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Well embedded Hilbert subspaces in C^*-algebras

by

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Abstract. A Hilbert subspace Y of a normed linear space X is a well embedded Hilbert subspace of X if there exists a linear subspace $Z \subset X$ such that $X = Y \oplus Z$ and that the linear operator $U \oplus I_Z$ (where I_Z is the identity operator on Z) is an isometry for every unitary operator U acting on Y. We characterize such subspaces in C^*-algebras.

1. Introduction. Recent advances in study of the structure of finite dimensional Banach spaces have brought the definition of a well embedded Hilbert subspace ([6], Definition 1 below). Let e.g. X be a finite dimensional complex Banach space with such a basis that the norm of any vector in X depends only on the absolute values of its components in this basis. Then X is a direct sum of well embedded Hilbert subspaces ([5] and [6]). We characterize completely well embedded Hilbert subspaces in C^*-algebras with identity, showing that they are rather "uncommon" in a sense explained below.

2. Definitions and preliminary results. Let X be a normed linear space (real or complex) and Y a linear subspace in X. We say that Y is a *Hilbert subspace* of X if it is a Hilbert space in the norm it inherits from the space X. A vector $x \in X$ is *orthogonal* to a vector $y \in X$ if $\|x+ay\| \geqslant \|x\|$ for all scalars a.

DEFINITION 1. A Hilbert subspace Y of a normed linear space X is a *well embedded Hilbert subspace* of X if there exists a linear subspace $Z \subset X$ such that $Y \oplus Z = X$ and that the linear operator $U \oplus I_Z$ (where I_Z is the identity operator on Z) is an isometry for every unitary operator U acting on Y.

Remark. If X is a Hilbert space, every closed subspace Y in X is a well embedded Hilbert subspace. In this case the space Z is the orthocomplement of the space Y.

LEMMA 1. *Let Y and Z be as in Definition* 1, *$y \in Y$, and $z \in Z$. Then y and z are mutually orthogonal.*

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Proof. Set $U = -I_Y$ in Definition 1. Thus y and z are eigenvectors corresponding to distinct eigenvalues of a linear isometry. Apply [2], Corollary 3.

Our approach to the problem is based upon Kadison's characterization of linear isometries between C^*-algebras with identities. We will use the following (weaker) result:

Let A be a C^*-algebra with identity 1 and T a linear isometry of A onto itself. Then $u = T1$ is a unitary element in A and $f = u^*T$ preserves the selfadjoints.

A nice elementary proof of Kadison's result is given in [3].

A non-zero idempotent g in a Banach algebra A is called *minimal* if the algebra gAg is one-dimensional.

3. Well embedded Hilbert subspaces in C^*-algebras.

THEOREM 1. *Every well embedded Hilbert subspace in a C^*-algebra with identity is necessarily one-dimensional and is spanned by a central minimal projection.*

Conversely, every central minimal projection in a C^*-algebra A spans a one-dimensional well embedded Hilbert subspace of A.

Proof. We prove the second statement first. Let $y \in A$ be a central minimal projection. Set $Y = \{x \mid x = yx\}$ and $Z = \{x \mid yx = 0\}$. Clearly, $X = Y \oplus Z$. If $x \in A$, then $yx = xy = yxy = ty$ for some scalar t. Hence Y is one-dimensional. In dimension 1 a unitary operator means multiplication by a complex number of modulus 1. Let $z \in Z$. Then

$$\|e^{it}y+z\|^2 = \|(e^{-it}y+z^*)(e^{it}y+z)\| = \|y+z^*z\| = \|y+z\|^2$$

for all real t and the proof is complete.

We return to the first statement. Let Y' be a well embedded Hilbert subspace in a C^*-algebra A with identity 1. Clearly, every closed linear subspace of Y' is also a well embedded Hilbert subspace of A. We may assume that $\dim A > 1$.

Let Y be an arbitrary one-dimensional linear subspace of Y' and Z a complementary subspace to Y such that Y and Z satisfy the requirements of Definition 1. For every real t, let $T_t\colon A \to A$ be the linear operator:

$$T_t(y+z) = e^{it}y+z \qquad (y \in Y,\ z \in Z).$$

The set $\{T_t\}$ is a one-parameter group of isometries. Write, as before, $u_t = T_t1$ and $f_t = u_t^*T_t$. We prove that $1 \notin Y$ and $1 \notin Z$. If $1 \in Y$, then $u_t = T_t1 = e^{it}1$. Let $z \in Z$ be an arbitrary non-zero element. We choose the scalar a so that $z^*-ay \in Z$ and compute

$$\begin{aligned} f_t(z^*) &= u_t^*T_t(ay+z^*-ay) = e^{-it}(e^{it}ay+z^*-ay) \\ &= ay+e^{-it}(z^*-ay). \end{aligned}$$

On the other hand,

$$f_t(z^*) = f_t(z)^* = (u_t^*T_tz)^* = (u_t^*z)^* = e^{it}z^*$$

for all real t. This implies readily that $z^* = 0$, a contradiction. If $1 \in Z$, then $u_t = T_t1 = 1$ and a similar argument leads to $Y = \{0\}$, a contradiction. Thus $1-y \neq 0$ and $1-y \in Z$ for some non-zero $y \in Y$. Since $u_t = T_t1 = e^{it}y+1-y$ and

$$1 = u_tu_t^* = 1+y-y^*+e^{it}(y-yy^*)+e^{-it}(y^*-yy^*)$$

for all real t, we see immediately that $y = y^* = y^2$.

Once more, let $z \in Z$ be arbitrary. Choose $a \in \mathbf{C}$ so that $z^*-ay \in Z$. Then

$$\begin{aligned} f_t(z^*) &= u_t^*T_t(ay+z^*-ay) \\ &= (e^{-it}y+1-y)(ae^{it}y+z^*-ay) \\ &= z^*+(ay-yz^*)+e^{-it}(yz^*-ay). \end{aligned}$$

Also

$$f_t(z^*) = f_t(z)^* = (u_t^*z)^* = z^*u_t = z^*-z^*y+e^{it}z^*y.$$

Comparing the two results we see that $z^*y = 0 = ay-yz^*$. But $ay = (ay)y = (yz^*)y = y(z^*y) = 0$. Thus $yz^* = 0 = yz = zy$. This proves that y is central and minimal.

Suppose now that $\dim Y' \geqslant 2$ (see the beginning of the proof). It follows from our results that there exist at least two linearly independent central minimal projections, say $y, y' \in Y'$. Since $yy' = yy'y = y'yy'$ and both y, y' are minimal, it follows that $yy' = 0$. Thus $\|y+y'\|^2 = \|(y+y')^2\| = \|y+y'\|$ and so $\|y+y'\| = \|y-y'\| = 1$. The parallelogram law fails.

Remark. Using the generalization in [4] of Kadison's theorem to arbitrary C^*-algebras, it is possible to prove Theorem 1 in the absence of an identity in a C^*-algebra, too. However, the only proof the author knows of is rather complicated (although elementary) and consequently it will not be published here.

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q-variate minimal stationary processes

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Abstract. A complete description of non full rank in general q-variate minimal stationary processes over discrete Abelian groups are given. This result subsumes the minimality theorems of various authors in special cases.

1. Introduction. In his fundamental paper [1] A. N. Kolmogorov introduced the important concept of minimal processes. Next the concept have been extended to the q-variate case (cf. [2] and [6], Section 10). The interpolation problem for q-variate stationary processes over groups was studied by H. Salehi and J. K. Scheidt [8] and by A. Weron [9], [10]. Furthermore in those papers characterizations of q-variate minimal processes are also given. In [8] a generalization of Masani's minimality theorem for full rank processes is obtained. Two characterizations of non-full rank processes are given in [10], but unfortunately one of which ([10], Theorem 5.7) contains an error. In this paper a counter example for this (see Example 5.3) and a correct statement of this theorem (see Theorem 4.6(d)) is given. Moreover, we will get a general theorem on characterizations of q-variate minimal (not necessary full rank) processes.

Section 2 is devoted to the preliminary results on the spaces $L_{2,F}$ — of square integrable matrix-valued functions and $H_{2,F}$ — of Hellinger square integrable matrix-valued measures. Section 3 treats on q-variate stationary processes over a discrete Abelian group. Using methods of the earlier work [10] on stationary processes over locally compact Abelian (LCA) groups, we obtain an analytical characterization of a subspace N_e which is important in the minimality problem. In Section 4 we discuss the minimality problem and give some characterizations of minimal processes. As a corollary we then deduce Kolmogorov's and Masani's minimality theorems. Finally in Section 5 we give several examples to show that conditions in the presented theorems are essential ones as well as to illustrate them.

2. $L_{2,F}$ and $H_{2,F}$ spaces. Let $\mathfrak{B}$ be a σ-algebra of subsets of a space Ω and let $\Phi = [\varphi_{ij}]$, $1 \leqslant i, j \leqslant q$, be a matrix-valued function on Ω. Troughout this paper all matrices have complex entries and C denotes