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The range of vector measures into Orlicz spaces

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Abstract. It is shown that the range of a σ-additive vector measure having values in an Orlicz space $L_\varphi(X, A, \mu)$, where φ is unbounded and satisfies condition Δ_2, is bounded. This implies that every scalar-valued, bounded measurable function can be integrated with respect to any vector measure taking values in such a space $L_\varphi(X, A, \mu)$. In the special case of the sequence spaces l^p, $0 < p < 1$, the range is relatively compact, and the closure is even convex and compact if the measure is nonatomic.

1. It is known that the range of every σ-additive vector measure with values in a locally pseudoconvex vector space is bounded (cf. [1]). On the other hand P. Turpin has shown in [11] that there exists a non-locally pseudoconvex F-space and a vector measure having unbounded range in that space. With regard to integration theory it would be important to know whether a vector measure has always bounded range in an Orlicz space $L_\varphi(X, A, \mu)$ (cf. [8]). P. Turpin states this question in [9] and [11].

In this note we answer the question positively for the class of Orlicz spaces $L_\varphi(X, A, \mu)$, where φ is unbounded and satisfies condition Δ_2. It is done by showing that every normbounded, convex set in $L_\varphi(X, A, \mu)$ is bounded and then using the fact that the convex hull of the range of such a vector measure is normbounded. The latter follows from an inequality for Orlicz spaces, which is essential for the proof that in these spaces unconditional convergence is equivalent to bounded multiplier convergence ([4], [10]).

As a consequence every scalar-valued, bounded measurable function can be integrated with respect to any vector measure taking values in such a space $L_\varphi(X, A, \mu)$.

In the special case of the sequence spaces l^p, $0 < p < 1$, the range is even relatively compact. When such a vector measure is also nonatomic, the closure of its range is compact and convex.

2. Throughout the paper, Ω will denote a set and Σ a σ-algebra of subsets. Let Y be an F-space (i.e. a complete metric topological linear space) with F-norm $\| \ \|$. By a *vector measure* m on Σ is understood a map $m: \Sigma \to Y$, which is countably additive. We set $m(\Sigma) := \{m(A): A \in \Sigma\}$.

A set $A \in \Sigma$ is called an *atom* of m if $m(A) \neq 0$ and if $E \in \Sigma$, $E \subset A$ imply $m(E) = 0$ or $m(E) = m(A)$.

By $v(m, E) := \sup\{\sum_n \|m(A_n)\|: A_n \in \Sigma,\ A \subset E$ and $A_i \cap A_j = \emptyset$ for $i \neq j\}$ we denote the total variation of m over the set E.

We use the notion $\mathrm{co}(A)$ for the convex hull of a set A.

$(X, \boldsymbol{A}, \mu)$ denotes a measure space (cf. [2]). Let $\varphi: [0, \infty) \to [0, \infty)$ be continuous, nondecreasing with $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ and φ satisfying condition Δ_2 (i.e. there is a $k > 1$ with $\varphi(2t) \leqslant k\varphi(t)$ for all $t > 0$). Then $L_\varphi(X, \boldsymbol{A}, \mu)$ is the linear space of all real or complex-valued μ-measurable functions on X with $J_\varphi(x) := \int_X \varphi(|x|)\,d\mu < \infty$. We identify functions which differ only on a set of measure zero.

$L_\varphi(X, \boldsymbol{A}, \mu)$ is called *Orlicz space*. It is an F-space with F-norm $\|x\| := \inf\{\varepsilon > 0: J_\varphi(x/\varepsilon) \leqslant \varepsilon\}$ (cf. [5], [6]). We list some well-known properties, which we need in the sequel:

(i) The sets $N_\varphi(\varepsilon) := \{x \in L_\varphi(X, \boldsymbol{A}, \mu): J_\varphi(x) \leqslant \varepsilon\}$ form a neighbourhood base of zero.

(ii) $\|x\| \leqslant 1$ implies $J_\varphi(x) \leqslant 1$.

(iii) J_φ is a modular with the property $J_\varphi(x \pm y) \leqslant k\big(J_\varphi(x) + J_\varphi(y)\big)$.

3. In a locally bounded Orlicz space $L_\varphi(X, \boldsymbol{A}, \mu)$ every normbounded set is also bounded [9]. This is obviously false in non-locally bounded spaces. However we have

THEOREM 1. *Let $M \subset L_\varphi(X, \boldsymbol{A}, \mu)$ be convex. Then M is bounded, whenever it is normbounded.*

Proof. For simplicity of notation we assume M to be in the unit ball of $L_\varphi(X, \boldsymbol{A}, \mu)$. After obvious changes the proof also works in general case.

We suppose that M is not bounded and lead this to a contradiction, namely we show that then there is a convex combination $\sum_{i=1}^N \lambda_i f_i$ of elements of M with $\|\sum_{i=1}^N \lambda_i f_i\| > 1$.

We distinguish between the following two cases.

(A) Let $(X, \boldsymbol{A}, \mu)$ be a finite positive measure space. For simplicity of notation again we assume $\mu(X) = 1$.

If M is not bounded, there is an $\varepsilon_0 > 0$ such that for every $\lambda > 0$ there exists some $f \in M$ with

$$\lambda f \notin N_\varphi(\varepsilon_0). \tag{1}$$

As M is normbounded, for every $\xi, \eta > 0$ there is a $\lambda > 0$ with

$$\mu\{t: \varphi(|\lambda f(t)|) \geqslant \xi\} \leqslant \eta \tag{2}$$

for all $f \in M$.

Let $k > 1$ be the real number defined by the Δ_2-condition of φ. We choose $N \in \boldsymbol{N}$ so that $N\varepsilon_0/2k > 1$.

Set $\xi_0 := \varepsilon_0/8Nk^{N+2}$ and λ_1 arbitrarily with $0 < \lambda_1 < 1$. From (1) we get that there is $f_1 \in M$ with $\int_X \varphi(|\lambda_1 f_1|)\,d\mu \geqslant \varepsilon_0$. Set

$$A_1 := \{t: \varphi(|\lambda_1 f_1(t)|) \geqslant \xi_0\}. \tag{3}$$

We can find a $\delta_1 > 0$ such that for all $B \in \boldsymbol{A}$ with $\mu(B) \leqslant \delta_1$ we have

$$\int_B \varphi(|\lambda_1 f_1|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+2}}. \tag{4}$$

Now we continue this construction inductively for all $j \in \{2, \ldots, N\}$. We set $\eta_j := \min\{\delta_1, \ldots, \delta_{j-1}\}$ and choose λ_j so that according to (2) we have for given ξ_0, η_j

$$\mu\{t: \varphi(|\lambda_j f(t)|) \geqslant \xi_0\} \leqslant \eta_j \tag{2*}$$

for all $f \in M$. (1) implies that there exists $f_j \in M$ with $\int_X \varphi(|\lambda_j f_j|)\,d\mu \geqslant \varepsilon_0$. Now we define

$$A_j := \{t: \varphi(|\lambda_j f_j(t)|) \geqslant \xi_0\}. \tag{3*}$$

We remark that this implies

$$\int_{A_j} \varphi(|\lambda_j f_j|)\,d\mu \geqslant \tfrac{7}{8}\varepsilon_0. \tag{5}$$

Then we choose real numbers $\delta_j > 0$, so that for all $B \in \boldsymbol{A}$ with $\mu(B) < \delta_j$ we have

$$\int_B \varphi(|\lambda_j f_j|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+2}}. \tag{4*}$$

Obviously we can suppose that $\sum_{j=1}^N \lambda_j \leqslant 1$.

Now we define $B_j := A_j \setminus \big(\bigcup_{i=j+1}^N A_i\big)$ for $j \in \{1, \ldots, N\}$. Setting $x := \sum_{i=1}^N \lambda_i f_i$, $y := \sum_{\substack{i=1\\ i\neq j}}^N \lambda_i f_i$, (iii) implies

$$\int_{B_j} \varphi\Big(\Big|\sum_{i=1}^N \lambda_i f_i\Big|\Big)\,d\mu \geqslant \frac{1}{k}\int_{B_j} \varphi(|\lambda_j f_j|)\,d\mu - \int_{B_j} \varphi\Big(\Big|\sum_{\substack{i=1\\ i\neq j}}^N \lambda_i f_i\Big|\Big)\,d\mu. \tag{6}$$

From the choice of η_i we have $\eta_i \leqslant \delta_j$ and $\mu(A_i) \leqslant \eta_i$ for all $i \in \{j+1, \ldots, N\}$, so by (4*) we get

$$\sum_{i=j+1}^{N}\int_{A_i}\varphi(|\lambda_j f_j|)\,d\mu \leqslant (N-j)\frac{\varepsilon_0}{8Nk^{N+2}} \leqslant \frac{\varepsilon_0}{8k}.$$

Hence by (5)

$$\text{(7)}\qquad \frac{1}{k}\int_{B_j}\varphi(|\lambda_j f_j|)\,d\mu = \frac{1}{k}\int_{A_j}\varphi(|\lambda_j f_j|)\,d\mu - \frac{1}{k}\int_{\bigcup_{i=j+1}^{N} A_i}\varphi(|\lambda_j f_j|)\,d\mu$$

$$\geqslant \frac{1}{k}\int_{A_j}\varphi(|\lambda_j f_j|)\,d\mu - \sum_{i=j+1}^{N}\frac{1}{k}\int_{A_i}\varphi(|\lambda_j f_j|)\,d\mu$$

$$\geqslant \frac{7\varepsilon_0}{8k} - \frac{\varepsilon_0}{8k} = \frac{3\varepsilon_0}{4k}.$$

From (iii) we get

$$\text{(8)}\qquad \int_{B_j}\varphi\Big(\Big|\sum_{\substack{i=1\\ i\neq j}}^{N}\lambda_i f_i\Big|\Big)\,d\mu \leqslant k\int_{B_j}\varphi\Big(\Big|\sum_{i=1}^{j-1}\lambda_i f_i\Big|\Big)\,d\mu + k\int_{B_j}\varphi\Big(\Big|\sum_{i=j+1}^{N}\lambda_i f_i\Big|\Big)\,d\mu.$$

Applying (iii) $j-2$ times for $j\geqslant 2$, we get

$$\text{(9)}\qquad k\int_{B_j}\varphi\Big(\Big|\sum_{i=1}^{j-1}\lambda_i f_i\Big|\Big)\,d\mu \leqslant \sum_{i=1}^{j-1}k^{i+1}\int_{B_j}\varphi(|\lambda_i f_i|)\,d\mu$$

$$\leqslant (j-1)\frac{\varepsilon_0}{8Nk} \leqslant \frac{\varepsilon_0}{8k},$$

since by the choice of η_j we have $\mu(B_j)\leqslant \delta_i$ and therefore

$$\int_{B_j}\varphi(|\lambda_i f_i|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+2}} \qquad \text{for all } i\in\{1,\ldots,j-1\}.$$

Similarly, by the definition of B_j, ξ_0 and by (3*)

$$\text{(10)}\qquad k\int_{B_j}\varphi\Big(\Big|\sum_{i=j+1}^{N}\lambda_i f_i\Big|\Big)\,d\mu \leqslant \sum_{i=j+1}^{N}k^{i+1}\int_{B_j}\varphi(|\lambda_i f_i|)\,d\mu$$

$$\leqslant \sum_{i=j+1}^{N}k^{i+1}\int_{X\setminus A_i}\varphi(|\lambda_i f_i|)\,d\mu \leqslant (N-j)k^{N+1}\frac{\varepsilon_0}{8Nk^{N+2}} \leqslant \frac{\varepsilon_0}{8k}.$$

So altogether we get by (6)–(10) the following estimation:

$$\int_X\varphi\Big(\Big|\sum_{i=1}^{N}\lambda_i f_i\Big|\Big)\,d\mu \geqslant \sum_{j=1}^{N}\int_{B_j}\varphi\Big(\Big|\sum_{i=1}^{N}\lambda_i f_i\Big|\Big)\,d\mu$$

$$\geqslant \sum_{j=1}^{N}\Big(\frac{1}{k}\int_{B_j}\varphi(|\lambda_j f_j|)\,d\mu - \int_{B_j}\varphi\Big(\Big|\sum_{\substack{i=1\\ i\neq j}}^{N}\lambda_i f_i\Big|\Big)\,d\mu\Big)$$

$$\geqslant \sum_{j=1}^{N}\Big(\frac{3\varepsilon_0}{4k} - \frac{\varepsilon_0}{4k}\Big) = \frac{N\varepsilon_0}{2k} > 1.$$

In view of (ii) this is a contradiction to our assumption.

(B) Let $(X, \boldsymbol{A}, \mu)$ be an arbitrary positive measure space.

By part (A) for all $E\in\boldsymbol{A}$ with $\mu(E)\leqslant\infty$,

$$\text{(11)}\qquad \{f\chi_E\colon f\in M\}\ \text{is bounded.}$$

As we assume that M is unbounded, there is an $\varepsilon_0>0$ with

$$\text{(12)}\qquad \{\lambda f\colon f\in M\}\not\subset N_\varphi(\varepsilon_0)$$

for all $\lambda>0$.

Choose $N\in\boldsymbol{N}$ so large that $N\varepsilon_0/2k>1$. Set $\lambda_1 := \frac{1}{2}$ and take $f_1\in M$ with $\int_X\varphi(|\lambda_1 f_1|)\,d\mu\geqslant\varepsilon_0$. Then there is $A_1\in\boldsymbol{A}$, $\mu(A_1)<\infty$, with

$$\int_{X\setminus A_1}\varphi(|\lambda_1 f_1|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+1}}.$$

From (11) we get that there is $\lambda_2\leqslant 1/2^2$ with

$$\int_{A_1}\varphi(|\lambda_2 f|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+1}} \qquad \text{for all } f\in M.$$

On the other hand, (12) implies the existence of $f_2\in M$ with $\int_X\varphi(|\lambda_2 f_2|)\,d\mu\geqslant\varepsilon_0$ There is $A_2'\in\boldsymbol{A}$ with $\mu(A_2')<\infty$ and

$$\int_{X\setminus A_2'}\varphi(|\lambda_2 f_2|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+1}}.$$

For $A_2 := A_2'\setminus A_1$ we have then

$$\int_{X\setminus A_2}\varphi(|\lambda_2 f_2|)\,d\mu \leqslant \frac{\varepsilon_0}{4Nk^{N+1}}.$$

For $i = 3, \ldots, N$ we proceed now inductively. We can choose $\lambda_i \leqslant 1/2^i$ such that

$$\int\limits_{\bigcup_{j=1}^{i-1} A_j} \varphi(|\lambda_i f|)\,d\mu \leqslant \frac{\varepsilon_0}{8Nk^{N+1}} \quad \text{for all } f \in M,$$

choose $f_i \in M$ with $\int\limits_X \varphi(|\lambda_i f_i|)\,d\mu \geqslant \varepsilon_0$ and $A_i \subset X \setminus \left(\bigcup_{j=1}^{i-1} A_j\right)$ with $\mu(A_i) < \infty$ and

$$\int\limits_{X \setminus A_i} \varphi(|\lambda_i f_i|)\,d\mu \leqslant \frac{\varepsilon_0}{4Nk^{N+1}}.$$

So we have for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$

$$\int\limits_{A_j} \varphi(|\lambda_j f_j|)\,d\mu \geqslant \frac{3\varepsilon_0}{4} \quad \text{and} \quad \int\limits_{A_j} \varphi(|\lambda_i f_i|)\,d\mu \leqslant \frac{\varepsilon_0}{4Nk^{N+1}}.$$

With this and inequality (iii) we get the following estimation:

$$\begin{aligned}
\int\limits_X \varphi\left(\left|\sum_{i=1}^N \lambda_i f_i\right|\right) d\mu &\geqslant \sum_{j=1}^N \int\limits_{A_j} \varphi\left(\left|\sum_{i=1}^N \lambda_i f_i\right|\right) d\mu \\
&\geqslant \sum_{j=1}^N \left(\frac{1}{k} \int\limits_{A_j} \varphi(|\lambda_j f_j|)\,d\mu - \int\limits_{A_j} \varphi\left(\left|\sum_{\substack{i=1\\ i\neq j}}^N \lambda_i f_i\right|\right) d\mu\right) \\
&\geqslant \sum_{j=1}^N \left(\frac{1}{k} \int\limits_{A_j} \varphi(|\lambda_j f_j|)\,d\mu - \sum_{\substack{i=1\\ i\neq j}}^N k^i \int\limits_{A_j} \varphi(|\lambda_i f_i|)\,d\mu\right) \\
&\geqslant \sum_{j=1}^N \left(\frac{3\varepsilon_0}{4k} - N\frac{\varepsilon_0}{4Nk}\right) \geqslant \frac{N\varepsilon_0}{2k} > 1,
\end{aligned}$$

which as in part (A) is a contradiction to the assumption.

In order to show the boundedness of the range of a vector measure, we need the following

PROPOSITION (Labuda). *Let R be a ring of sets, $m\colon R \to L_\varphi(X, \boldsymbol{A}, \mu)$ an additive set function. If $m(R)$ is normbounded, then* $\operatorname{co}\big(m(R)\big)$ *is normbounded.*

Proof. For each $(x_i)_{i=1}^n \subset L_\varphi(X, \boldsymbol{A}, \mu)$ and each $(\lambda_i)_{i=1}^n$ with $|\lambda_i| \leqslant 1$ we have

$$\left\|\sum_{i=1}^n \lambda_i x_i\right\| \leqslant 8 \max\left\{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\| : \varepsilon_i \in \{1, -1\}\right\}.$$

This inequality is proved by I. Labuda in [4] and by P. Turpin in [10].

Every convex combination $\sum_{i=1}^n \delta_i m(E_i)$, $\sum_{i=1}^n \delta_i = 1$, may be represented as a simple function $\sum_{i=1}^n \lambda_i m(A_i)$, where $|\lambda_i| \leqslant 1$ and the sets $A_i \in R$ are pairwise disjoint. Hence if $m(R)$ is bounded by a constant C, the inequality above implies that there are some $(\varepsilon_i)_{i=1}^n$, $\varepsilon_i = \pm 1$, with

$$\begin{aligned}
\left\|\sum_{i=1}^n \delta_i m(E_i)\right\| = \left\|\sum_{i=1}^n \lambda_i m(A_i)\right\| &\leqslant 8\left\|\sum_{i=1}^n \varepsilon_i m(A_i)\right\| \\
&= 8\,\|m(F) - m(G)\| \leqslant 16C,
\end{aligned}$$

where F is the union of all A_i with $\varepsilon_i = 1$ and G the union of all A_i with $\varepsilon_i = -1$.

As the range of a vector measure m is normbounded in $L_\varphi(X, \boldsymbol{A}, \mu)$ [1], the following theorem follows immediately from Theorem 1 and the Proposition.

THEOREM 2. *The range of a vector measure $m\colon \Sigma \to L_\varphi(X, \boldsymbol{A}, \mu)$ is bounded.*

Using the terminology of the integration theory, which P. Turpin developed in ([8], Chap. VII), we get the following

COROLLARY. *Let $m\colon \Sigma \to L_\varphi(X, \boldsymbol{A}, \mu)$ be a vector measure. Then every scalar-valued, bounded measurable function is integrable.*

The proof follows immediately from Theorem 2 and the fact that unconditional convergence and bounded multiplier convergence of series are equivalent in $L_\varphi(X, \boldsymbol{A}, \mu)$ ([4], [10]).

4. If one considers a vector measure m with bounded total variation and values in a Banach space X, which is either reflexive or a separable dual space, then the range of m is precompact in the norm topology of X and moreover, if m is nonatomic, the closure of the range of m is compact and convex (cf. [12]). We will show now that vector measures into the sequence spaces l^p $(0 < p < 1)$ have still the same property, although we have the following

LEMMA. *The total variation of a non-trivial vector measure $m\colon \Sigma \to l^p$ $(0 < p < 1)$ is unbounded, whenever m is not purely atomic.*

Proof. By assumption there is a set $A \in \Sigma$ with $m(A) \neq 0$, and A contains no atom of m. We can suppose that we have $m(A) = (x_i)$ with $x_1 = 1$. Now we consider the scalar-valued measure $\lambda_1 \circ m\colon \Sigma \to \boldsymbol{R}$, where $\lambda_1\colon l^p \to \boldsymbol{R}$ is defined by $\lambda_2\big((x_i)\big) := x_1$. As $\lambda_1 \circ m$ is nonatomic, there exists a partition of A with $\lambda_1 \circ m(A_i) = 1/n$ for all $1 \leqslant i \leqslant n$. From the definition of the p-norm in l^p we get $v(m, A) \geqslant \sum_{i=1}^n \|m(A_i)\| \geqslant \sum_{i=1}^n |\lambda_1 \circ m(A_i)|^p = n/n^p = n^{1-p}$, which proves the lemma.

We need the following notations. By $\bar{A}^p$ ($\bar{A}^1$, $\bar{A}^w$) we denote the closure of a set A in the p-norm topology of l^p (in the norm topology of l^1, in the weak topology of l^1, resp.).

THEOREM 3. *The range of a vector measure $m\colon \Sigma \to l^p$ $(0<p<1)$ is relatively compact. Moreover, if m is nonatomic, the closure of the range is compact and convex.*

Proof. We first consider the case where m is nonatomic. As the inclusion of l^p into l^1 is continuous, we can regard m as a vector measure from Σ into l^1. From [3] it follows that $\overline{m(\Sigma)}^w = \overline{\mathrm{co}\,(m(\Sigma))}^1$ and the range $m(\Sigma)$ is relatively weakly compact. Since weak and strong convergence of sequences coincide in l^1 (cf. [2], Cor. IV. 8.14.), $\overline{m(\Sigma)}^w$ is compact in the norm topology, and therefore we have $\overline{m(\Sigma)}^1 = \overline{m(\Sigma)}^w$.

The range of a vector measure is bounded in l^p [1] and therefore $\overline{m(\Sigma)}^1$ is still contained and bounded in l^p. Since $\overline{m(\Sigma)}^1$ is a closed, bounded, convex set in l^p, it follows from [7] that $\overline{m(\Sigma)}^1$ is compact in l^p. As the inclusion of l^p into l^1 is continuous, we have $\overline{m(\Sigma)}^p = \overline{m(\Sigma)}^1$, which proves the theorem if m is nonatomic.

As the range of a vector measure is the sum of the ranges of a nonatomic and a purely atomic vector measure, and the latter is compact, the assertion of the theorem is proved.

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