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Almost everywhere convergence of Walsh Fourier series of $\mathscr{H}^1$-functions

by

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Abstract. Let $S_n f$ denote the nth partial sum of the Walsh Fourier series of a function $f \in L^1(0, 1)$. We say that f is in $\mathscr{H}^1$ if $\sqrt{\sum_{n=1}^{\infty} (S_{2^n} f - S_{2^{n-1}} f)^2}$ is in L^1. This corresponds to the definition of $\mathscr{H}^1$ given by Garsia [6] for martingales. We prove that for $f \in \mathscr{H}^1$ and $\{n_k\}_{k\geqslant 0}$ a lacunary sequence of positive integers, $S_{n_k} f$ converges a.e.; whereas, there exists a function in $\mathscr{H}^1$ whose full sequence of partial sums diverges. The space $\mathscr{H}^1$, our results, and their proofs are all analogous to the classical trigonometric case.

Introduction. For $f \in L^1(0, 1)$, let $S_n f$ denote the nth partial sum of its Walsh Fourier series. We say that f is in $\mathscr{H}^1$ if the corresponding square function $Sf = \sqrt{\sum_{n=1}^{\infty} (S_{2^n} f - S_{2^{n-1}} f)^2}$ is in L^1. Note that $\{S_{2^n} f\}_{n\geqslant 0}$ is a martingale, and so our definition is a special case of the definition of $\mathscr{H}^1$ given by Garsia [6] for martingales.

In this paper we prove the following theorems.

THEOREM 1. *Let $f \in \mathscr{H}^1$ and $\{n_k\}_{k\geqslant 0}$ be a lacunary sequence of positive integers. Then $S_{n_k} f(x) \to f(x)$ $(k \to \infty)$ for a.e. $x \in (0, 1)$.*

THEOREM 2. *There exists an $f \in \mathscr{H}^1$ such that its full sequence of partial sums, $S_n f$, diverges everywhere.*

The space $\mathscr{H}^1$, our results, and their proofs are all analogous to the classical trigonometric case.

We recall that in the trigonometric case, a function f belongs to H^1 if and only if f and its conjugate $\tilde{f}$ are both in L^1. According to Fefferman and Stein [4], this is equivalent to the Littlewood–Paley function $g(f)$ being in L^1. The square function Sf is analogous to $g(f)$. (See Littlewood and Paley [7].) Moreover, let φ be the characteristic function of the unit interval, sufficiently smoothed out. Define

$$\varphi_t(x) = \frac{1}{t}\varphi\left(\frac{x}{t}\right) \quad \text{and} \quad f^+ = \sup_{t>0} |\varphi_t * f|.$$

Fefferman and Stein [4] have shown that f is in H^1 if and only if f^+ is in L^1. $\varphi_t * f$ is nearly an average of f. In the Walsh system, $S_{2^n}f$ is exactly an average of f. According to Davis [2], $f \in \mathscr{H}^1$ if and only if

$$f^* = \sup_{n\geq 0} |S_{2^n}f| \text{ is in } L^1.$$

The trigonometric analogue of Theorem 1 is known. (See Zygmund [10], Vol. II, pp. 234–239.) The Littlewood–Paley function $g(f)$ plays an important role in the proof. The underlying idea of our proof is similar. We use the square function Sf in place of $g(f)$. To handle Sf, we follow arguments of Fefferman and Stein [3]. Our proof is not as complicated as in the trigonometric case because of the special nature of the Walsh functions.

An example of a function in H^1 with a.e. divergent Fourier series can be obtained by modifying Kolmogorov's L^1 example. His example was constructed as a sum of non-overlapping polynomials φ_k. By multiplying each φ_k by a suitable exponential $e^{i\mu_k x}$, we obtain a function $g \in H^1$ with a.e. divergent Fourier series (see Zygmund [10], Vol. 1). We follow the same line of argument. For the Walsh system, Moon [8] modified a construction of Stein [9] and gave an example of a integrable function h whose Walsh Fourier series diverges everywhere. This function is a sum of non-overlapping Walsh-polynomials, ψ_j, as in Kolmogorov's example. Our example is also obtained by multiplying each ψ_j by a Walsh function.

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Definitions and properties. We recall some definitions and properties of Walsh functions.

ω will denote a dyadic subinterval of $(0,1)$, and we will write $|E|$ for Lebesgue measure of a set E.

Let r_n be the nth Rademacher function. For any nonnegative integer n, with $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$, $\varepsilon_j = 0$ or 1, the nth *Walsh function* is defined by

$$w_n = \prod_{j=0}^{\infty} (r_j)^{\varepsilon_j}.$$

For $n < 2^N$, w_n is constant on intervals ω, with $|\omega| \leq 2^{-N}$.

If $x = \sum_{i=0}^{\infty} \xi_i 2^{-i-1}$ and $t = \sum_{i=0}^{\infty} \eta_i 2^{-i-1}$, $\xi_i, \eta_i = 0$ or 1, let $x+t = \sum_{i=0}^{\infty} |\xi_i - \eta_i| 2^{-i-1}$. Then $w_n(x+t) = w_n(x) \cdot w_n(t)$.

For $f \in L^1(0,1)$, we have

$$(S_n f)(x) = \sum_{k=0}^{n-1} c_k(f) w_k(x) = \int_0^1 f(t) D_n(x+t)\, dt,$$

where $c_k(f) = \int_0^1 f(t) w_k(t)\, dt$ and $D_n = \sum_{k=0}^{n-1} w_k$ is the nth *Dirichlet kernel*. We will need to use the fact that,

$$\limsup_{n\to\infty} \frac{\int_0^1 |D_n(t)|\, dt}{\log n} > 0 \quad (\text{see Fine [5]}).$$

Let

$$\delta_j^*(t) = 2^j \{\chi_{(0,2^{-j-1})}(t) - \chi_{(2^{-j-1},2^{-j})}(t)\}, \quad j = 0, 1, 2, \ldots$$

The *modified Dirichlet kernel* is defined as

$$D_n^*(t) = \sum_{j=0}^{\infty} \varepsilon_j \delta_j^*(t), \quad \text{where} \quad n = \sum_{j=0}^{\infty} \varepsilon_j 2^j, \ \varepsilon_j = 0 \text{ or } 1.$$

It can be shown that $D_n^*(t) = w_n(t) D_n(t)$. Let

$$(S_n^* f)(x) = \int_0^1 f(t) D_n^*(x+t)\, dt.$$

Note that,

$$(S_n^* f)(x) = w_n(x)(S_n w_n f)(x).$$

By Bessel's inequality, we have

$$\|S_n^* f\|_2 \leq \|f\|_2.$$

For $f \in L^1(0,1)$, set

$$f^* = \sup_{n\geq 0} |S_{2^n} f| \quad \text{and} \quad Sf = \sqrt{\sum_{n=0}^{\infty} [S_{2^{n+1}} f - S_{2^n} f]^2}.$$

Davis [2] has shown that there exist positive constants c and C such that

$$c\|Sf\|_1 \leq \|f^*\|_1 \leq C\|Sf\|_1.$$

We say that $f \in \mathscr{H}^1$ if $Sf \in L^1$, or equivalently if $f^* \in L^1$, and we write

$$\|f\|_{\mathscr{H}^1} = \|Sf\|_1.$$

Proof of theorems.

Proof of Theorem 1. We show that for any lacunary sequence $\{n_k\}_{k\geq 0}$

$$|\{x \in (0,1) : \sup_{k\geq 0} |S_{n_k} f(x)| > y\}| \leq \frac{c}{y} \|f\|_{\mathscr{H}^1} \tag{1}$$

for all $y > 0$, $f \in \mathscr{H}^1$. Since $S_{2^n} f$ converges to f in $\mathscr{H}^1$ norm, Theorem 1 will follow from (1) by the usual density argument.

It is well known that every lacunary sequence, $\{n_k\}_{k\geqslant 0}$ can be split into a finite number of lacunary subsequences $\{n_k^j\}_{k\geqslant 0}$ with $n_{k+1}^j \geqslant 2n_k^j$, and hence, we may assume $2^k \leqslant n_k < 2^{k+1}$. Let $d_k = S_{2^{k+1}}f - S_{2^k}f$. Then $S_{n_k}f = S_{2^k}f + S_{n_k}d_k$. Since

$$|\{x\in(0,1):\ \sup_{k\geqslant 0}|S_{2^k}f(x)|>y\}| \leqslant \frac{1}{y}\int_0^1 f^*(x)\,dx \leqslant \frac{c}{y}\|f\|_{\mathscr{H}^1},$$

it is sufficient to prove

(2) $$|\{x\in(0,1):\ \sup_{k\geqslant 0}|S_{n_k}d_k(x)|>y\}| \leqslant \frac{c}{y}\|f\|_{\mathscr{H}^1}.$$

Applying the Calderón–Zygmund lemma [1] to $(\sum_{k=0}^{\infty}|d_k|^2)^{1/2}$, we obtain a collection of disjoint intervals, $\{\omega_j\}_{j\geqslant 0}$, such that

(a) $$\Big(\sum_{k=0}^{\infty}|d_k(x)|^2\Big)^{1/2} \leqslant y \quad \text{for a.e. } x\notin\Omega \equiv \bigcup_{j\geqslant 0}\omega_j,$$

(b) $$y < \frac{1}{|\omega_j|}\int_{\omega_j}\Big(\sum_{k=0}^{\infty}|d_k(x)|^2\Big)^{1/2}dx \leqslant 2y \quad \text{for} \quad j\geqslant 0,$$

and

(c) $$\sum_{j=0}^{\infty}|\omega_j| \leqslant \frac{1}{y}\int_0^1\Big(\sum_{k=0}^{\infty}|d_k(x)|^2\Big)^{1/2}dx.$$

For each $k\geqslant 0$, we write $d_k = g_k + b_k$, where

$$g_k(x) = \begin{cases} w_{n_k}(x)\dfrac{1}{|\omega_j|}\displaystyle\int_{\omega_j} w_{n_k}(t)\,d_k(t)\,dt & \text{if} \quad x\in\omega_j,\\ d_k(x) & \text{if} \quad x\notin\Omega,\end{cases}$$

and $b_k = d_k - g_k$.

We will prove

(3) $$\Big\|\Big(\sum_{k=0}^{\infty}|S_{n_k}^*(w_{n_k}g_k)|^2\Big)^{1/2}\Big\|_2^2 \leqslant cy\Big\|\Big(\sum_{k=0}^{\infty}|d_k|^2\Big)^{1/2}\Big\|_1.$$

To deduce (2) from (3) we note that $\int_{\omega_j} w_{n_k}(t)\,b_k(t)\,dt = 0$, and if $x\notin\omega_j$, $D_n^*(x+t)$ is constant as t varies over ω_j. Hence, for $x\notin\Omega$,

$$S_{n_k}d_k(x) = w_{n_k}(x)S_{n_k}^*(w_{n_k}d_k)(x) = w_{n_k}(x)S_{n_k}^*(w_{n_k}g_k)(x).$$

By inequalities (c) and (3) we have

$$\begin{aligned}
&|\{x\in(0,1):\ \sup_{k\geqslant 0}|S_{n_k}d_k(x)|>y\}|\\
&\leqslant |\Omega| + \big|\{x\notin\Omega:\ \sup_{k\geqslant 0}|S_{n_k}^*(w_{n_k}g_k)(x)|>y\}\big|\\
&\leqslant \frac{1}{y}\int_0^1\Big(\sum_{k=0}^{\infty}|d_k(x)|^2\Big)^{1/2}dx + \Big|\Big\{x\in(0,1):\ \Big(\sum_{k=0}^{\infty}|S_{n_k}^*(w_{n_k}g_k)(x)|^2\Big)^{1/2}>y\Big\}\Big|\\
&\leqslant \frac{1}{y}\Big\|\Big(\sum_{k=0}^{\infty}|d_k|^2\Big)^{1/2}\Big\|_1 + \frac{1}{y^2}\Big\|\Big(\sum_{k=0}^{\infty}|S_{n_k}^*(w_{n_k}g_k)|^2\Big)^{1/2}\Big\|_2^2\\
&\leqslant \frac{c}{y}\Big\|\Big(\sum_{k=0}^{\infty}|d_k|^2\Big)^{1/2}\Big\|_1 = \frac{c}{y}\|f\|_{\mathscr{H}^1}.
\end{aligned}$$

To prove (3), we proceed as follows.

For $x\in\omega_j$,

$$\begin{aligned}
\Big(\sum_{k=0}^{\infty}|g_k(x)|^2\Big)^{1/2} &= \Big(\sum_{k=0}^{\infty}\Big|\frac{1}{|\omega_j|}\int_{\omega_j}d_k(t)\,w_{n_k}(t)\,dt\Big|^2\Big)^{1/2}\\
&\leqslant \frac{1}{|\omega_j|}\int_{\omega_j}\Big(\sum_{k=0}^{\infty}|d_k(t)|^2\Big)^{1/2}dt \leqslant 2y,
\end{aligned}$$

by the vector-valued form of Minkowski's inequality and (b). For a.e. $x\notin\Omega$,

$$\Big(\sum_{k=0}^{\infty}|g_k(x)|^2\Big)^{1/2} = \Big(\sum_{k=0}^{\infty}|d_k(x)^2|\Big)^{1/2} \leqslant y,$$

by inequality (a). Therefore,

$$\begin{aligned}
\Big\|\Big(\sum_{k=0}^{\infty}|g_k|^2\Big)^{1/2}\Big\|_2^2 &= \sum_{j\geqslant 0}\int_{\omega_j}\Big[\Big(\sum_{k=0}^{\infty}|g_k(x)|^2\Big)^{1/2}\Big]^2dx + \int_{(0,1)-\Omega}\Big[\Big(\sum_{k=0}^{\infty}|g_k(x)|^2\Big)^{1/2}\Big]^2dx\\
&\leqslant (2y)^2\sum_{j\geqslant 0}|\omega_j| + y\int_0^1\Big(\sum_{k=0}^{\infty}|d_k(x)|^2\Big)^{1/2}dx \leqslant 5y\Big\|\Big(\sum_{k=0}^{\infty}|d_k|^2\Big)^{1/2}\Big\|_1.
\end{aligned}$$

By Bessel's inequality

$$\|S_{n_k}^*(w_{n_k}g_k)\|_2^2 \leqslant \|g_k\|_2^2,$$

and so

$$\Big\|\Big(\sum_{k=0}^{\infty}|S_{n_k}^*(w_{n_k}g_k)|^2\Big)^{1/2}\Big\|_2^2 \leqslant \Big\|\Big(\sum_{k=0}^{\infty}|g_k|^2\Big)^{1/2}\Big\|_2^2 \leqslant 5y\Big\|\Big(\sum_{k=0}^{\infty}|d_k|^2\Big)^{1/2}\Big\|_1.$$

This completes the proof of (3), and hence of Theorem 1.

Proof of Theorem 2. Let $2^{N-1} \leqslant n < 2^N$, $N \geqslant 1$, and $\omega_{N,j} = \big(j2^{-N}, (j+1)2^{-N}\big)$, $0 \leqslant j < 2^N$. Note that, $D_n(x+t)$ is constant as x and t range over $\omega_{N,k}$ and $\omega_{N,j}$, respectively. For $0 \leqslant k < 2^N$, define

$$E_{N,k} = \{t \in (0,1):\ w_{2^N+k}(t) = \operatorname{sgn} D_n(x+t),\ \forall x \in \omega_{N,k}\}.$$

Then $|E_{N,k}| = \frac{1}{2}$ and $E_{N,k}$ is equidistributed over the intervals $\omega_{N,j}$, $0 \leqslant j < 2^N$.

Let

$$E_N = \bigcap_{k=0}^{2^N-1} E_{N,k} \quad \text{and} \quad \omega_j = E_N \cap \omega_{N,j}.$$

Clearly, $|E_N| = 2^{-2^N}$, and ω_j is an interval of length 2^{-2^N-N}.

We denote $\chi_{E_N}(t) \cdot w_{2^{N+2^N}}(t)$ by $h_N(t)$ and $2^{N+2^N} + 2^{N+k}$ by $l(N,k)$, $0 \leqslant k < 2^N$. Then

(I) $c_m(h_N) = 0$ *unless* $2^{N+2^N} \leqslant m < 2^{N+2^N+1}$ and

(II) $|(S_{l(N,k)+n} h_N)(x) - (S_{l(N,k)} h_N)(x)| = |E_N| \cdot \int_0^1 |D_n(x+t)|\,dt$, $\forall x \in \omega_{N,k}$.

(I) is clear.

To prove (II), first observe that, for $0 \leqslant k < 2^N$,

$$D_{l(N,k)+n}(t) - D_{l(N,k)}(t) = w_{l(N,k)}(t) D_n(t).$$

Hence, for $x \in \omega_{N,k}$,

$$\begin{aligned} |(S_{l(N,k)+n} h_N)(x) - (S_{l(N,k)} h_N)(x)| &= \Big| \sum_{j=0}^{2^N-1} \int_{\omega_j} w_{2^N+k}(t) D_n(x+t)\,dt \Big| \\ &= \sum_{j=0}^{2^N-1} \int_{\omega_j} |D_n(x+t)|\,dt \\ &= |E_N| \cdot \int_0^1 |D_n(x+t)|\,dt. \end{aligned}$$

The last equality holds because

$$|\omega_j| = 2^{-2^N-N} = |E_N| \cdot |\omega_{N,j}|.$$

Choosing n such that

$$\int_0^1 |D_n(x+t)|\,dt = \int_0^1 |D_n(t)|\,dt \geqslant K \cdot N, \qquad K > 0,$$

we obtain

$$|(S_{l(N,k)+n} h_N)(x) - (S_{l(N,k)} h_N)(x)| \geqslant K \cdot N \cdot |E_N|, \quad \text{for} \quad x \in \omega_{N,k}.$$

Let $\{N_i\}_{i \geqslant 0}$ be any increasing sequence of positive integers such that $\sum_{i=0}^{\infty} N_i^{-1} < \infty$, and set

$$f(x) = \sum_{i=0}^{\infty} N_i^{-1} 2^{2^{N_i}} h_{N_i}(x).$$

Then

$$\int_0^1 |f(x)|\,dx \leqslant \sum_{i=0}^{\infty} N_i^{-1} < \infty, \quad \text{so} \quad f \in L^1.$$

$$c_m(f) = \begin{cases} N_i^{-1} 2^{2^{N_i}} c_m(h_{N_i}) & \text{if} \quad 2^{N_i+2^{N_i}} \leqslant m < 2^{N_i+2^{N_i}+1}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence, for every $x \in (0,1)$ and every i, there is a $k_i = k_i(x)$ such that $x \in \omega_{N_i,k_i}$ and

$$\begin{aligned} &|(S_{l(N_i,k_i)+n_i} f)(x) - (S_{l(N_i,k_i)} f)(x)| \\ &\qquad = N_i^{-1} 2^{2^{N_i}} |(S_{l(N_i,k_i)+n_i} h_{N_i})(x) - (S_{l(N_i,k_i)} h_{N_i})(x)| \geqslant K > 0. \end{aligned}$$

Thus $(S_n f)(x)$ cannot converge. Finally, to show $f \in \mathscr{H}^1$, we note that

$$(S_{2^{p+1}} f)(t) - (S_{2^p} f)(t) = \begin{cases} (S_{2^{N_i+2^{N_i}+1}} f)(t) - (S_{2^{N_i+2^{N_i}}} f)(t) & \text{if} \quad p = 2^{N_i+2^{N_i}} \\ 0 & \text{otherwise}, \end{cases}$$

and

$$(S_{2^{N_i+2^{N_i}+1}} f)(t) - (S_{2^{N_i+2^{N_i}}} f)(t) = N_i^{-1} \cdot 2^{2^{N_i}} \cdot h_{N_i}(t) \quad \text{a.e.}$$

Therefore,

$$\begin{aligned} \|f\|_{\mathscr{H}^1} &= \int_0^1 (Sf)(t)\,dt = \int_0^1 \sqrt{\sum_{p=0}^{\infty} [(S_{2^{p+1}} f)(t) - (S_{2^p} f)(t)]^2}\,dt \\ &= \int_0^1 \sqrt{\sum_{i=0}^{\infty} [N_i^{-1} \cdot 2^{2^{N_i}} h_{N_i}(t)]^2}\,dt \quad \int_0^1 \sum_{i=0}^{\infty} N_i^{-1} \cdot 2^{2^{N_i}} \cdot \chi_{E_{N_i}}(t)\,dt \\ &= \sum_{i=0}^{\infty} N_i^{-1} < \infty. \end{aligned}$$

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On parabolic Marcinkiewicz integrals

by

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Abstract. Throughout this paper it is studied the existence of parabolic Marcinkiewicz integrals of the type:

$$J_\lambda(f)(x) = \int_{\mathbf{R}^n} \frac{\delta^\lambda(y)}{\varrho(|x-y|)^{\lambda+|a|}} f(y)\varphi(x-y)\,dy$$

where $\lambda > 0$, $\varrho(|x|)$ stands for the parabolic distance from x to the origin, $|a| = \sum_{i=1}^n a_i$, where $a_i > 1$, f belongs to $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, and φ is a function satisfying:

(i) $\varepsilon^{-|a|} \int_{\varrho(|x|)<\varepsilon} |\varphi(x)|\,dx \leqslant M$;

(ii) $\int_{\varrho(|x|)>4\varrho(|h|)} \frac{|\varphi(x+h)-\varphi(x)|}{\varrho(|x|)^{|a|}}\,dx < C$.

0. Introduction. Let $\varrho(|x|)$ be the parabolic metric in $\mathbf{R}^n$, namely:

$$0.1.\quad \sum_{i=1}^n \left(\frac{x_i}{\varrho^{a_i}}\right)^2 = 1,\ a_i \geqslant 1,\ i = 1, 2, \ldots, n,$$

where $\varrho(|x|)$ is the only positive root of the above equation (see [4]). In [6], E. M. Ostrow and E. M. Stein introduced the folowing type of integrals:

$$0.2.\quad T_\lambda(x) = \int_{\mathbf{R}^n} \frac{f(y)\,\delta^\lambda(y)}{|x-y|^{n+\lambda}+\delta^{n+\lambda}(x)}\varphi(x-y)\,dy$$

where $\lambda > 0$, $f \in L^1(\mathbf{R}^n)$ and $\varepsilon^{-n}\int_{|y|<\varepsilon} |\varphi(y)|\,dy < C$. Here, $\delta(x)$ stands for the Euclidean distance from x to a closed subset F of $\mathbf{R}^n$. In the above paper, the authors prove the existence a.e. of the integral 0.2. It has been pointed out by A. Zygmund in [8], that T_λ maps continuously $L^1(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$ leaving as an open question whether if T_λ maps continuously $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ for $p > 1$. The purpose of this paper is to give an answer to that problem when assuming an extra condition on φ, namely:

$$0.3.\quad \int_{|x|>2|h|} \frac{|\varphi(x+h)-\varphi(x)|}{|x|^n}\,dx < C.$$

Here, C does not depend on h.

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