore

$$\left[\frac{P(\partial)}{\mu(\partial)}F\right]_0 = [Q(\partial)F]_0 + \int_{-\infty}^{0} G(-x)[tF](x)\,dx. \tag{9.18}$$

9.18.1. From 9.17.1 we see that (9.17) belongs to $\mathfrak{B}$ whenever $\deg(Q) = 0$ and $F\in\mathfrak{S}[G]$; since (9.17) is a particular solution, it is easily seen that the general solution also belongs to $\mathfrak{B}$ (when $Q = 0$).

9.18.2. If F and $G\otimes F$ are functionable, it results from (9.17) and (8.3) that

$$\frac{P(\partial)}{\mu(\partial)}F = Q(\partial)F + \left\{\int_{-\infty}^{t} G(t-x)F(x)\,dx\right\}; \tag{9.19}$$

in particular, if $F\in\mathfrak{S}[G]$ and $P(\partial)F\in\mathscr{K}$, it follows from (9.11) that (9.19) defines a solution of the equation $\mu(\partial)y = P(\partial)F$.

9.19.1. Let μ, P and Q be the polynomials

$$\mu(z) = a_0 + a_1 z + \ldots + a_m z^m,$$
$$P(z) = b_0 + b_1 z + \ldots + b_n z^n,$$

and

$$Q(z) = Q_0 + Q_1 z + \ldots + Q_r z^r.$$

Suppose that $F\in\mathscr{C}^n$ and $F\in\mathfrak{S}[G]$; Definition 5.1 gives $P(\partial)F\in\mathscr{C}^0$: if we set $y = [P(\partial)/\mu(\partial)]F$ it therefore follows from (9.11) that $\mu(\partial)y = P(\partial)F\in\mathscr{C}^0$; from 5.7 it therefore results that $y\in\mathscr{C}^m$ and the equation

$$y(\tau) = \sum_{k=0}^{r} Q_k F^{(k)}(\tau) + \int_0^\infty G(u)F(\tau-u)\,du \tag{9.20}$$

(for all τ in Ω) follows from 5.7 and (9.19); since $y\in\mathscr{C}^m$, it implies that

$$a_0 y(\tau) + a_1 y'(\tau) + \ldots + a_m y^{(m)}(\tau) = b_0 F(\tau) + \ldots + b_n F^{(n)}(\tau)$$

for all τ in Ω.

9.21. The classical cases. Let $\mathscr{S}_1$ be the space of all linear combinations of operators of the form $\{e^{at}t^m\}$, where $m\geqslant 0$ and $\operatorname{Re}(a)\geqslant 0$. Let $\mathscr{S}_0$ be the space of all the operators F in $\mathfrak{B}$ such that

$$0 = \lim e^{\varepsilon\tau}[tF](\tau) \quad (\text{as } \tau\downarrow -\infty)$$

for every $\varepsilon > 0$. Note that $\mathscr{S}_0$ contains all functions of slow growth [20], p. 104. In this Section 9.21 we suppose that μ is a stable polynomial (see 8.6). It is easily seen that

$$\mathscr{S}_1 \subset \mathscr{S}_0 \subset \mathfrak{S}[D\lambda(D)/\mu(D)].$$

In the very special case where $F(t) = e^{at}$ with $\operatorname{Re}(a)\geqslant 0$, we have $F\in\mathscr{S}_1$; Equation (9.20) becomes

$$\left[\frac{P(\partial)}{\mu(\partial)}\{e^{at}\}\right](\tau) = e^{a\tau}\left[Q(a) + \int_0^\infty e^{-au}G(u)\,du\right],$$

where $G = D\lambda(D)/\mu(D)$; from (8.7) therefore

$$\frac{P(\partial)}{\mu(\partial)}\{e^{at}\} = e^{at}\left[Q(a) + \frac{\lambda(a)}{\mu(a)}\right] = \frac{P(a)}{\mu(a)}\{e^{at}\}:$$

the last equation is from (9.9). Obviously, the general solution of the equation $\mu(\partial)y = P(\partial)\{e^{at}\}$ differs from $P(a)\{e^{at}\}/\mu(a)$ by a transient term.

9.22. If $\omega\neq 0$ it follows easily (from the above) that

$$\frac{P(\partial)}{\mu(\partial)}\{\sin\omega t\} = \left|\frac{P(i\omega)}{\mu(i\omega)}\right|\left\{\sin\left(\omega t + \arg\frac{P(i\omega)}{\mu(i\omega)}\right)\right\}.$$

9.23. The frequency response. When $a = i\omega$ with $-\infty < \omega < \infty$ then $P(a)e^{at}/\mu(a)$ is called the *frequency response*; the case $P = 1$ is treated in Doetsch [5], pp. 54–56 and p. 174; the case $\deg(Q) = 0$ is found in Wunsch [19], p. 996.

9.24. Equilibrium at time $= -\infty$. When $F(-\infty) = 0$ the steady-state solution of $\mu(\partial)y = P(\partial)F$ can sometimes be obtained by imposing the condition $y(-\infty) = 0$ (neither of these limits exist in 9.22). For example, the steady-state solution of the equation

$$(1) \qquad \partial^2 y + 4\partial y + 3y = 2\partial^2 F + 6\partial F + 6F$$

with $F(t) = 1/(1+t^2)^m$ has been obtained in Doležal [6], pp. 111–112, by imposing the condition $y(-\infty) = 0$. Instead, let us suppose only that F is a functionable operator such that $tF(t) \in [\mathscr{K}_2]$ (which, by 7.13, guarantees that the right-hand side of (1) is definable); from 9.18.2 and (1) it follows that

$$y = \left[2 - \frac{2\partial}{\partial^2 + 4\partial + 3}\right] F = 2F - \left\{\int_{-\infty}^{t} G(t-x)F(x)\,dx\right\},$$

where, by (9.16),

$$G = \left\{\mathfrak{L}^{-1}\left(\frac{2s}{s^2+4s+3}\right)\right\} = \{-e^{-t} + 3e^{-3t}\},$$

so that

$$y(t) = 2F(t) + e^{-t}\int_{-\infty}^{t} e^{x}F(x)\,dx - 3e^{-3t}\int_{-\infty}^{t} e^{3x}F(x)\,dx,$$

which is the result that the Doležal procedure would yield in case $F(-\infty) = 0$. To ensure that this is indeed a solution, we require that

$$\left|\int_{-\infty}^{0} e^{x}F(x)\,dx\right| + \left|\int_{-\infty}^{0} e^{3x}F(x)\,dx\right| < \infty,$$

which is precisely the condition $F \in \mathfrak{S}[G]$ (see 9.7).

10. FOUR EXAMPLES

10.1. Given two numbers L and R, the equation

$$(10.2) \qquad L\partial^2 y + R\partial y + y = \partial F$$

governs a simple electric circuit; suppose that $F \in \mathfrak{B}$ and $tF(t) \in [\mathscr{K}_1]$ (this ensures that ∂F is definable and belongs to $\mathscr{K}$: see 7.13). In case $L = 1$ and $R = 2$, the steady-state solution of (10.2) is given by Definition (9.10):

$$(10.3) \qquad y = \frac{\partial}{(\partial+1)^2} F = \frac{DF}{(D+1)^2} + G \otimes F,$$

where

$$G = \left\{\mathfrak{L}^{-1}\left(\frac{s}{(s+1)^2}\right)\right\} = \{(1-t)e^{-t}\}.$$

To ensure that (10.3) is a solution of (10.2) we require that $F \in \mathfrak{S}[G]$, which means that

$$|[\{t^n e^{-t}\} \otimes F](0)| < \infty \quad \text{for} \quad n = 0, 1:$$

see 9.7. Our hypotheses $F \in \mathfrak{S}[G]$ and $tF(t) \in [\mathscr{K}_1]$ are satisfied when

$$(10.4) \qquad F = f + \sum_{k=0}^{\infty} \partial^k (D\mathsf{T}_k) = f + \sum_{k=0}^{\infty} D^k D\mathsf{T}_k,$$

where $f \in \mathscr{C}^1$ and $f \in \mathscr{S}_1$ (see 9.21); operator (10.4) is not functionable and does not correspond to a distribution of finite order. In case F is functionable, (10.3) becomes

$$y(t) = \int_{-\infty}^{t} [1-(t-x)]e^{-t+x}F(x)\,dx \qquad \text{(see 9.18.2)},$$

in view of 9.7, the existence of the integrals

$$\int_{-\infty}^{0} x^n e^{x} F(x)\,dx \qquad (\text{for } n = 0, 1)$$

is sufficient (and necessary) to ensure that y is a solution of our equation (10.2). Compare with [14], p. 383.

10.5. The case $L = 0$ and $R = 1$. In view of 9.15–(9.17) and (9.9), the steady-state solution of the equation $\partial y + y = \partial F$ is

$$(1) \qquad y = \frac{\partial}{\partial+1} F = F - \frac{1}{\partial+1} F = F - \frac{DF}{D+1} - G \otimes F,$$

where

$$G = \left\{\mathfrak{L}^{-1}\left(\frac{1}{s+1}\right)\right\} = \{e^{-t}\}$$

— provided that $|[\{e^{-t}\} \otimes F](0)| < \infty$. If F is functionable, then (1) and (9.19) give

$$(2) \qquad y(t) = F(t) - e^{-t}\int_{-\infty}^{t} e^{x}F(x)\,dx,$$

and the condition $|[\{e^{-t}\}\otimes F](0)|<\infty$ means simply that $e^tF(t)$ is integrable over $(-\infty, 0)$.

Finally, consider the case $F(t)=l(t)e^{at}+c\mathsf{T}_0(t)e^{bt}$ with $a>0$, $b>0$, and c is a complex number. We now have $|[\{e^{-t}\}\otimes F](0)|<\infty$ (as required); also, both F and ∂F belong to $\mathfrak{B}$; Equation (2) yields $y(0+)-y(0-)=c$ and

$$y(t)=\frac{a}{A}l(t)e^{at}+\mathsf{T}_0(t)\frac{bc}{B}e^{bt}-\mathsf{T}_0(t)\frac{ac-bc}{AB}e^{-t},$$

where $A=a+1$ and $B=b+1$; in case $b=2$ and $a=c=1$ the Fourier transform procedure gives no answer, because the "inversion step" fails nor does the inverse two-sided Laplace transform exist: see [12], p. 255).

10.6. The anti-derivative. To find the steady-state solution of $\partial y=F$ (when $F\in\mathfrak{B}$), set $P=1$ and $\mu(\partial)=\partial$ in (9.9)–(9.10):

$$\partial^{-1}F\overset{\text{def}}{=}\frac{1}{\partial}F=\frac{F}{D}+\{1\}\otimes F=D^{-1}F+\{1\}\otimes F;$$

moreover, from (9.18) it follows that

$$[\partial^{-1}F]_0=[\{1\}\otimes F](0)=\int_{-\infty}^{0}[lF](x)\,dx; \tag{3}$$

recall that $\partial(\partial^{-1}F)=F$ when $F\in\mathfrak{S}[\{1\}]$, which means that $|[\{1\}\otimes F](0)|<\infty$, which in turn means that the integral in (3) exists. In particular, if F is the unit impulse $D\mathsf{T}_0$, then (3) gives $\partial^{-1}F=\mathsf{T}_0$ (since $lF=0$). This clears up a question which was very unsatisfactorily treated in the textbook [9], pp. 146–152.

10.7. Example. Suppose that $F\in\mathfrak{B}$ and $lF(t)\in[\mathscr{K}_2]$ (see 7.13). The steady-state solution of the equation $\partial y=\partial^2F+F$ is given by

$$y=\frac{\partial^2+1}{\partial}F=\partial F+\partial^{-1}F;$$

it is required that $F\in\mathfrak{S}[\{1\}]$ (as in 10.6). For instance, F could be operator (10.4) when $f\in\mathscr{C}^2$ and $f(t)\in L^1(-\infty, 0)$.

11. A GENERALIZATION OF DUHAMEL'S INTEGRAL

11.1. Theorem. *Let F and G be functionable operators. If the operator $G\otimes F$ is functionable, then*

$$G\wedge F+G\otimes F=\Big\{\int_{-\infty}^{t}G(t-x)F(x)\,dx\Big\}. \tag{11.2}$$

Proof. Since $lF(x)=F(x)$ for $x<0$ (by 7.8), it follows from Definition (8.1) that

$$G\otimes F=\Big\{\int_{-\infty}^{0}G(t-x)F(x)\,dx\Big\};$$

from 2.20 therefore,

$$G\wedge F+G\otimes F=\Big\{\int_{0}^{t}G(t-x)F(x)\,dx\Big\}+\Big\{\int_{-\infty}^{0}G(t-x)F(x)\,dx\Big\};$$

Conclusion (11.2) is now an immediate consequence of the linearity of the mapping $f\mapsto\{f\}$ (see 2.6).

11.3. Theorem. *Suppose that $F\in\mathfrak{B}$. If both G and $G\otimes F$ are functionable, then*

$$l[G\wedge F+G\otimes F]=\Big\{l(t)\int_{-\infty}^{t}G(t-x)[lF](x)\,dx\Big\} \tag{11.4}$$

and

$$[G\wedge F+G\otimes F](\tau)=\int_{-\infty}^{\tau}G(\tau-x)[lF](x)\,dx\quad\textit{for}\quad\tau<0. \tag{11.5}$$

Proof. From 7.9 it follows that

$$G\wedge lF=l[G\wedge lF]=l\Big\{\int_{0}^{t}G(t-x)[lF](x)\,dx\Big\}; \tag{4}$$

the last equation is from 2.20. From (4) and Definition (8.1) therefore

$$l[G\wedge F+G\otimes F]=l\Big\{\int_{0}^{t}G(t-x)[lF](x)\,dx\Big\}+l\Big\{\int_{-\infty}^{0}G(t-x)[lF](x)\,dx\Big\}.$$

Conclusion (11.4) is now immediate from (7.6); Conclusion (11.5) comes from (11.4) and (7.8).

12. SOME PROOFS

We shall begin by proving 4.4 in the special case $n=1$; the proof generalizes easily.

12.1. Theorem. *Let B be an operator. If $B\supset h\in[\mathscr{K}_1]$, then $\partial B\supset h'$.*

Proof. Since $h\in[\mathscr{K}_1]$, there exist two numbers $x_k<0$ such that $h^{(k)}$ is continuous in $(x_k, 0)$ (for $k=0,1$). Since $B\supset h$, there exists a number $\alpha<0$ such that both h and h' have continuous extensions to the closed interval $[\alpha, 0]$ and B agrees with the function h on $(\alpha, 0)$: for any v in W and any t in $(\alpha, 0)$ we have (since $v' \in W$)

$$B\cdot v'(t) = -\int_t^0 h(t-x)v''(x)\,dx \qquad \text{(see 3.1)};$$

integrating by parts, we obtain

$$B\cdot v'(t) - v'(t)h(0-) = -\int_t^0 h'(t-x)v'(x)\,dx. \tag{5}$$

Since the left-hand side of (5) equals $[DB-h(0-)D]\cdot v(t)$, we have proved that ∂B agrees with h' on the interval $(\alpha, 0)$.

12.2. Suppose that $f_k \in L^{\rm loc}(\Omega)$ (for $k=1,2$). The function

$$f_{12} \overset{\rm def}{=} \int_0^t f_1(t-x)f_2(x)\,dx \tag{12.3}$$

belongs to $L^{\rm loc}(\Omega)$ (see [10], 0.19) and

$$\{f_1\}\wedge\{f_2\} = \{f_{12}\} \qquad \text{(see [10], 2.16)}. \tag{12.4}$$

12.5. Next, *to prove that* $B_1\wedge B_2 \in \mathscr{K}$ *whenever* $B_k \in \mathscr{K}$ (for $k=1,2$).

12.6. Suppose that $h_k \in [\mathscr{K}]$ (for $k=1,2$): there exists a number $x_{12} < 0$ such that both h_1 and h_2 have a continuous extension to the closed interval $[x_{12}, 0]$. Let f_k be the function defined by $f_k(u) = h_k(u)$ for $x_{12} < u < 0$, while $f_k(0) = h_k(0-)$ and $f_k(u) = 0$ for $u \leqslant x_{12}$ or $u > 0$. Note that $f_k \in L^{\rm loc}(\Omega)$; the relations

$$f_{12} \in \mathscr{K} \tag{12.7}$$

and

$$f_{12}(0) = 0 \tag{12.8}$$

are easily obtained by noting that the usual reasonings (as in [4], p. 258 or [7], pp. 34–43) apply equally well when the interval $[0, \infty)$ is replaced by an interval of the form $[x_{12}, 0]$ with $x_{12} < 0$.

12.9. In particular, if h_1 is the unit constant, we have

$$f_{12} = \int_0^t h_2(x)\,dx.$$

12.10. Lemma. *If* $h_k \in [\mathscr{K}]$ *and* $h_k \subset B_k \in \mathscr{A}$ *(for* $k=1,2$*), then*

$$f_{12} \subset B_1\wedge B_2. \tag{12.11}$$

Proof. In view of Definition 3.1, our hypotheses imply the existence of a number $a_{12} < 0$ such that B_k agrees with h_k in the interval $(a_{12}, 0)$: let a be the largest of the two negative numbers a_{12} and x_{12} (see 12.6).

Let f_k be the function defined in 12.6; clearly, $f_k(\tau) = h_k(\tau)$ for $\tau \in (a, 0)$ and therefore B_k agrees with f_k on $(a, 0)$: from 3.2 it therefore follows that the equation

$$[\{f_k\}\cdot w_k](\tau) = [B_k\cdot w_k](\tau) \qquad (\text{when } a < \tau < 0) \tag{1}$$

holds for any w_k in W. Let w be any element of W; since $\wedge\{f_2\} \in \mathscr{A}$, the equation

$$w_1 = \wedge\{f_2\}\cdot w \tag{2}$$

defines an element w_1 of W; substituting $k=1$ and (2) into (1), we obtain

$$[\{f_1\}\cdot(\wedge\{f_2\}\cdot w)](\tau) = [B_1\cdot(\wedge\{f_2\}\cdot w)](\tau);$$

from (2.3) it therefore follows that

$$[(\{f_1\}\wedge\{f_2\})\cdot w](\tau) = [(B_1\wedge\{f_2\})\cdot w](\tau) \quad \text{for} \quad a < \tau < 0,$$

so that (12.4) and (2.5) give

$$[\{f_{12}\}\cdot w](\tau) = [(\{f_2\}\wedge B_1)\cdot w](\tau). \tag{3}$$

On the other hand, the equation

$$w_2 = \wedge B_1\cdot w \tag{4}$$

defines an element w_2 of W; substituting (4) and $k=2$ in (1):

$$[\{f_2\}\cdot(\wedge B_1\cdot w)](\tau) = [B_2\cdot(\wedge B_1\cdot w)](\tau); \tag{5}$$

combining (3) and (5), we can use (2.3) to obtain

$$[\{f_{12}\}\cdot w](\tau) = [B_2\wedge B_1\cdot w](\tau) \qquad (\text{for } a < \tau < 0). \tag{6}$$

Since w is an arbitrary element of W, Conclusion (12.11) follows from (6) (using commutativity: see 2.5).

12.12. Theorem. *If* $B_k \in \mathscr{K}$ *(for* $k=1,2$*), then* $B_1\wedge B_2$ *also belongs to* $\mathscr{K}$*; moreover,*

$$[B_1\wedge B_2]_0 = 0 \quad \textit{and} \quad \partial[B_1\wedge B_2] = D[B_1\wedge B_2] = B_1B_2. \tag{12.13}$$

Proof. In view of 3.4, the hypotheses $B_k \in \mathscr{K}$ implies the existence of h_k in $[\mathscr{K}]$ such that $h_k \subset B_k$ (for $k=1,2$); from 12.10 it therefore follows that $f_{12} \subset B_1\wedge B_2$; the conclusion $B_1\wedge B_2 \in \mathscr{K}$ is now immediate from (12.7) and Definition 3.4; further, from (12.11) and 3.5 we see that

$$[B_1\wedge B_2]_0 = f_{12}(0) \doteq 0$$

(the last equation is from (12.8)); from Definition (3.6) therefore $\partial[B_1\wedge B_2] = DB_1\wedge B_2 = B_1B_2$ (the last equation is from 2.18).

12.14. Remarks. If B_1 is the operator $\{\mathbf{1}\}$ of the unit constant, then $B_1 \supset \mathbf{1}$ and the relations

$$\bigwedge B_2 = \{\mathbf{1}\} \bigwedge B_2 \supset \int_0^t h_2(x)\,dx \tag{12.15}$$

follow from (2.16.1), 12.10, and 12.9 (of course, we suppose that $B_2 \supset h_2 \in \mathscr{K}$). From (12.15) it easily results that

$$\bigwedge B_2 \in \mathscr{K}_1. \tag{12.15.1}$$

From (12.13) we also have

$$\partial(\bigwedge B_2) = B_2. \tag{12.16}$$

If B_2 is functionable, it follows from (12.4) that

$$\bigwedge B_2 = \Big\{\int_0^t B_2(x)\,dx\Big\}. \tag{12.17}$$

12.18. If ν is an integer $\geqslant 1$, then

$$\bigwedge^\nu = \{t^\nu/\nu!\}. \tag{12.19}$$

The proof (by induction) is based on the equations

$$\bigwedge^{n+1} = \bigwedge \bigwedge^n = \Big\{\int_0^t \frac{x^n}{n!}\,dx\Big\} = \Big\{\frac{x^{n+1}}{(n+1)!}\Big\}:$$

the middle equation is from (12.4)–(12.3).

12.20. *If $0 \leqslant r \leqslant \nu$, then $\partial^r \bigwedge^\nu = D^r \bigwedge^\nu = \bigwedge^{\nu-r}$.*

Indeed, since $\bigwedge^\nu \in \mathscr{C}^r$ (see 5.1), the conclusion is immediate from (12.19) and 5.3.

12.21. Lemma. *If c_r $(0 \leqslant r \leqslant n)$ is a finite family of complex numbers such that*

$$\sum_{r=0}^{n} c_r D^r = 0, \tag{12.22}$$

then $c_r = 0$ for $0 \leqslant r \leqslant n$.

Proof. Right-multiplying by $\bigwedge^{r+1}$ both sides of (12.22), it follows from 12.20 that

$$\sum_{r=0}^{n} c_r D^r \bigwedge^{n+1} = \sum_{r=0}^{n} c_r \bigwedge^{n+1-r} = 0,$$

whence $c_n t + c_{n-1} t^2/2 + \ldots + c_0 t^{n+1}/(n+1)! = 0$; the conclusion is now at hand.

12.23. Lemma. *If $n > 1$, if $y \in \mathscr{K}_n$ and $\partial^n y = D^n y$, then $0 = [\partial^k y]_0$ for $0 \leqslant k < n$.*

Proof. Immediate from 4.6 and 12.21.

12.24. Lemma. *Suppose that $R_2 \in \mathscr{K}$. If m is an integer $\geqslant 1$, then*

$$\bigwedge^m R_2 \in \mathscr{K}_m; \tag{12.25}$$

moreover,

$$\partial^m(\bigwedge^m R_2) = R_2 \tag{12.26}$$

and

$$[\partial^k(\bigwedge^m R_2)]_0 = 0 \quad \text{for} \quad 0 \leqslant k < m. \tag{12.27}$$

Proof. Since $\bigwedge^m R_2 \in \mathscr{K}_m$ for $m = 1$ (by (12.15.1)), we proceed by induction. If $\bigwedge^k R_2 \in \mathscr{K}_k$ for $k \geqslant 1$, then $\bigwedge^k R_2 \supset h_2 \in [\mathscr{K}_k]$, so that (12.15) (with $B_2 = \bigwedge^k R_2$) gives

$$\bigwedge^{k+1} R_2 = \bigwedge[\bigwedge^k R_2] \supset \int_0^t h_2(x)\,dx,$$

whence $\bigwedge^{k+1} R_2 \in \mathscr{K}_{k+1}$. This completes the induction proof of (12.25). To prove (12.26), suppose $k \geqslant 1$. Since $\bigwedge^k \in \mathscr{C}^k \subset \mathscr{K}_k$, it follows from (12.13) that

$$\partial(\bigwedge^{k+1} R_2) = \partial(\bigwedge^k \bigwedge R_2) = \bigwedge^k R_2;$$

therefore

$$\partial^{k+1}(\bigwedge^{k+1} R_2) = \partial^k[\partial(\bigwedge^{k+1} R_2)] = \partial^k[\bigwedge^k R_2]. \tag{7}$$

Thus, if

$$\partial^k(\bigwedge^k R_2) = R_2 \tag{8}$$

it follows from (7) that $\partial^{k+1}(\bigwedge^{k+1} R_2) = R_2$; since (8) holds for $k = 1$ (by (12.16)), this completes the induction proof of (12.26).

In view of 12.20, we can write (12.26) in the form

$$\partial^m(\bigwedge^m R_2) = D^m(\bigwedge^m R_2). \tag{9}$$

Conclusion (12.27) is now direct from (9), 12.23, and (12.25).

12.28. Remark. In view of (12.19), equation (12.26) can be written

$$R_2 = \partial^m(\bigwedge^{m-1} \bigwedge R_2) = \partial^m\Big(\Big\{\frac{t^{m-1}}{(m-1)!}\Big\} \bigwedge R_2\Big):$$

recall that $R_2 \in \mathscr{K}$. If R_2 is also functionable, we can use (2.21) to write the preceding equation in the form

$$R_2 = \partial^m\Big\{\int_0^t \frac{(t-x)^{m-1}}{(m-1)!} R_2(x)\,dx\Big\}.$$

13. AN EXISTENCE THEOREM

Henceforth, μ is a fixed polynomial

$$\mu(z) = a_0 + a_1 z + \ldots + a_m z^m,$$

where $a_m \neq 0$ and $m \geqslant 1$. Our aim is to prove the existence of a solution of the equation $\mu(\partial)y = B$, where B is a given element of $\mathscr{K}$; of course, B need not be functionable. In case B is functionable (and such that $B(t)$ is continuous in Ω), this existence theorem is well known. Presumably, the theorem could be derived from distribution theory, but it would still remain to prove that any solution belongs to $\mathscr{K}$.

We shall denote by $\mathscr{C}^\infty$ the space of all the elements g of $\mathscr{C}^0$ such that the function $g(t)$ is infinitely differentiable. Thus, $\mathscr{C}^\infty \subset \mathscr{C}^m$ for any integer $m \geqslant 0$.

In this section, g denotes the element of $\mathscr{C}^\infty$ such that

$$(13.1)\qquad \mu(\partial)g = 0 \quad \text{and} \quad g^{(k)}(0) = \begin{cases} 0 & \text{for} \quad k < m-1, \\ a_m^{-1} & \text{for} \quad k = m-1; \end{cases}$$

from (5.3) therefore

$$(13.2)\qquad [\partial^k g]_0 = \begin{cases} 0 & \text{for} \quad k < m-1, \\ a_m^{-1} & \text{for} \quad k = m-1. \end{cases}$$

13.3. Remark. If p is a polynomial

$$(13.4)\qquad p(z) = b_0 + b_1 z + \ldots + b_m z^m$$

it follows from (5.18) and (13.2) that

$$(13.5)\qquad p(\partial)g = p(D)g - a_m^{-1} b_m D.$$

13.6. Theorem. *The operator $\mu(D)$ is an invertible element of the algebra $\mathscr{A}$; its inverse belongs to $\mathscr{K}$. If λ is a polynomial of degree $\nu < m$, the equation*

$$(13.7)\qquad G \stackrel{\text{def}}{=} \frac{D\lambda(D)}{\mu(D)}$$

implies $G \in \mathscr{C}^\infty$,

$$(13.8)\qquad G = \lambda(\partial)g,$$

and

$$(13.9)\qquad \mu(\partial)G = 0.$$

Proof. Set $p = \lambda$ in 13.3: since $\nu < m$, it follows that $b_m = 0$; therefore, (13.5) gives

$$(13.10)\qquad \lambda(\partial)g = \lambda(D)g.$$

Set $p = \mu$ in 13.3: therefore, $b_m = a_m$, whence (13.5) implies

$$(13.11)\qquad \mu(\partial)g = \mu(D)g - D.$$

Since $\mu(\partial)g = 0$ (by (13.1)), equation (13.11) implies that

$$(13.12)\qquad \mu(D)g = D.$$

Left-multiplying by $\wedge$ both sides of (13.12), we obtain $\mu(D)[\wedge g] = \{1\}$; since $\wedge g \in \mathscr{K}$ (by (12.15.1)), the operator $\mu(D)$ is invertible and belongs to $\mathscr{K}$; from (13.12) therefore,

$$(13.13)\qquad g = \frac{D}{\mu(D)} \stackrel{\text{def}}{=} D[\mu(D)]^{-1}.$$

Combining (13.13) with (13.10):

$$(13.14)\qquad \lambda(\partial)g = \frac{D\lambda(D)}{\mu(D)} = G:$$

the last equation is from (13.7). Since $g \in \mathscr{C}^\infty$, it follows from 5.7 that $G \in \mathscr{C}^\infty$. In view of (13.14), it only remains to prove (13.9); to that effect, note that, by (13.14),

$$\mu(\partial)G = \mu(\partial)[\lambda(\partial)g] = \lambda(\partial)[\mu(\partial)g] = 0:$$

the middle equation is from (5.9), while the last equation is from (13.1).

13.15. Theorem. *Let μ be a polynomial of degree $m \geqslant 1$. If $B \in \mathscr{K}$, then*

$$(13.16)\qquad \frac{B}{\mu(D)} \in \mathscr{K}_m$$

moreover,

$$(13.17)\qquad \left[\partial^k \left(B/\mu(D)\right)\right]_0 = 0 \quad \textit{for} \quad 0 \leqslant k < m$$

and

$$(13.18)\qquad \mu(\partial)\left(B/\mu(D)\right) = B;$$

also,

(13.19) *if $\mu(\partial)y$ is definable and belongs to $\mathscr{K}$, then $y \in \mathscr{K}_m$.*

Proof. Setting $p(\partial) = \partial^m$ in 13.3, we obtain

$$(1)\qquad \partial^m g = D^m g - a_m^{-1} D = D^m \frac{D}{\mu(D)} - a_m^{-1} D:$$

the last equation is from (13.13). Since $g \in \mathscr{C}^\infty$, it follows from 5.3 and (1) that

$$(2)\qquad g^{(m)} = -a_m^{-1} D + \frac{1}{\mu(D)} D^{m+1}.$$

Left-multiplying by $B\wedge^{m+1}$ both sides of (2):

$$B\wedge^{m+1}g^{(m)} = -a_m^{-1}BD\wedge^{m+1}+\frac{B}{\mu(D)}D^{m+1}\wedge^{m+1};$$

from 12.20 and 2.18 therefore,

$$\wedge^m B\wedge g^{(m)} = -a_m^{-1}B\wedge^m+\frac{B}{\mu(D)};$$

that is,

$$(3)\qquad \frac{B}{\mu(D)} = \wedge^m B_2,\quad \text{where}\quad B_2 = a_m^{-1}B+B\wedge g^{(m)}.$$

Since $B\in\mathscr{K}$ (by hypothesis), and since $g^{(m)}\in\mathscr{K}$, it follows from 12.12 that $B_2\in\mathscr{K}$: conclusion (13.16) now comes from (3) and (12.25). Since $B_2\in\mathscr{K}$, conclusion (13.17) results from (3) and (12.27).

We can set $p=\mu$ and $y=B/\mu(D)$ in 5.17 to obtain (13.18) from (13.17): note that $\mu(\partial)y$ is definable (by (13.16) and 4.4).

It remains to prove (13.19). By hypothesis (see Definition 5.4) we have $\partial^k y\in\mathscr{K}$ for $0\leqslant k\leqslant m$; from 5.17–5.16 it follows the existence of a polynomial λ of degree less than m such that

$$\mu(D)y-D\lambda(D) = \mu(\partial)y,$$

whence

$$(4)\qquad y = \frac{\mu(\partial)y}{\mu(D)}-\frac{D\lambda(D)}{\mu(D)} = \frac{B}{\mu(D)}-G:$$

the last equation is obtained by setting $B=\mu(\partial)y$ and $G=D\lambda(D)/\mu(D)$. Since $B\in\mathscr{K}$ (from our hypothesis $\partial^k y\in\mathscr{K}$) and since $G\in\mathscr{C}^\infty$ (from 13.6), the conclusion $y\in\mathscr{K}_m$ now comes from (13.16).

13.20. THEOREM. *Suppose that $B\in\mathscr{K}$ and let c_k $(0\leqslant k<m)$ be a sequence of complex numbers. The initial-value problem*

$$\mu(\partial)y = B\quad \textit{with}\quad [\partial^k y]_0 = c_k\quad (\textit{for } 0\leqslant k<m)$$

has a solution y in $\mathscr{K}_m$.

Proof. Let F be the element of $\mathscr{C}^\infty$ such that

$$(5)\qquad \mu(\partial)F = 0\quad\text{and}\quad [\partial^k F]_0 = c_k;$$

recall that $[\partial^k F]_0 = F^{(k)}(0)$ (see 5.3). Set

$$(6)\qquad y \overset{\text{def}}{=} \frac{B}{\mu(D)}+F.$$

From (6) and (13.18) therefore

$$\mu(\partial)y = \mu(\partial)\frac{B}{\mu(D)}+0 = B.$$

On the other hand, the equations

$$[\partial^k y]_0 = \left[\partial^k\frac{B}{\mu(D)}\right]_0+[\partial^k F]_0 = 0+c_k = c_k$$

are from (6), (5), and (13.17). The conclusion $y\in\mathscr{K}_m$ comes from (6) and (13.16).

13.21. THEOREM. *If $B\in\mathscr{K}$ and ∂B is functionable, then $B\in\mathscr{C}^0$ and $B'(t) = [\partial B](t)$. Let μ be a polynomial of degree $m\geqslant 1$; if $\mu(\partial)y$ is definable and belongs to $\mathscr{C}^0$, then $y\in\mathscr{C}^m$.*

Proof. Set $F=\partial B$; from Definition 3.6 it follows that $F = D[B-\{[B]_0\mathbf{1}\}]$; left-multiplying by $\wedge$ both sides of this equation, we have

$$\wedge F = B-\{[B]_0\mathbf{1}\}\quad (\text{from } 2.18);$$

from (12.17) therefore,

$$B = \left\{[B]_0\mathbf{1}+\int_0^t F(x)\,dx\right\};$$

ince $F(t)\in L^{\text{loc}}(\Omega)$, it now results from 2.12 that

$$B(t) = [B]_0+\int_0^t F(x)\,dx,$$

whence $B'(\tau) = F(\tau) = [\partial B](\tau)$ for almost-every τ in Ω.

Set $B=\mu(\partial)y$: our hypothesis implies that $B\in\mathscr{C}^0$; as is well known, there exists an operator y_1 in $\mathscr{C}^m$ such that (in view of 5.6)

$$\mu(\partial)y_1 = B\quad\text{and}\quad y_1^{(k)}(0) = [\partial^k y_1]_0 = [\partial^k y]_0\quad (\text{for } 0\leqslant k<m);$$

rom 5.17 it therefore follows that $\mu(D)y = \mu(D)y_1$, which implies $y=y_1$: ince $y_1\in\mathscr{C}^m$, we have our conclusion $y\in\mathscr{C}^m$.

13.22. THEOREM. *Let h be a non-zero polynomial of degree $<m$, where m is the degree of μ. If $F\in\mathscr{K}$ and if $h(\partial)F$ is definable and belongs to $\mathscr{K}$, then*

$$(13.23)\qquad \mu(\partial)\left[\frac{h(D)F}{\mu(D)}\right] = h(\partial)F.$$

Proof. From 5.17–5.16 it follows the existence of a polynomial λ such that $\deg(\lambda)<\deg(h)$ and

$$h(\partial)F = h(D)F-D\lambda(D);$$

consequently,

$$(13.24)\qquad \frac{h(D)F}{\mu(D)} = \frac{h(\partial)F}{\mu(D)}+G,\quad\text{where}\quad G = \frac{D\lambda(D)}{\mu(D)}.$$

Since $\deg(\lambda) < \deg(h) < m$, it results from (13.24) and (13.9) that

$$\mu(\partial)\left[\frac{h(D)F}{\mu(D)}\right] = \mu(\partial)\left[\frac{h(\partial)F}{\mu(D)}\right]+0 = h(\partial)F;$$

the last equation is obtained by setting $B = h(\partial)F$ in (13.18).

14. CONCLUDING SECTION

It remains to prove that the operator (9.10) satisfies the differentia equation (9.11). As in Sections 7–11, the open interval Ω has the form $(-\infty, \beta)$; further, μ is a polynomial of degree $\geqslant 1$.

14.1. THEOREM. *Let λ be a polynomial whose degree is smaller than the degree of μ. Set*

(14.2) $$G = \frac{D\lambda(D)}{\mu(D)}.$$

If $F \in \mathfrak{S}[G]$, then $G\otimes F \in \mathscr{C}^\infty$ and

(14.3) $$\mu(\partial)[G\otimes F] = 0.$$

Proof. Let Z be the set of zeros of μ; as in 9.2, let $\pi(a)$ $(a\in Z)$ be the family of integers $\pi(a) \geqslant 0$ such that

(14.4) $$G = \sum_{a\in Z}\sum_{m=0}^{\pi(a)} c_a^m\{e^{at}t^m\},$$

where the c_a^m are complex numbers. Our hypothesis $F\in\mathfrak{S}[G]$ means that $F\in\mathfrak{B}$ and

(14.5) $$\sum_{a\in Z}\sum_{n=0}^{\pi(a)}\left|\int_{-\infty}^{0} e^{-ax}x^n[\iota F](x)\,dx\right| < \infty.$$

Take any a in Z and let $0\leqslant m\leqslant \pi(a)$. Set $g = e^{at}t^m$; consequently,

(8) $$g(\tau) = e^{a\tau}\tau^m \quad \text{for} \quad \tau\in\Omega.$$

If $k\geqslant 0$ it follows from Definition (8.1) that

(9) $$g^{(k)}\otimes F(t) = \int_{-\infty}^{0} g^{(k)}(t-x)[\iota F](x)\,dx.$$

In view of (9)–(8):

(10) $$g\otimes F(t) = \int_{-\infty}^{0}(t-x)^m e^{at}e^{-ax}[\iota F](x)\,dx;$$

therefore,

(11) $$g\otimes F(t) = \sum_{\nu=0}^{m}\binom{m}{\nu}e^{at}t^m\Big[\int_{-\infty}^{0} e^{-ax}(-x)^{m-\nu}[\iota F](x)\,dx\Big];$$

from (8) and (11) we may therefore conclude that

(12) $$\{e^{at}t^m\}\otimes F \quad \textit{belongs to } \mathscr{C}^\infty.$$

Since $0\leqslant m-\nu\leqslant m\leqslant \pi(a)$, the existence of the number

(13) $$F(a, m-\nu) \overset{\text{def}}{=} \int_{-\infty}^{0} e^{-ax}(-x)^{m-\nu}[\iota F](x)\,dx$$

follows from (14.5). Equation (11) can now be written

(14) $$\{e^{at}t^m\}\otimes F = \sum_{\nu=0}^{m}\binom{m}{\nu}F(a, m-\nu)\{e^{at}t^m\}.$$

On the other hand, it follows from (9) that

$$g^{(k)}\otimes F(t) = \int_{-\infty}^{0}\left(\frac{\partial}{\partial t}\right)^k g(t-x)[\iota F](x)\,dx;$$

from (8) therefore,

(15) $$g^{(k)}\otimes F(t) = \sum_{\nu=0}^{m}\binom{m}{\nu}\int_{-\infty}^{0}\left[\left(\frac{\partial}{\partial t}\right)^k e^{at}t^m\right]e^{-ax}(-x)^{m-\nu}[\iota F](x)\,dx.$$

From (15) and (13) therefore

$$g^{(k)}\otimes F(t) = \sum_{\nu=0}^{m}\binom{m}{\nu}\left[\left(\frac{\partial}{\partial t}\right)^k e^{at}t^m\right]F(a, m-\nu),$$

whence, by (8) and 5.3,

(16) $$[\partial^k\{e^{at}t^m\}]\otimes F = \sum_{\nu=0}^{m}\binom{m}{\nu}F(a, m-\nu)[\partial^k\{e^{at}t^m\}].$$

From (14) and (16) we may now conclude that

(17) $$\partial^k[\{e^{at}t^m\}\otimes F] = [\partial^k\{e^{at}t^m\}]\otimes F.$$

Now for the conclusion. From (14.4) we see that

(18) $$\partial^k[G\otimes F] = \sum_{a,m} c_a^m\partial^k[\{e^{at}t^m\}\otimes F];$$

from (17) therefore,

$$\partial^k[G\otimes F] = \sum_{a,m} c_a^m [\partial^k\{e^{at}t^m\}]\otimes F$$
$$= \partial^k\Big[\sum_{a,m} c_a^m \{e^{at}t^m\}\Big]\otimes F;$$

another application of (14.4) therefore gives

$$\partial^k[G\otimes F] = [\partial^k G]\otimes F \quad \text{for any integer } k\geqslant 0,$$

whence

$$\mu(\partial)[G\otimes F] = [\mu(\partial)G]\otimes F = 0:$$

the last equation is from (13.9). This gives Conclusion (14.3). The conclusion $G\otimes F\in\mathscr{C}^\infty$ comes from (12) by setting $k=0$ in (18).

14.6. THEOREM. *Let P be a non-zero polynomial. Let Q and λ be the polynomials such that*

(14.7) $$P/\mu = Q+\lambda/\mu \quad \text{and} \quad \deg(\lambda)<\deg(\mu).$$

Suppose that $F\in\mathfrak{S}[D\lambda(D)/\mu(D)]$; if $P(\partial)F$ is definable and belongs to $\mathscr{K}$, then

(14.8) $$\mu(\partial)\left[\frac{P(\partial)}{\mu(\partial)}F\right] = P(\partial)F.$$

Proof. Let n be the degree of P, let m be the degree of μ, and let r be the degree of Q. Since $P(\partial)F$ is definable and belongs to $\mathscr{K}$, it follows from 5.5 that

(19) $$F\in\mathscr{K}_n.$$

Since $\deg(\lambda)<m\leqslant n$, it results from (19) that $F\in\mathscr{K}_i$, where i is the degree of λ: therefore, $\lambda(\partial)F$ is definable and belongs to $\mathscr{K}$ (see 5.5): consequently, we may set $h=\lambda$ in 13.22 to obtain

(20) $$\mu(\partial)\left[\frac{\lambda(D)F}{\mu(D)}\right] = \lambda(\partial)F.$$

On the other hand, (14.7) gives

(21) $$P = \mu Q+\lambda \quad \text{and} \quad n = m+r;$$

it therefore follows from (5.10) that

(22) $$P(\partial)F = \mu Q(\partial)F+\lambda(\partial)F = \mu(\partial)[Q(\partial)F]+\lambda(\partial)F:$$

the last equation is from (19), $n=\deg(\mu Q)$, 5.5 and (5.9). From Definition (9.10) we see that

$$\mu(\partial)\left[\frac{P(\partial)}{\mu(\partial)}F\right] = \mu(\partial)[Q(\partial)F]+\mu(\partial)\left[\frac{\lambda(D)F}{\mu(D)}\right]+\mu(\partial)[G\otimes F]:$$

from (20) and (14.3) therefore

$$\mu(\partial)\left[\frac{P(\partial)}{\mu(\partial)}F\right] = \mu(\partial)[Q(\partial)F]+\lambda(\partial)F+0 = P(\partial)F:$$

the last equation is from (22). This concludes the proof of (14.8).

15. APPENDIX

This section is devoted to proving (7.6) and 7.10. Let B be a functionable operator. The equation

(15.1) $$\mathsf{T}_0 B = \{\mathsf{T}_0(t)B(t)\}$$

is proved in [10], (6.2). From (7.2) it follows that

$$\{l(t)B(t)\} = \{B(t)-\mathsf{T}_0(t)B(t)\} = B-\mathsf{T}_0B = [\{1\}-\mathsf{T}_0]B:$$

the middle equation is from (7.5) and (15.1); the last equation comes from (2.16.1); Conclusion (7.6) is now immediate from (7.1). Thus, we have also proved that

(15.2) $$B(t)l(t) = lB(t).$$

It remains to prove 7.10.

15.3. Recall that $A\in\mathscr{A}$ if (and only if) $A\{w\} = \{A\cdot w\}$ for all w in W (see Definition 2.2).

15.4. LEMMA. *If $R\in\mathscr{A}$ and $w\in W$, then*

(15.5) $$[R\cdot w](\tau)l(\tau) = [lR\cdot w](\tau) \quad \textit{for} \quad \tau\in\Omega.$$

Proof. From 15.3 we see that

$$\{R\cdot w\}(\tau)l(\tau) = [R\{w\}](\tau)l(\tau) = [lR\{w\}](\tau) = \{lR\cdot w\}(\tau):$$

the middle equation is from (15.2); the last equation is from 15.3 (with $A=lR$). Conclusion (15.5) now comes by observing that

$$\{A\cdot w\}(\tau) = [A\cdot w](\tau) \quad \text{for any operator } A \text{ (see 2.12).}$$

15.6. THEOREM. *Suppose that $A\in\mathscr{A}$ and $f\in L^{\mathrm{loc}}(\Omega)$. If A agrees with f on $(-\infty,0)$, then*

(15.7) $$lA = l\{f\}.$$

Proof. Take any w in W. By hypothesis,

(15.8) $$[A\cdot w](\tau) = [\{f\}\cdot w](\tau) \quad \text{for} \quad \tau<0:$$

see 3.2. From (15.8) and (7.2) it follows that

$$[A\cdot w](\tau)\mathfrak{l}(\tau) = [\{f\}\cdot w](\tau)\mathfrak{l}(\tau) \quad \text{for} \quad \tau \in \Omega,$$

whence a double application of (15.5) now gives

$$[\mathfrak{l}A\cdot w](\tau) = [\mathfrak{l}\{f\}\cdot w](\tau):$$

since $\tau\in\Omega$ and $w\in W$, we have obtained (15.7).

15.9. Proof of 7.10. In view of 15.6, it suffices to verify that (15.7) implies (15.8). To that effect, note that (15.5) gives

$$[R\cdot w](\tau) = [\mathfrak{l}R\cdot w](\tau) \quad \text{for} \quad \tau<0. \tag{15.10}$$

Therefore, for $\tau<0$,

$$[A\cdot w](\tau) = [\mathfrak{l}A\cdot w](\tau) = [\mathfrak{l}\{f\}\cdot w](\tau) = [\{f\}\cdot w](\tau);$$

the middle equation is from (15.7); the last equation is from (15.10). We have obtained (15.8) as a consequence of (15.7).

References

[1] L. Berg, *Operatorenrechnung*, I. *Algebraische Methoden*, VEB Deutscher Verlag der Wiss., Berlin 1972.
[2] H. Bremermann, *Distributions, complex variables, and Fourier transforms*, Reading 1965.
[3] E. A. Coddington, *An introduction to ordinary differential equations*, Englewood Cliffs, 1961.
[4] R. V. Churchill, *Operational mathematics*, New York 1958.
[5] G. Doetsch, *Guide to the applications of the Laplace and Z-transforms*, London 1971.
[6] V. Doležal, *Dynamics of linear systems*, Groningen, Prague 1967.
[7] D. L. Holl, C. G. Maple and B. Vinograde, *Introduction to the Laplace transform*, New York 1959.
[8] W. Kaplan, *Operational methods for linear systems*, Reading 1962.
[9] G. Krabbe, *Operational calculus*, New York 1970.
[10] — *An algebra of generalized functions on an open interval: two sided operational calculus*, Pacific J. Math. (42) 2 (1972), pp. 381–411.
[11] — *An algebra of generalized functions on an open interval: two-sided operational calculus*, Bull. Amer. Math. Soc. (77) 1 (1971), pp. 78–84.
[12] B. P. Lathi, *Signals, systems and communication*, New York 1966.
[13] B. J. Leon, *Lumped systems*, New York 1968.
[14] — and P. A. Wintz, *Basic linear networks for electrical and electronics engineers*, New York 1970.
[15] B. Van Der Pol and H. Bremmer, *Operational calculus based on the two-sided Laplace integral*, Cambridge 1955.
[16] F. M. Reza and S. Seely, *Modern network analysis*, New York 1959.
[17] R. J. Schwartz and B. Friedland, *Linear systems*, New York 1965.
[18] H. S. Shultz, *An algebra of distributions on an open interval*, Trans. Amer. Math. Soc. 169 (1972), pp. 163–181.

[19] G. Wunsch, *Die Heavisidesche Operatorenrechnung in neuer Begründung*, Wissensch. Zeitschr. Techn. Hochschule Dresden (9) 4 (1959/60), pp. 991–998.
[20] A. H. Zemanian, *Distribution theory and transform analysis, an introduction to generalized functions, with applications*, New York 1965.

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