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On parabolic Marcinkiewicz integrals

by

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Abstract. Throughout this paper it is studied the existence of parabolic Marcinkiewicz integrals of the type:

$$J_\lambda(f)(x) = \int_{\mathbf{R}^n} \frac{\delta^\lambda(y)}{\varrho(|x-y|)^{\lambda+|a|}} f(y)\varphi(x-y)\,dy$$

where $\lambda > 0$, $\varrho(|x|)$ stands for the parabolic distance from x to the origin, $|a| = \sum_{i=1}^n a_i$, where $a_i > 1$, f belongs to $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, and φ is a function satisfying:

(i) $\varepsilon^{-|a|} \int_{\varrho(|x|)<\varepsilon} |\varphi(x)|\,dx \leqslant M$;

(ii) $\displaystyle\int_{\varrho(|x|)>4\varrho(|h|)} \frac{|\varphi(x+h)-\varphi(x)|}{\varrho(|x|)^{|a|}}\,dx < C$.

0. Introduction. Let $\varrho(|x|)$ be the parabolic metric in $\mathbf{R}^n$, namely:

0.1. $$\sum_{i=1}^n \left(\frac{x_i}{\varrho^{a_i}}\right)^2 = 1,\quad a_i \geqslant 1,\ i = 1, 2, \ldots, n,$$

where $\varrho(|x|)$ is the only positive root of the above equation (see [4]). In [6], E. M. Ostrow and E. M. Stein introduced the folowing type of integrals:

0.2. $$T_\lambda(x) = \int_{\mathbf{R}^n} \frac{f(y)\,\delta^\lambda(y)}{|x-y|^{n+\lambda}+\delta^{n+\lambda}(x)}\varphi(x-y)\,dy$$

where $\lambda > 0$, $f \in L^1(\mathbf{R}^n)$ and $\varepsilon^{-n} \int_{|y|<\varepsilon} |\varphi(y)|\,dy < C$. Here, $\delta(x)$ stands for the Euclidean distance from x to a closed subset F of $\mathbf{R}^n$. In the above paper, the authors prove the existence a.e. of the integral 0.2. It has been pointed out by A. Zygmund in [8], that T_λ maps continuously $L^1(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$ leaving as an open question whether if T_λ maps continuously $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ for $p > 1$. The purpose of this paper is to give an answer to that problem when assuming an extra condition on φ, namely:

0.3. $$\int_{|x|>2|h|} \frac{|\varphi(x+h)-\varphi(x)|}{|x|^n}\,dx < C.$$

Here, C does not depend on h.

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The nature of condition 0.3 is such that it allows us to solve the same problem in the parabolic case where techniques like the "method of rotation" are not available.

The main tool in the proofs of the results in this paper is a result on theory of differentiation, which in the elliptic case is due to Robert Fefferman [5]. When φ is a homogeneous function of degree 0. (Non-isotropic differentiation.)

Incidentally, the method used here differs from those by R. Fefferman and it is closely related to the classic theory of Singular Integrals.

1. Statement of the results.

1.1. Let F be a closed set in $\boldsymbol{R}^n$; $\delta(x, F)$, or simply $\delta(x)$, will denote the parabolic distance from x to F, that is,

(1.1.1) $$\delta(x) = \inf_{y\in F} \varrho(|x-y|).$$

1.2. *Assumptions on* φ. Throughout this paper, φ is going to be a real valued and measurable function defined on $\boldsymbol{R}^n$ and satisfying:

(1.2.1) $$\int_{\varrho(|x|)<\varepsilon} |\varphi(x)|\,dx < M_0\varepsilon^{-|a|}, \qquad |a| = \sum_1^n a_i,$$

(1.2.2) $$\int_{\varrho(|x|)>4\varrho(|h|)} |\varphi(x+h)-\varphi(x)|\frac{1}{\varrho(|x|)^{|a|}}\,dx < M_1.$$

Here, M_0 does not depend on ε and M_1 does not depend on h.

1.3. Let f be a function in $L^p(\boldsymbol{R}^n)$ or $L^p(\boldsymbol{R}^n - F)$ where F is a fixed closed subset of $\boldsymbol{R}^n$ and consider the operators:

(1.3.1) $$T_\lambda(f)(x) = \int_{\boldsymbol{R}^n} \frac{\delta^\lambda(y)f(y)}{\varrho(|x-y|)^{|a|+\lambda}+\delta(x)^{|a|+\lambda}}\varphi(x-y)\,dy,$$

(1.3.2) $$J_\lambda(f)(x) = \int_{\boldsymbol{R}^n} \frac{\delta^\lambda(y)}{\varrho(|x-y|)^{|a|+\lambda}} f(y)\varphi(x-y)\,dy,$$

where $\lambda > 0$, $\delta(x)$ is the parabolic distance from x to F.

The operators above defined satisfy the following inequalities:

1.4. Theorem A.

(i) $\|T_\lambda(f)\|_p < C_p\|f\|_p$, $\quad 1 \leqslant p < \infty$.

(ii) *If* $|f| \leqslant 1$ *and it is supported on a sphere* S, *we have*

$$\int_S \exp\{\gamma|T_x(f)|\}\,dx < A|S|$$

for small enough γ. *Here,* C_p, γ, *and* A *are independent from* f.

(iii) *If* $f \in L^p(\boldsymbol{R}^n)$ *and* $1 \leqslant p < \infty$, *we have*

$$\int_F |J_\lambda(f)|^p\,dx \leqslant C_p \int_{\boldsymbol{R}^n} |f|^p\,dx.$$

(iv) *If* $|f| \leqslant 1$ *and it is supported on* S *(sphere),*

$$\int_{F\cap S} \exp\{\gamma|J_\lambda(f)|\}\,dx \leqslant A|S|.$$

Here, γ small enough, and C_p, γ, and A independent from f.

2. Auxiliary results on differentiation theory. From now on, C is always going to denote a constant, not necessarily the same at each occurrence.

2.1. Lemma. *Let* $K(x)$ *be a real valued, non-negative measurable function defined on* $\boldsymbol{R}^n$. *Suppose that* $K(x)$ *satisfies the following properties.*

(i) $\displaystyle \operatorname*{Sup}_{B>0} \int_{B<\varrho(x)<2B} K(x)\,dx \leqslant M_0$.

(ii) $\displaystyle \int_{\varrho(|x|)>4\varrho(|h|)} |K(x+h)-K(x)|\,dx < C$.

Call

$$\overset{*}{K}(f)(x) = \operatorname*{Sup}_{\varepsilon>0}\Big|\int_{\varepsilon<\varrho(|y|)<2\varepsilon} K(y)f(x-y)\,dy\Big|.$$

Then $\overset{*}{K}(f)$ *satisfies:*

(a) $\big|E(\overset{*}{K}(f) \gtrdot \lambda)\big| < \dfrac{C_1}{\lambda}\|f\|_1$;

(b) $\|\overset{*}{K}(f)\|_p < C_p\|f\|_p$ *if* $p > 1$;

$C_p < C\dfrac{p}{p-1}$. *Here,* C, C_1, C_p *do not depend on* f.

Proof. The method to be employed here, follows the pattern of the corresponding ones in [1], [2], [4] and [7].

The novelty consists in the fact that we are dealing with a differentiation operator.

Take $f \geqslant 0$ in $L^1(\boldsymbol{R}^n)$ and choose $\lambda > 0$; then it is possible to select an at most denumerable family of rectangles depending on a parameter t such that:

(1) The edges of the rectangles are parallel to the coordinate axes.

(2) The size and the shape of the rectangles R_k are determined by the value that the parameter t takes for R_k, say t_k. The size of the jth edge is given by t^{a_j}, thus, for the R_k rectangle that value is going to be $t_k^{a_j}$. Here, the a_j are the parameters of the parabolic metric ϱ.

(3) The rectangles R_k are non-overlapping, that is:

$$\mathring{R}_l \cap \mathring{R}_k = \emptyset, \quad l \neq k.$$

(4) $f(x) \leqslant \lambda$ a.e. in $\boldsymbol{R}^n - \bigcup_1^\infty R_k$ and

$$\lambda < \frac{1}{|R_k|} \int_{R_k} f\,dt < B\lambda, \quad k = 1, 2, \ldots$$

Consequently, $G_\lambda = \bigcup_1^\infty R_k$ has measure less than

$$\frac{1}{\lambda} \int_{\boldsymbol{R}^n} f\,dt.$$

Here, B is a constant that depends on the a_j-s and the dimension only.

(5) Let $\beta > 0$ be the only positive solution of $\sum_1^n \beta^{-2a_j} = 1$. If we call $S_{\beta t_k}(x_k) = \{y;\ \varrho(|y - x_k|) < \beta t_k\}$, where x_k is the center of R_k, then we have

$$S_{\beta t_k}(x_k) \supset R_k, \quad k = 1, 2, \ldots;$$

on the other hand,

$$|S_{5\beta t_k}(x_k)| < C|R_k|, \quad k = 1, 2, \ldots$$

Here, C depends on n, β and the a_j-s only.

Consequently:

$$\Big|\bigcup_1^\infty S_{5\beta t_k}(x_k)\Big| \leqslant \frac{C}{\lambda} \|f\|_1.$$

For a proof of (1) to (5) see [4].

Consider the following partition for f; $f = f_1 + f_2$, where

$$\begin{aligned} & f_1 = f \quad \text{a.e. in } \boldsymbol{R}^n - G_\lambda, \\ (2.1.1) \qquad & f_1 = 0 \quad \text{if} \quad x \in \bigcup_1^\infty R_k, \\ & f_2 = \sum_1^\infty f(x)\varphi_k(x), \end{aligned}$$

where $\varphi_k(x)$ is the characteristic function of R_k.

Let us take $x \in \boldsymbol{R}^n - \bigcup_1^\infty S_{5\beta t_k}(x_k)$ and consider the integral

$$(2.1.2) \qquad \int\limits_{\varepsilon < \varrho(|x-y|) < 2\varepsilon} K(x-y) f_2(y)\,dy = \int\limits_{\varepsilon < \varrho(|x-y|) < 2\varepsilon} K(x-y) \Big(\sum_{k'} f(y)\varphi_k(y)\Big)\,dy$$

where the sum is extended over those k for which

$$(2.1.3) \qquad R_{k'} \cap \{y;\ \varepsilon < \varrho(|x-y|) < 2\varepsilon\} \neq \emptyset.$$

On account of the fact that $\varrho(|x - y_k|) > 5\beta t_k$, there exists a fixed constant $r > 0$, depending on 5β and the a_j-s only, such that

$$(2.1.4) \qquad R_{k'} \subset \{y;\ r\varepsilon < \varrho(|x-y|) < r^{-1}2\varepsilon\}.$$

Consequently:

$$(2.1.5) \qquad \int\limits_{\varepsilon < \varrho(|x-y|) < 2\varepsilon} K(x-y) f_2(y)\,dy \leqslant \sum_{k'} \int\limits_{r\varepsilon < \varrho(|x-y|) < 2r^{-1}\varepsilon} K(x-y) f(y) \varphi_k(y)\,dy = \sum_{k'} \int_{\boldsymbol{R}^n} K(x-y) f(y)\varphi_k(y)\,dy.$$

Call μ_k to the mean value of f over R_k. The latter integral in (2.1.5) can be written as

$$(2.1.6) \qquad \sum_{k'} \int_{\boldsymbol{R}^n} K(x-y)\big(f(y) - \mu_k\big)\varphi_k(y)\,dy + \sum_{k'} \int_{\boldsymbol{R}^n} K(x-y)\mu_k\varphi_k(y)\,dy.$$

On account of (i) and (4) we have

$$(2.1.7) \qquad \sum_{k'} \int_{\boldsymbol{R}^n} K(x-y)\mu_k\varphi_k(y)\,dy \leqslant B \int\limits_{\varepsilon\gamma < \varrho(|x-y|) < 2\gamma^{-1}\varepsilon} K(x-y)\,dy.$$

On the other hand, on view of the fact that the $\big(f(y) - \mu_k\big)\varphi_k(y)$ have mean value zero we could write

$$\begin{aligned} (2.1.8) \qquad & \Big|\sum_{k'} \int_{\boldsymbol{R}^n} K(x-y)[f(y) - \mu_k]\varphi_k(y)\,dy\Big| \\ & = \Big|\sum_{k'} \int_{\boldsymbol{R}^n} [K(x-y) - K(x-x_k)][f(y) - \mu_k]\varphi_k(y)\,dy \\ & \leqslant \sum_1^\infty \int_{\boldsymbol{R}^n} |K(x-y) - K(x-y_k)|\big(|f(y)| + \mu_k\big)\varphi_k(y)\,dy. \end{aligned}$$

Call $\theta(x)$ to the right-hand member of inequality (2.1.8). On account of part (ii) of the hypothesis we have

$$(2.1.9) \qquad \int\limits_{\boldsymbol{R}^n - \bigcup_1^\infty S_{5\beta t_k}(x_k)} \theta(x)\,dx < C\|f\|_1.$$

On view of (5) and (2.1.9) we have

$$(2.1.10) \qquad \big|E\big(\theta(x) > \lambda\big)\big| \leqslant \frac{C}{\lambda}\|f\|_1.$$

So far, we have proved the following inequality

$$(2.1.11) \qquad \int\limits_{\varepsilon < \varrho(|x-y|) < 2\varepsilon} K(x-y) f_2(y)\,dy < \theta(x) + C\lambda.$$

Since the right-hand member in the above inequality does not depend on ε, we have

$$\overset{*}{K}(f_2) < \theta(x) + C\lambda. \tag{2.1.12}$$

Therefore, if we select $D > 2C$ we have

$$E\big(\overset{*}{K}(f_2) > D\lambda\big) \subset E\left\{\theta(x) > \frac{D}{2}\lambda\right\} \cup E\left\{C\lambda > \frac{D}{2}\lambda\right\}. \tag{2.1.13}$$

Observe that

$$E\left\{x;\ C\lambda > \frac{D}{2}\lambda\right\} = \emptyset. \tag{2.1.14}$$

Consequently:

$$|E\{\overset{*}{K}(f_2) > D\lambda\}| < \left|E\left\{\theta(x) > \frac{D}{2}\lambda\right\}\right|. \tag{2.1.15}$$

On the other hand,

$$\left|E\left\{\theta(x) > \frac{D}{2}\lambda\right\}\right| < \frac{C}{D}2\frac{1}{\lambda}\|f\|_1 + \Big|\bigcup_1^\infty S_{5\beta t_k}(x_k)\Big| < \frac{H}{\lambda}\|f\|_1. \tag{2.1.16}$$

Let us return to f_1. On account of the fact that $f_1 < \lambda$ a.e. and from (i) it follows:

$$\overset{*}{K}(f_1) \leqslant M_0\lambda. \tag{2.1.17}$$

Therefore,

$$E\{\overset{*}{K}(f) > L\lambda\} \subset E\{\overset{*}{K}(f_1) > \tfrac{1}{2}\lambda\} \cup U\left\{\overset{*}{K}(f_2) > \frac{L}{2}\lambda\right\}; \tag{2.1.18}$$

if we select $L > \max(2M_0, 4C)$ we obtain

$$\big|E\big(\overset{*}{K}(f) > L\cdot\lambda\big)\big| < \frac{N}{\lambda}\|f\|_1. \tag{2.1.19}$$

Property (i) shows that $\overset{*}{K}$ maps continuously L^∞ into L^∞; consequently, by interpolating (2.1.19) and the L^∞ result, we get the thesis and also the sizes of the type constants.

2.2. LEMMA. *Let $\varphi(x)$ be a real valued measurable function defined on R^n satisfying the conditions stated in* 1.2. *Define*:

(i) $$\overset{*}{f}_\varphi(x) = \operatorname*{Sup}_{\varepsilon>0}\Big|\varepsilon^{-|a|}\int_{\varrho(|x-y|)<\varepsilon}\varphi(x-y)f(y)\,dy\Big|.$$

Then, we have

(a) $\|\overset{*}{f}_\varphi\|_p < C_p\|f\|_p$; $C_p < C\cdot p/(p-1)$, $p > 1$.

If $p = 1$, we have instead

(b) $|E(\overset{*}{f}_\varphi > \lambda)| < \dfrac{C_0}{\lambda}\|f\|_1$.

Here, the constants C, C_p, and C_0 do not depend on f.

Proof. Observe that the kernel $K(x) = |\varphi(x)|/\varrho(|x|)^{|a|}$ satisfies the conditions of Lemma (2.1). In fact,

$$\int_{\varepsilon<\varrho(|y|)<2\varepsilon}\frac{|\varphi(y)|}{\varrho(|y|)^{|a|}}dy \leqslant \varepsilon^{-|a|}\int_{\varepsilon<\varrho(|y|)<2\varepsilon}|\varphi(y)|\,dy \leqslant C(2\varepsilon)^{-|a|}\int_{\varrho(|y|)<2\varepsilon}|\varphi(y)|\,dy \leqslant C_0. \tag{2.2.1}$$

Now, we have to check the smoothness condition

$$\begin{aligned}\left|\frac{|\varphi(x+h)|}{\varrho(|x+h|)^{|a|}} - \frac{|\varphi(x)|}{\varrho(|x|)^{|a|}}\right| &= \left|\frac{|\varphi(x+h)|}{\varrho(x+h)^{|a|}} - \frac{|\varphi(x+h)|}{\varrho(|x|)^{|a|}} + \frac{|\varphi(x+h|}{\varrho(|x|)^{|a|}} - \frac{|\varphi(x)|}{\varrho(|x|)^{|a|}}\right| \\ &\leqslant |\varphi(x+h)|\left|\frac{1}{\varrho(x+h)^{|a|}} - \frac{1}{\varrho(|x|)^{|a|}}\right| + \frac{\big||\varphi(x+h)|-|\varphi(x)|\big|}{\varrho(|x|)^{|a|}}.\end{aligned} \tag{2.2.2}$$

On account of the fact that $\big||\varphi(x+h)|-|\varphi(x)|\big| \leqslant |\varphi(x+h)-\varphi(x)|$ it follows that

$$\int_{\varrho(|x|)>4\varrho(|h|)}\frac{\big||\varphi(x+h)|-|\varphi(x)|\big|}{\varrho(|x|)^{|a|}}dx < C. \tag{2.2.3}$$

On the other hand,

$$\left|\frac{1}{\varrho(|x+h|)^{|a|}} - \frac{1}{\varrho(|x|)^{|a|}}\right| \leqslant C\frac{1}{\varrho(|x|)^{|a|}}\sum_1^n\left|\frac{\varrho(|h|)}{\varrho(|x|)}\right|^{a_i}. \tag{2.2.4}$$

See [4], pp. 27 and 28.

Incidentally, we know that $\varrho(|h|) < \frac{1}{4}\varrho(|x|)$, thus the right-hand member of (2.2.4) is dominated by

$$C\frac{1}{\varrho(|x|)^{|a|+1}}\varrho(|h|). \tag{2.2.5}$$

Applying the above estimate to the integral:

$$\begin{aligned}\int_{\varrho(|x|)>4\varrho(|h|)}|\varphi(x+h)|\frac{1}{|\varrho^{|a|}(|x+h|)-\varrho^{|a|}(|x|)|}dx &\leqslant C\varrho(|h|)\int_{\varrho(|x|)>\varrho(|h|)}\frac{|\varphi(x+h)|}{\varrho(|x|)^{|a|+1}}dx \\ &\leqslant CM_a(|\varphi|)(0),\end{aligned} \tag{2.2.6}$$

where

$$M_a(|\varphi|)(0) = \operatorname*{Sup}_{\varepsilon>0} \frac{1}{\varepsilon^{|a|}} \int_{\varrho(|x|)<\varepsilon} |\varphi(x)|\, dx.$$

The estimate (2.2.6) together with (2.2.1) shows that $K(x)$ satisfies the hypothesis of Lemma (2.1). Call

$$\overset{*}{T}(f)(x) = \operatorname*{Sup}_{\varepsilon>0} \int_{\varepsilon<\varrho(|x-y|)<2\varepsilon} \frac{|\varphi(x-y)|}{\varrho(|x-y|)^{|a|}} |f(y)|\, dy.$$

Now, fix $\varepsilon > 0$ and consider the integer k such that $2^k \leqslant \varepsilon < 2^{k+1}$. We have the following estimate:

(2.2.7) $$\varepsilon^{-|a|} \int_{\varrho(|y|)<\varepsilon} |\varphi(y)|\,|f(x-y)|\, dy$$

$$\leqslant \varepsilon^{-|a|} \int_{2^k<\varrho(|y|)<2^{k+1}} |\varphi(y)|\,|f(x-y)|\, dy + \varepsilon^{-|a|} \sum_{j\leqslant k} \int_{2^{j-1}\leqslant\varrho(|y|)\leqslant 2^j} |\varphi(y)|\,|f(x-y)|\, dy$$

$$\leqslant C \sum_{j\leqslant k} \varepsilon^{-|a|} 2^{j|a|} \int_{2^{j-1}<\varrho(|y|)<2^j} \frac{|\varphi(y)|}{\varrho(|y|)^{|a|}} |f(x-y)|\, dy \leqslant C_0 \overset{*}{T}(f)(x).$$

This last estimate finishes the proof.

2.3. COROLLARY. *Let $a_i = 1$, $i = 1, 2, \ldots, n$, and $\varphi(x)$ be a homogeneous function of degree zero, absolutely integrable over the unit sphere. Call $\omega(\delta)$ to the integral modulus of continuity of φ on the unit sphere, that is:*

(2.3.1) $$\operatorname*{Sup}_{h;|h|<\delta} \int_{\Sigma} |\varphi(x+h) - \varphi(x)|\, d\sigma = \omega(\delta),$$

where $\Sigma = \{x;\ |x| = 1\}$; $d\sigma$ is the "area" element on Σ, and h stands for a "vector" on Σ, $|h|$ its magnitude.

If

$$\int_0^2 \frac{\omega(\delta)}{\delta}\, d\delta < \infty$$

then

(2.3.2) $$\overset{*}{K}(f) = \operatorname*{Sup}_{\varepsilon>0} \varepsilon^{-n} \int_{|y|<\varepsilon} |\varphi(y)|\,|f(x-y)|\, dy$$

satisfies the inequalities of Lemma 2.2.

In the first place, the Dini condition on $\omega(\delta)$ implies

(2.3.3) $$\int_{|x|>2|h|} \frac{|\varphi(x+h) - \varphi(x)|}{|x|^n}\, dx \leqslant C_0.$$

for some C_0. See Theorem 1 in [1].

On the other hand, the homogeneity of φ and its integrability gives

(2.3.4) $$\varepsilon^{-n} \int_{|x|<\varepsilon} |\varphi(x)|\, dx \leqslant C \int_{\Sigma} |\varphi(x)|\, dx.$$

This result is due to R. Fefferman [5].

2.4. Now, we are going to return to the elliptic case. Let $\varphi(x)$ be a function satisfying the following conditions:

(2.4.1) $$\varepsilon^{-n} \int_{|x|<\varepsilon} |\varphi(x)|\, dx \leqslant C.$$

Let

(2.4.2) $$\Gamma(x) = \operatorname*{Sup}_{h;|h|<|x|/2,\,|h|<2/3} \big|\operatorname{Log}|h|\big|\,|\varphi(x+h) - \varphi(x)|\cdot|x|^{-n}$$

and suppose that $\Gamma(x)$ satisfies the following inequality:

(2.4.3) $$\big|E\big(\Gamma(x) > \lambda\big)\big| < \frac{C}{\lambda}.$$

Then, we have the following

2.5. THEOREM B. *Suppose that φ satisfies conditions* (2.4.1), (2.4.2) *and* (2.4.3). *Call*

$$\overset{*}{f}_\varphi = \operatorname*{Sup}_{1>\varepsilon>0} \Big|\varepsilon^{-n} \int_{|y|<\varepsilon} \varphi(y) f(x-y)\, dy\Big|$$

Then $\overset{*}{f}_\varphi$ satisfies the following estimates:

(i) $\|\overset{*}{f}_\varphi\|_p \leqslant C_p \|f\|_p$, $1 < p \leqslant \infty$, $C_p \leqslant C\dfrac{p}{p-1}$;

(ii) $|E(\overset{*}{f}_\varphi > \lambda)| < \dfrac{C_0}{\lambda} \|f\|_1$.

Here, as before, C_0, C_p, and C do not depend on f.

Proof. The proof of this theorem follows very closely the corresponding ones of Lemmas 2.1 and 2.2 and we shall avoid unnecessary repetitions. Let $f \geqslant 0$ be in $L^1(\boldsymbol{R}^n)$ and fix $\lambda > 0$. Consider now a denumerable family of non-overlapping cubes $\{I_k\}$ that have the following properties.

(2.5.1) (a) *If $G_\lambda = \bigcup_1^\infty I_k$; then $f \leqslant \lambda$ in $\boldsymbol{R}^n - G_\lambda$.*

(b) *If $\mu_k = \dfrac{1}{|I_k|} \displaystyle\int_{I_k} f dt$, then $\lambda < \mu_k \leqslant 2^n \lambda$.*

(c) *The I_k have edges parallel to the coordinate axes.*

(d) *If $5I_k$ denotes a dilation of I_k five times about its center, we define $5G_\lambda = \bigcup_1^\infty 5I_k$.*

Clearly:

$$|5G_\lambda| < \frac{5^n}{\lambda} \int_{I_k} f\,dy.$$

Now we decompose f as f_1+f_2; where $f_1 = f$ in $\boldsymbol{R}^n - G_\lambda$ and zero otherwise and $f_2 = \sum_1^\infty f(y)\varphi_k(y)$ where the $\varphi_k(y)$ are the characteristic functions of the I_k.

Now, f_1 is handled in the same way as in Lemma 2.1. If $x \in \boldsymbol{R}^n - 5G_\lambda$, then we have that

(2.5.2) $$(f_2)_\varphi^*(x) \leqslant \sum_1^\infty \int_{I_k} \left| \frac{\varphi(x-y)}{|x-y|^n} - \frac{\varphi(x-y_k)}{|x-y_k|^n} \right| |f-\mu_k|\varphi_k(y)\,dy$$

where y_k denotes the center of I_k.

Notice also:

(2.5.3) $$\left| \frac{\varphi(x-y)}{|x-y|^n} - \frac{\varphi(x-y_k)}{|x-y_k|^n} \right| \leqslant |\varphi(x-y)| \left| \frac{1}{|x-y|^n} \overset{(*)}{-} \frac{1}{|x-y_k|^n} \right| + \\ + \frac{|\varphi(x-y) \overset{(**)}{-} \varphi(x-y_k)|}{|x-y_k|^n}.$$

The contribution of $(*)$ in (2.5.3) is handled in the same way as in Lemma (2.1). Therefore we are going to turn our attention to the contribution produced by $(**)$, only, that is for:

(2.5.4) $$\sum_1^\infty \int_{I_k} \frac{|\varphi(x-y)-\varphi(x-y_k)|}{|x-y_k|^n} |f(y)-\mu_k|\varphi_k(y)\,dy.$$

To begin with, we may assume that all diameters $d(I_k)$ are less than $\frac{2}{3}$. That is not a restriction because when considering $x \in \boldsymbol{R}^n - 5G_\lambda$ and $\varepsilon < 1$ the cubes whose diameters are bigger than $\frac{2}{3}$ give no contribution. Likewise, we may assume that $\Gamma(x) \geqslant 1$ a.e. and supported on a ball B about the origin, since we are studying the behavior of $\varphi(x)$ for values of x such that $|x| \leqslant 1$.

On account of the preceding remark, if we call $D(s)$ to the distribution function of $\Gamma(x)$, we have

(2.5.6) $$\begin{aligned} D(S) &\leqslant |B| \quad \text{if} \quad S < 1, \\ D(S) &< C/S \quad \text{if} \quad S > 1. \end{aligned}$$

Besides the exceptional set $5G_\lambda$ we are going to construct a second exceptional set H_λ, depending on $\Gamma(x)$, and λ. Call d_k to $d(I_k)$ and let H_k be the set where $\Gamma(x) > 1/|I_k|$.

Clearly we have

(2.5.7) $$|H_k| < C|I_k|,$$

H_λ is going to be $\bigcup_1^\infty \{H_k - y_k\}$, where $H_k - y_k$ is the translation of H_k in y_k. Thus

(2.5.8) $$|H_\lambda| \leqslant \sum_1^\infty |H_k| \leqslant \frac{C}{\lambda} \int_{\boldsymbol{R}^n} f\,dy.$$

Consider now $x \in \boldsymbol{R}^n - 5G_\lambda - H_\lambda$ and let us return to the integral

(2.5.9) $$\sum_1^\infty \int_{\boldsymbol{R}^n} \frac{|\varphi(x-y)-\varphi(x-y_k)|}{|x-y_k|^n} |f-\mu_k|\varphi_k(y)\,dy.$$

Observe that

$$\frac{|\varphi(x-y)-\varphi(x-y_k)|}{|x-y_k|^n} \leqslant \Gamma(x-y_k)\frac{1}{|\log d_k|} \leqslant C\Gamma(x-y_k)\frac{1}{|\log|I_k||}.$$

Therefore the integral (2.5.4) is dominated by

$$C\sum_1^\infty \Gamma(x-y_k)\frac{1}{|\log|I_k||} 2\int_{I_k} f(y)\,dy.$$

On account of 5.6 we get the following inequality

(2.5.11)

$$\int_{\boldsymbol{R}^n - 5G_\lambda - H_\lambda} \sum_1^\infty \Gamma(x-y_k)\frac{1}{|\log|I_k||}\int_{I_k} f(y)\,dy \leqslant C\sum_1^\infty \int_{I_k} f(y)\,dy \leqslant C\|f\|_1.$$

This finishes the proof.

2.6. Remark. The technique in the proof of the above theorem is an adaptation to this particular case of Lemma 2.3 in [3].

2.7. Remark. If we replace conditions (2.4.2) and (2.4.3) by the following one

(2.7.1) $$\operatorname*{Sup}_{h;|h|\leqslant\varepsilon} \int_{\boldsymbol{R}^n - B_\varepsilon} \frac{|\varphi(x+h)-\varphi(x)|}{|x|^n}\,dx < C$$

(here, the B_ε is a family of measurable sets satisfying $|B_\varepsilon| < C_0\varepsilon^{-n}$), then we have in this case the same result of Theorem B. The proof follows the same lines. The exceptional set in this case is $5G_\lambda \cup H_\lambda$; $H_\lambda = \bigcup_1^\infty \{B_{d(I_k)} - y_k\}$.

3. Proof of Theorem A. We shall follow the type of proof introduced in [8]. In the first place, $J_\lambda(x) = T_\lambda(x)$ when $x \in F$, that shows that the boundedness of $T_\lambda(x)$ in $L^p(\boldsymbol{R}^n)$ implies the boundedness of $J_\lambda(x)$ in $L^p(F)$ (see [8], pp. 252 and 253).

We shall consider instead of $T_\lambda(x)$ a modification, namely

$$K_\lambda(f)(x) = \int_{R^n} \frac{\delta^\lambda(y) f(y) |\varphi(x-y)|}{\varrho(|x-y|)^{|\alpha|+\lambda} + \delta(y)^{|\alpha|+\lambda}} dy. \tag{3.1.1}$$

If $f \geqslant 0$, there exist two constants C_1 and C_2 independent from f, such that

$$\begin{aligned} K_\lambda(x) &\leqslant C_1 \int_{R^n} \frac{\delta^\lambda(y) f(y) |\varphi(x-y)|}{\varrho(|x-y|)^{|\alpha|+\lambda} + \delta(x)^{|\alpha|+\lambda}} dy, \\ K_\lambda(x) &\geqslant C_2 \int_{R^n} \frac{\delta^\lambda(y) f(y)}{\varrho(|x-y|)^{|\alpha|+\lambda} + \delta(x)^{|\alpha|+\lambda}} |\varphi(x-y)| dy. \end{aligned} \tag{3.1.2}$$

To show inequalities (3.1.2), consider first the inequality

$$|\delta(x) - \delta(y)| \leqslant \varrho(|x-y|). \tag{3.1.3}$$

(Whose proof is exactly the same as in the Euclidean metric case.) On the other hand, on view of the inequality $\delta(y) \leqslant \varrho(|x-y|) + \delta(x)$, we have, by Jensen's inequality,

$$\delta(y)^{|\alpha|+\lambda} \leqslant 2^{|\alpha|+\lambda-1} \{\varrho(|x-y|)^{|\alpha|+\lambda} + \delta(x)^{|\alpha|+\lambda}\} \tag{3.1.4}$$

and a similar inequality with x and y interchanged. From this fact we deduce immediately the inequalities (3.1.2).

From Lemmas 2.1 and 2.2 we have that

$$\overset{*}{g}_\varrho = \operatorname*{Sup}_{\varepsilon>0} \varepsilon^{-|\alpha|} \int_{\varrho(x-y|<\varepsilon} |\varphi(x-y)| g(y) dy \tag{3.1.5}$$

satisfies

$$\|\overset{*}{g}_\varphi\|_p < C \frac{p}{p-1} \|g\|_p, \quad p > 1. \tag{3.1.6}$$

Here, C independent from g. Consider now $f \in L^p(\boldsymbol{R}^n)$, $g \in L^{\frac{p}{p-1}}(\boldsymbol{R}^n)$ and the integral

$$\int_{R^n} g(x) K_\lambda(f)(x) dx. \tag{3.1.7}$$

After interchanging the order of integration we get

$$\int_{R^n} f(y) \int_{R^n} \frac{\delta^\lambda(y) g(x) |\varphi(x-y)|}{\varrho(|x-y|)^{|\alpha|+\lambda} + \delta(y)^{|\alpha|+\lambda}} dx dy. \tag{3.1.8}$$

The inner integral can be dominated by

$$\begin{aligned} &\delta(y)^{-|\alpha|} \int_{\varrho(|x-y|)<\delta(y)} |g(x)| |\varphi(x-y)| dx + \\ &+ \delta^\lambda(y) \sum_{k; 2^k > \delta(y)} 2^{-(|\alpha|+\lambda)(k-1)} \int_{2^{k-1} < \varrho(|x-y|) < 2^k} |g(x)| |\varphi(y-x)| dx \leqslant C \overset{*}{g}_\varphi(y). \end{aligned} \tag{3.1.9}$$

Consequently, (3.1.7) is dominated by

$$C \int f(y) \overset{*}{g}_\varphi(y) dy. \tag{3.1.10}$$

That shows that:

$$\|K_\lambda(f)\|_p < Cp \cdot \|f\|_p, \quad 1 \leqslant p < \infty. \tag{3.1.11}$$

Here, C does not depend on f.

Finally, the growth of $C \cdot p$ gives the exponential estimate (see argument in [8], p. 254).

This finishes the proof of Theorem A.

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