Contents of volume LIX, number 2

STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, W. Orlicz** (*Editor-in-Chief*), **A. Pełczyński, W. Żelazko**

The journal prints original papers in English, French, German, and Russian, mainly on functional analysis, abstract methods of mathematical analysis and on the theory of probabilities. Usually 3 issues constitute a volume.

The papers submitted should be typed on one side only and they should be accompanied by abstracts, normally not exceeding 200 words. The authors are requested to send two copies, one of them being the typed, not Xerox copy. Authors are advised to retain a copy of the paper submitted for publication.

Manuscripts and the correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA
ul. Śniadeckich 8
00–950 Warszawa, Poland

Correspondence concerning exchange should be addressed to

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ul. Śniadeckich 8
00–950 Warszawa, Poland

The journal is available at your bookseller or at

"ARS POLONA"
Krakowskie Przedmieście 7
00–068 Warszawa, Poland

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

Weak star and bounded weak star continuity of Banach algebra products

by

CARLOS E. KENIG* (Chicago)

and

HORACIO PORTA (Urbana)

Contents

1. Introduction. A variety of common Banach algebras are dual Banach spaces. For example, a well-known result of Sakai (see [20]) states that von Neumann algebras are characterized, among all C^*-algebras, by this property. In Sections 6 through 11 of this paper we consider several instances of this phenomenon, notably, convolution measure algebras, L^p-spaces, H^∞ spaces of plane domains, operator algebras in Hilbert space and second duals of Banach algebras under Arens products.

The question we consider in such a Banach algebra is whether the product is continuous for the weak* convergence or for the bounded weak* convergence, in each variable, or jointly.

The natural set up for this problem is considered in the first part of the paper, where we study the following situation. Let A be a (real or complex) Banach space with dual $B = A^*$, so that each $b \in B$ acts on A as a linear functional. Suppose that a second action is defined making

* The first author is a Victor J. Andrew Fellow at the University of Chicago; the research of the second author was partly supported by NSF grants.

each $b \in B$ act on A as an operator $A \to A$. Under these hypotheses, a product bb' can be defined on B by just taking the composition of the maps b and b', where b is considered as an operator and b' as a functional: $A \xrightarrow{b} A \xrightarrow{b'} R$ (or C as the case may be). Of course such a product need not be associative in general. When it is, it makes a Banach algebra of B. In all the examples mentioned above, the Banach algebra products are obtained in this way. In Sections 3, 4 we determine some relations between continuity properties of the product and operator properties of the action of B on A.

We use the notations of [7] and [4]. In particular, if $a \in A$, $b \in A^*$, $\langle a, b\rangle$ is the duality map. For X, Y Banach spaces, $L(X, Y)$ is the Banach space of all bounded linear operators $T: X \to Y$, and T^t: $Y^* \to X^*$ is the adjoint of T. The sequence spaces are denoted, as usual, by l^p, c_0 and their non-separable analogues are denoted by $l^p(S)$, $c_0(S)$, S an arbitrary set. Inner products in Hilbert spaces are denoted by $(x|y)$.

Finally, the weak* topology on a dual space is denoted by w^* and the bounded weak* topology by lw^*. We will say that $b_\alpha \to 0$ for the bounded weak* convergence (or for lw^*) when $b_\alpha \to 0$ weak* and moreover $\|b_\alpha\|$ is bounded. (We use the symbol lw^* rather than the more traditional bw^* to conform with the general set up of [14].)

Some of these results were announced in [10]. We received encouragement or comments from Felix Rumatione, Ana Roth, and Lee A. Rubel, and want to express here our gratitude.

2. Bilinear maps. The notations and abbreviations introduced here will be used throughout.

We shall denote by A an arbitrary Banach space and by B its dual $B = A^*$; $a, a', \ldots$ will be typical elements of A and $b, b', \ldots, c, c', \ldots$ typical elements of B. We shall consider bilinear bounded transformations F: $B \times A \to A$. We think on them as "mixed products" and in fact the notation $F(b, a) = b \vdash a$, emphasizing this attitude, will be used. We can dualize $b \vdash a$ to obtain a bilinear map $B \times B \to B$. More precisely,

2.1. Definition. The product associated to $(b, a) \mapsto b \vdash a$ is the *bilinear transformation* $(b, b') \mapsto b \cdot b'$ from $B \times B$ into B defined by $\langle a, b \cdot b'\rangle = \langle b \vdash a, b'\rangle$.

It is clear that $b \cdot b'$ is also bounded and bilinear. A bounded bilinear map $B \times B \to B$ arising in this fashion will be called a $\vdash$*-product*.

Consider the family of all operators $T: B \to A$ of the form $b \mapsto b \vdash a_0$, where a_0 varies in A. We define the properties $(w^*\text{-}w)$, (r), $(lw^*\text{-}n)$, (wk), (k) and (f) for $\vdash$ by requiring that all T have the corresponding operator property described below:

$(w^*\text{-}w)$ weak*-weak continuous;
(r) the range of the transpose T^t: $B \to A^{**}$ is contained in A;
$(lw^*\text{-}n)$ lw^*-norm continuous;

(wk) weakly compact;
(k) compact;
(f) finite rank.

In order to clarify the mutual relations between various types of continuity of bilinear maps, we shall introduce a list of abbreviations. Let Z be any (real or complex) linear topological space and let F: $B \times B \to Z$ be a bilinear transformation. Consider the following continuity properties of F:

$[J, (x_0, y_0)]$ $F(x, y) \to F(x_0, y_0)$ when $x \to x_0$ and $y \to y_0$;
$[J]$ F has $[J, (x_0, y_0)]$ for all $(x_0, y_0) \in X \times Y$;
$[L]$ for each x_0, y_0, $F(x, y_0) \to F(x_0, y_0)$ when $x \to x_0$;
$[R]$ for each x_0, y_0, $F(x_0, y) \to F(x_0, y_0)$ when $y \to y_0$;

where, depending on the case, $x \to x_0$ and $y \to y_0$ will stand for weak* or bounded weak* convergence.

2.2. Proposition. *For the weak* convergence,* $[J, (0, 0)]$ *and* $[J]$ *are equivalent. For the bounded weak* convergence,* $[J, (0, 0)] + [L] + [R] \Leftrightarrow [J]$ *and none of the three properties on the left is redundant in general.*

It is perhaps somewhat disappointing that $[J, (0, 0)]$ is not equivalent to $[J]$. The proof of the equivalence in Proposition 2.2 is elementary. The redundance statement will be easily established after some examples are introduced (see Section 5 below).

In the sequel, properties in square brackets refer to continuity properties of a $\vdash$-product in the convergence alluded to.

3. The w^*-convergence.

(3.1) Theorem. *Let $b \cdot c$ be the product associated to $b \vdash a$. Then for the weak* convergence:*

(3.1.i) $[R]$ *always holds;*
(3.1.ii) $[L]$, (r) *and* $(w^*\text{-}w)$ *are equivalent;*
(3.1.iii) $[J, (0, 0)]$, $[J]$, $(f) + [L]$ *and* $(f) + (w^*\text{-}w)$ *are equivalent;*
(3.1.iv) $[L]$ *holds if A is reflexive; moreover* $[L]$ *holds for all* $\vdash$*-products if and only if A is reflexive.*

Proof. (3.1.i) follows trivially from the formula $\langle a_0, b_0 \cdot c\rangle = \langle b_0 \vdash a_0, c\rangle$. The equivalence between $[L]$ and $(w^*\text{-}w)$ is proved as follows: assume that $b_\beta \xrightarrow{w^*} 0$; then $b_\beta \cdot c \xrightarrow{w^*} 0$ iff $\langle a, b_\beta \cdot c\rangle = \langle b_\beta \vdash a, c\rangle \to 0$ iff $b_\beta \vdash a \xrightarrow{w} 0$. The equivalence between $[L]$ and (r) is proved as follows. First, let T: $B \to A$ be the operator $Tb = b \vdash a_0$. Then $\langle b, T^t c\rangle = \langle Tb, c\rangle = \langle b \vdash a, c\rangle = \langle a, b \cdot c\rangle$ so that $[L]$ holds iff $T^t c$ is a weak* continuous functional on B, i.e., an element of A. This concludes the proof of (3.1.ii). We want to prove now that $[J, (0, 0)]$ implies (f). Assume that $[J, (0, 0)]$ holds and that $\dim\{b \vdash a_0;\ b \in B\} = \infty$ for a suitable $a_0 \in A$. Consider the w^*-

neighborhood of $0 \in B$ defined by $V = \{b \in B,\ |\langle a_0, b\rangle| < 1\}$. Let now U, W be arbitrary w^*-neighborhoods of $0 \in B$. We are going to show that there are elements $b_0 \in U$, $c_0 \in W$ such that $b_0 \cdot c_0 \notin V$. First, there is a finite family $a_1, a_2, \ldots, a_m$ in A such that $\langle a_j, b\rangle = 0$, $j = 1, 2, \ldots, m$, implies $b \in U$ and a finite family, which we can call $a_{m+1}, a_{m+2}, \ldots, a_n$, in A such that $\langle a_j, b\rangle = 0$, $j = m+1, \ldots, n$, implies $b \in W$. Consider now the subspace $S = \{b \in B;\ \langle a_j, b\rangle = 0,\ 1 \leqslant j \leqslant n\}$. Clearly, $S \subset U \cap W$ and S has finite codimension in B, and so $\dim(S \vdash a_0) = +\infty$. Hence, there is a $b_0 \in S$ such that $b_0 \vdash a_0$ does not belong to the linear span of $a_1, a_2, \ldots, a_n$. Therefore, by Hahn–Banach, there is a $c_0 \in B$ such that $\langle a_j, c_0\rangle = 0$, $1 \leqslant j \leqslant n$ and $\langle b_0 \vdash a_0, c_0\rangle = 1$. Thus $b_0, c_0 \in S$ and a fortiori, $b_0 \in U$, $c_0 \in W$ while $b_0 \cdot c_0 \notin V$. This proves that $[J, (0,0)]$ implies (f). In order to prove that $[J, (0,0)]$ implies $[L]$ we shall need:

(3.2) *For an arbitrary Banach space X, an element $x^{**} \in X^{**}$ belongs to X iff for each net $x_\beta^* \xrightarrow{w^*} 0$, there is β_0 with $\{\langle x_\beta^*, x^{**}\rangle,\ \beta \geqslant \beta_0\}$ bounded.*

The "only if" part of (3.2) is trivial. Assume now that $x^{**} \notin X$; a standard argument applies to show that for each finite set $F = \{x_1, \ldots, x_k\}$ there is $x_F^* \in X^*$ such that $\langle x_j, x_F^*\rangle = 0$, $1 \leqslant j \leqslant k$ and $\langle x_F^*, x^{**}\rangle = k$. Then $x_F^* \xrightarrow{w^*} 0$ as F increases while $\{\langle x_F^*, x^{**}\rangle\}$ is unbounded for each tail $F \supset F_0$.

Assume $[J, (0,0)]$ holds for $b \cdot c$. Then (f) holds and so for each $a_0 \in A$ there are linearly independent $a_1, a_2, \ldots, a_n \in A$ and $b_1^*, b_2^*, \ldots, b_n^* \in B^*$ such that $b \vdash a_0 = \sum \langle b, b_k^*\rangle a_k$. We will use now (3.2) to show that each b_j^* belongs to A. Fix j with $1 \leqslant j \leqslant n$. Assume that $b_\beta \xrightarrow{w^*} 0$. If $\{\langle b_\beta, b_j^*\rangle,\ \beta \geqslant \beta_0\}$ is unbounded for each β_0, then for an appropriate subnet, still denoted b_β, we have $0 < |\langle b_\beta, b_j^*\rangle| \to \infty$. Pick now $c_j \in B$ with $\langle a_k, c_j\rangle = \delta_{kj}$ for $1 \leqslant k \leqslant n$ and define $c_\beta = \langle b_\beta, b_j^*\rangle^{-1} c_j$. Clearly, $\|c_\beta\| \to 0$ so that $c_\beta \xrightarrow{w^*} 0$. But then $\langle a_0, b_\beta \cdot c_\beta\rangle = \langle b_\beta \vdash a_0, c_\beta\rangle = \sum_k \langle b_\beta, b_k^*\rangle \langle a_k, c_\beta\rangle = \sum_k \langle b_\beta, b_k^*\rangle \langle b_\beta, b_j^*\rangle^{-1} \langle a_k, c_j\rangle = 1$, which contradicts $[J, (0,0)]$. Thus $b_j^* \in A$ for each $j = 1, \ldots, n$, and so relabelling $b_j^* = \bar{a}_j$, we have $b \vdash a_0 = \sum \langle \bar{a}_k, b\rangle a_k$ and $\langle a_0, b \cdot c\rangle = \sum \langle \bar{a}_k, b\rangle \langle a_k, c\rangle$. Clearly $[L]$ follows.

Now the fact that $[J, (0,0)]$ implies $[J]$ follows from this and (3.1.i) using Proposition (2.1). If A is reflexive, (r) holds trivially and according to (3.1.ii), $[L]$ follows. The second half of (3.1.iv) will follow from Example 5.1. This completes the proof of 3.1.

Remark. The equivalent conditions in (3.1.vii) obviously imply the equivalent conditions in (3.1.ii). For the fact that the converse is false in general, see Section 5.

4. The lw^*-convergence.

4.1. THEOREM. *Let $b \cdot c$ be the product associated to $b \vdash a$. Then for the bounded weak* convergence:*

(4.1.i) *$[J, (0,0)]$ is equivalent to (k);*

(4.1.ii) *$[J]$, $[J, (0, c)]$ (for all c), $[J, (0,0)] + [L]$, $(k) + [L]$, and $(lw^*\text{-}n)$ are equivalent;*

(4.1.iii) *$[L]$ implies (wk).*

Proof. We begin with $(k) \Rightarrow [J, (0,0)]$, so let us assume that (k) holds and let $b_\beta \xrightarrow{w^*} 0$, $c_\gamma \xrightarrow{w^*} 0$, $\|b_\beta\| \leqslant 1$, $\|c_\gamma\| \leqslant 1$. Consider the compact subset $K \subset A$ defined by $K = \text{closure}\{b \vdash a_0;\ b \in B,\ \|b\| \leqslant 1\}$. From the fact that $c_\gamma \to 0$ pointwise on K and that $\{c_\gamma\}$ are norm bounded, and hence equicontinuous, we conclude that $c_\gamma \to 0$ uniformly on K. Then, $\langle b_\beta \vdash a_0, c_\gamma\rangle \to 0$ uniformly on β as γ increases; hence $b_\beta \cdot c_\gamma \xrightarrow{w^*} 0$.

Conversely, assume that $[J, (0,0)]$ holds. Suppose there is an element $a_0 \in A$ with $b \mapsto b \vdash a_0$ not compact, and let $M = \{b \vdash a_0,\ b \in B,\ \|b\| \leqslant 1\}$. M is bounded but not precompact. Hence there is an $r > 0$ such that for each finite dimensional subspace $Y \subset A$, there is an $m \in M$ with $\operatorname{dist}(m, Y) \geqslant r$. For each finite subset $F = \{a_1, \ldots, a_n\}$ of A, let Y_F be the linear span of F and let $m_F = b_F \vdash a_0$, with $\|b_F\| \leqslant 1$, satisfy $\operatorname{dist}(m_F, Y_F) \geqslant r$. From a corollary of Hahn–Banach (see [7], Lemma II. 3.12) there is a $c_F \in B$ satisfying: $c_F = 0$ on Y_F, $\langle m_F, c_F\rangle = 1$ and $\|c_F\| = 1/\operatorname{dist}(m_F, Y_F) \leqslant 1/r$. Clearly, $c_F \xrightarrow{lw^*} 0$. Moreover, $\langle b_F \vdash a_0, c_F\rangle = \langle m_F, c_F\rangle = 1$. By w^*-compactness there is a w^*-convergent subnet $b_{F_i} \xrightarrow{lw^*} b_\infty$, and so $\langle a_0, (b_{F_i} - b_\infty) \cdot c_{F_i}\rangle = \langle b_{F_i} \vdash a_0, c_{F_i}\rangle - \langle b_\infty \vdash a_0, c_{F_i}\rangle = 1 - \langle b_\infty \vdash a_0, c_{F_i}\rangle \to 1$, a contradiction with $[J, (0,0)]$.

In order to prove (4.1.ii) it will suffice to show that $[J, (0, c)]$ (for all c) $\Rightarrow (lw^*\text{-}n) \Rightarrow (k) + [L]$ and to use (2.1) and (3.1.i).

Assume then that $[J, (0, c)]$ holds for all $c \in B$ and let $b_\beta \xrightarrow{lw^*} 0$. Pick $a \in A$ and let $\{b_\gamma\}$ be an arbitrary subnet of $\{b_\beta\}$. Clearly for each γ there is a $c_\gamma \in B$ with $\langle b_\gamma \vdash a, c_\gamma\rangle = \|b_\gamma \vdash a\|$, $\|c_\gamma\| = 1$. By the w^*-compactness of $\|b\| \leqslant 1$, there is a w^*-convergent subnet $c_\delta \xrightarrow{lw^*} c_\infty$. Then, by $[J, (0, c_\infty)]$, $\|b_\delta \vdash a\| = \langle b_\delta \vdash a, c_\delta\rangle = \langle a, b_\delta \cdot c_\delta\rangle \to 0$, and $(lw^*\text{-}n)$ follows.

Clearly $(lw^*\text{-}n) \Rightarrow (k)$. We show that $(lw^*\text{-}n) \Rightarrow [L]$ as follows: for $b_\beta \xrightarrow{lw^*} 0$, $|\langle a, b_\beta \cdot c\rangle| = |\langle b_\beta \vdash a, c\rangle| \leqslant \|b_\beta \vdash a\| \|c\| \to 0$.

(4.1.iii) is immediate from the w^*-compactness of $\|b\| \leqslant 1$.

Remark. Clearly the equivalent conditions in (4.1.ii) imply the equivalent conditions in (4.1.i). The fact that, in general, the converse of this and of (4.1.iii) fail will be shown by appropriate examples in the next section.

5. Examples.

5.1. Let A be a non-reflexive Banach space, $B = A^*$, and pick $a_0^{**} \in A^{**}$ with $a_0^{**} \notin A$. Define $b \vdash a = \langle b, a_0^{**}\rangle a$. Clearly, $b \cdot c = \langle b, a_0^{**}\rangle c$. Since $[L]$ does not hold for w^*, this completes the proof of (3.1.iv).

5.2. Let A be any real Hilbert space, $B = A$ and pick $0 \neq a_0 \in A$. Define $b \vdash a = (a\,|\,a_0)b$, so that $b \cdot c = (b\,|\,c)a_0$.

Using $+$ and $-$ to mean "it holds" and "it does not hold", respectively, these examples satisfy, for the bounded weak* convergence:

	(f)	(k)	(wk)	[L]	[J, (0, 0)]	[J]
5.1	+	+	+	−	+	−
5.2	−	−	+	+	−	−

6. Convolution algebras. Let G be a locally compact group. We denote by $C_0(G)$ the Banach space of all complex continuous functions vanishing at infinity and by $M^1(G)$ the space of all bounded regular measures on G. $C_0(G)$ and $M^1(G)$ are Banach spaces under the sup norm $\|\ \|_\infty$ and the total variation norm $\|\mu\| = |\mu|(G)$, respectively, and the pairing $\langle f, \mu\rangle = \int_G f(s)\mu(ds)$ identifies $M^1(G)$ with the dual of $C_0(G)$. In the sequel, the "w^*-topology on $M^1(G)$" refers to the weak* topology of $M^1(G)$ as the dual of $C_0(G)$ (for the relationships with other standard measure topologies — e.g., the vague topology —, see [3], Chapter VIII, Section 3, Example 11).

If $f \in C_0(G)$, $\mu \in M^1(G)$ and $s \in G$, we define $(\mu \vdash f)(s) = \int_G f(ts)\mu(dt)$, i.e.,

(6.1) $(\mu \vdash f)(s) = \langle \gamma(s)f, \mu\rangle$, where $\gamma(s)\colon C_0(G) \to C_0(G)$ is the regular (right) representation $\big(\gamma(s)f\big)(t) = f(ts)$.

We have

(6.2.i) $$\mu \vdash f \in C_0(G),$$

(6.2.ii) $$\|\mu \vdash f\|_\infty \leqslant \|f\|_\infty \|\mu\|,$$

(6.2.iii) $\mu, f \mapsto \mu \vdash f$ is a bilinear map $M^1(G) \times C_0(G) \to C_0(G)$.

For (6.2.i) see [3], dépliant II; (6.2.ii) follows from $|(\mu \vdash f)(s)| = |\langle \gamma(s)f, \mu\rangle| \leqslant \|\gamma(s)f\|_\infty \|\mu\| = \|f\|_\infty \|\mu\|$ and (6.2.iii) is obvious.

This shows that we are in the situation described in Section 2 with $A = C_0(G)$, $B = M^1(G)$ and $\vdash$ defined by (6.1) above.

Let now $\mu, \nu \in M^1(G)$. Then for $f \in C_0(G)$, $\langle f, \mu \cdot \nu\rangle = \langle \mu \vdash f, \nu\rangle = \int_G \big(\int_G f(ts)\mu(dt)\big)\nu(ds) = \int\int_{G\times G} f(ts)\mu(dt)\nu(ds)$. This shows that the product associated to $\vdash$ is the ordinary *convolution of measures* $\mu \cdot \nu = \mu * \nu$.

The following theorem extends Example 1.4 in [6] and Corollary 1 on page 284 of [21] (cf. also Theorem 3 of [21]):

(6.3) Theorem. *Let G be a locally compact group, $M^1(G)$ the Banach space of regular bounded measures on G, and $\mu * \nu$ the convolution of the measures $\mu, \nu \in M^1(G)$. Then:*

(6.3.i) *$\nu \mapsto \mu * \nu$ is w^*-continuous for each μ;*

(6.3.ii) *$\mu \mapsto \mu * \nu$ is w^*-continuous for each ν;*

(6.3.iii) *$\mu, \nu \mapsto \mu * \nu$ is jointly w^*-continuous if and only if G is finite;*

(6.3.iv) *$\mu, \nu \mapsto \mu * \nu$ is jointly lw^*-continuous if and only if G is compact.*

Proof. (6.3.i) follows from (3.1.i). Let $f^{(-)}(s) = f(s^{-1})$ and define $\mu^{(-)}$ by $\langle f, \mu^{(-)}\rangle = \langle f^{(-)}, \mu\rangle$. Then $(\mu * \nu)^{(-)} = \nu^{(-)} * \mu^{(-)}$ and therefore $\langle f, \mu * \nu\rangle = \langle (\nu^{(-)} \vdash f^{(-)})^{(-)}, \mu\rangle$, which proves (6.3.ii).

From 1.4.3 in [6] follows that if G is compact, then $\mu * \nu$ is jointly lw^*-continuous (the proof of 1.4.3 in [6] actually shows that $\mu \mapsto \mu \vdash f$ is a compact operator for each $f \in C_0(G)$). Conversely, assume that G is not compact and for each compact set $K \subset G$ pick μ_K to be a unit point mass measure with support $\{t_K\}$ outside K. Clearly $\mu_K \xrightarrow{lw^*} 0$. For the same reasons, if ν_K is the unit point mass measure with support $\{t_K^{-1}\}$, then also $\nu_K \xrightarrow{lw^*} 0$. However, $\mu_K * \nu_K = \varepsilon$ is the unit point mass with support $\{e\}$ (e = identity of G) for all K. Hence, $\mu * \nu$ is not lw^*-jointly-continuous at $(0, 0)$, and (6.3.iv) follows.

Assume now that convolution is jointly w^*-continuous. Then it is also jointly lw^*-continuous and, from (6.3.iv), G is compact. Then 1.4.2 in [6] applies and G is finite. The converse is obvious, so that (6.3.iii) follows and the theorem is proved.

Observe that (6.3.iv) together with (4.1.i) and the fact that $C(G)$ has the approximation property, imply the Peter–Weyl Theorem (as formulated, for instance, in [7], p. 940).

We close this section with two remarks. First, we point out that the results in Theorem 6.3 actually hold for an arbitrary invariant closed subspace A of $C_0(G)$ and its dual.

Second, minor modifications will also apply to the following situation. Let $S \subset G$ be an open semigroup of G, $A = C_0(S)$, $B = M^1(S)$. Here again the product associated to $(\mu \vdash f)(s) = \int_S f(ts)\mu(dt)$ is the convolution $\mu * \nu$ in $M^1(S)$. The conclusion is: if S is not finite then the convolution product is not jointly w^*-continuous and if S is not compact, then it is not jointly lw^*-continuous. This requires a different version of 1.4.1 in [6], namely: *if G is locally compact and not compact, for each $f \in C_0(G)$ not identically zero there are infinitely many linearly independent translates of f* ("the anti-Peter–Weyl lemma"). This fact can in turn be proven as follows: assume that all translates $\gamma(s)f$ of f are linear combinations of n linearly independent translates $f_j = \gamma(s_j)f$. For each $s \in G$, there are $\beta_s \geqslant 0$ and $a_j^{(s)}$, $j = 1, \ldots, n$ with $\beta_s + \sum |a_j^{(s)}| = 1$ and $\beta_s \gamma(s)f + \sum a_j^{(s)} f_j = 0$. Let "$s \to \infty$" so that $\big(\gamma(s)f\big)(t) \to 0$ for each t and pick a subnet $s' \to \infty$ such that $\beta_{(s')} \to \beta_\infty$ and $a_j^{(s')} \to a_j^{(\infty)}$, $j = 1, 2, \ldots, n$. Then

$$\Big[\sum a_j^{(\infty)} f_j\Big](t) = \lim[-\beta_{s'}\gamma(s')f](t) = 0$$

so that $a_j^{(\infty)} = 0$ for each j. Then $1 = \beta_\infty + \sum |a_j^{(\infty)}|$ implies $\beta_\infty = 1$. But now, taking $t = (s')^{-1}a$, we get

$$\beta_{(s')}f(a) = \beta_{(s')}[\gamma(s')f]\big((s')^{-1}a\big) = -\sum [a_j^{(s')}f_j]\big((s')^{-1}a\big)$$

and therefore

$$|f(a)| = |\lim \beta_{(s')}f(a)| = \lim \Big|\sum [a_j^{(s')}f_j]\big((s')^{-1}a\big)\Big| \leqslant \lim \|f\| \sum a_j^{(s')}| = 0,$$

as claimed.

A typical application of the second generalization is the case $A = c_0$, $B = l^1$, with $G = Z$ and $S = N$.

7. The space $L^\infty(\mu)$**.** In [21], J. Shapiro proved that $L^\infty(T)$, T the unit circle, is not a topological algebra under pointwise multiplication and the lw^* topology (which has an intrinsic meaning since $L^\infty(T)$, as any other W^*-algebra, is the dual of only one Banach space — namely, $L^1(T)$). We aim to explain this result in terms of measure theoretic properties (see (7.3.i) below).

All measures considered below are positive measures. We recall that a measure space (X, Σ, μ) is *localizable* if for each family $\{U_a\}$ of measurable sets, there is a measurable set U such that: (1) $\mu(U_a - U) = 0$ for all a, and (2) if $\mu(U_a - V) = 0$ for all a, then $\mu(U - V) = 0$ also; a measure space (X, Σ, μ) has the *finite subset property* if any set of positive measure has a subset of finite positive measure (see [23] for both definitions). The following important fact can be found in Theorem 4, Section 50, Chapter 12 of [23].

(7.1) *Let (X, Σ, μ) be a measure space with $\mu \geqslant 0$. Necessary and sufficient for $L^\infty(\mu)$ to be the dual of $L^1(\mu)$ is that the measure space be localizable and have the finite subset property.*

Under these conditions, we can take $A = L^1(\mu)$, $B = L^\infty(\mu)$ and define $g \vdash f \in L^1(\mu)$ for $g \in L^\infty(\mu)$, $f \in L^1(\mu)$ by $(f \vdash g)(s) = f(s)g(s)$. The $\vdash$-product is the ordinary pointwise product in $L^\infty(\mu)$.

We shall adopt the following definition: an *atom* in a measure space (X, Σ, μ) is a measurable set E with $0 < \mu(E) < +\infty$ such that for no $F \subset E$ we have $0 < \mu(F) < \mu(E)$. The measure space is *purely atomic* if for each measurable set $A \subset X$, the set $A' = A - \bigcup\{E;\ E \subset A$ and E is an atom$\}$ is measurable and $\mu(A') = 0$. With these definitions we have (see [13]):

(7.2) *If (X, Σ, μ) is purely atomic, $L^1(\mu)$ is isometric to $l^1(S)$ for an appropriate set S, and an isometry T can be picked to also map pointwise products $f \cdot g$ (when f, g and $f \cdot g$ are in $L^1(\mu)$) into $\{(Tf)_a(Tg)_a W_a\}$ for an appropriate weight $\{W_a\}_{a \in S}$, $0 < W_a < +\infty$.*

The set S is the set of atoms and $T\colon L^1(\mu) \to l^1(S)$ is the map defined by: for each $a \in S$, $(Tf)_a = f(a)\mu(a)$, where $f(a)$ is the common value $f(t)$, $t \in a$; also $W_a = \mu(a)$.

We can now state our result:

(7.3) Theorem. *Let (X, Σ, μ) be a localizable measure space with the finite subset property (so that $L^\infty(\mu)$ is the dual of $L^1(\mu)$). Then for the pointwise product in $L^\infty(\mu)$,*

(7.3.i) *$[J, (0, 0)]$ holds for lw^* if and only if $(X, \Sigma, \mu,)$ is purely atomic;*

(7.3.ii) *$[J, (0, 0)]$ holds for w^* if and only if $\dim L^\infty(\mu) < +\infty$ (i.e., (X, Σ, μ) is purely atomic and it has only finitely many atoms).*

Proof. The "if" part in (7.3.ii) is obvious and the "if" part in (7.3.i) reduces, in view of (7.2), to the statement in (8.4) below.

Assume now that (X, Σ, μ) is not purely atomic. Since (X, Σ, μ) is localizable and has the finite subset property we can pick a measurable set T with $0 < \mu(T) < +\infty$ and containing no atoms. Let $\{T_{\varepsilon(1)\varepsilon(2)\ldots\varepsilon(k)}\}$ be a dyadic partition of T, i.e., for each 0–1 word $\varepsilon(1)\,\varepsilon(2) \ldots \varepsilon(k)$, $T_{\varepsilon(1)\ldots\varepsilon(k)}$ is a measurable subset of T such that: (1) $T = T_0 \cup T_1$, $T_0 = T_{00} \cup T_{01}$, $T_1 = T_{10} \cup T_{11}, \ldots, T_{\varepsilon(1)\ldots\varepsilon(k)} = T_{\varepsilon(1)\ldots\varepsilon(k)0} \cup T_{\varepsilon(1)\ldots\varepsilon(k)1}$, etc.; (2) $T_{\varepsilon(1)\ldots\varepsilon(k)} \cap \cap T_{\delta(1)\ldots\delta(k)} = \emptyset$ if $\varepsilon(j) \neq \delta(j)$ for some $j = 1, \ldots, k$, and (3) $\mu(T_{\varepsilon(1)\ldots\varepsilon(k)\varepsilon(k+1)}) = \frac{1}{2}\mu(T_{\varepsilon(1)\ldots\varepsilon(k)})$. For each $k = 1, 2, \ldots$ define the function φ_k on X by: $\varphi_k = 0$ iff T and if $x \in T_{\varepsilon(1)\ldots\varepsilon(k)}$, let $q = \sum_{j=1}^{k} 2^{\varepsilon(j)}$ and set $\varphi_k(x) = \mu(T)^{1/2}(-1)^q$. Clearly, $\{\varphi_k\}$ is an orthonormal system in $L^2(\mu)$ and therefore $\varphi_k \to 0$ for lw^* in $L^\infty(\mu)$. However, $\varphi_k \cdot \varphi_k = \mu(T)\chi_T$ (χ_T = characteristic function of T), so that $\varphi_k \cdot \varphi_k$ does not converge to 0 for lw^* and this proves that the product is not jointly lw^*-continuous at $(0, 0)$.

Assume finally that the product in $L^\infty(\mu)$ is jointly w^*-continuous at $(0, 0)$. In particular it is jointly lw^*-continuous at $(0, 0)$ and so $L^\infty(\mu) = l^\infty(S)$. Then (8.5) applies to show that S is finite and $\dim L^\infty(\mu) < +\infty$.

Remark. It is clear that the failure to joint lw^*-continuity in $L^\infty(\mu)$ follows in our proof from the existence of an orthonormal system $\{\varphi_k\}$ with φ_k^2 = some fixed constant on a set of positive measure. Such a system can be defined easily on any cube $[0, 1]^Q$ (Q an arbitrary set) by just taking $\varphi_k\big((x_q)_{q \in Q}\big) = \exp(ikx_{q_0})$, where q_0 is a distinguished index. But then the existence of similar systems for arbitrary (X, Σ, μ) follows from Maharam's theorem ([12]). This provides an alternative proof of (7.3.i).

In view of (7.3.i) and (4.1.i), we conclude that, for (X, Σ, μ) as above,

(7.4) *All the operators $L^\infty(\mu) \to L^1(\mu)$ defined by $f \mapsto f \cdot g_0$, with $g_0 \in L^1(\mu)$ fixed, are compact if and only if (X, Σ, μ) is purely atomic.*

8. The spaces $L^p(\mu)$ **with** μ **purely atomic.** Assume in this section that (X, Σ, μ) is a purely atomic space. For each p with $1 < p \leqslant +\infty$, the dual of $A = L^q(\mu)$ is $B = L^p(\mu)$, where $(1/p)+(1/q) = 1$ (and $q = 1$ if $p = \infty$). When $1 < p < +\infty$ (as always for reflexive spaces) all products (i.e., bounded bilinear maps $B\times B\to B$) on $B = L^p(\mu)$ are $\vdash$ - products. Hence, from Theorem 3.1 we get:

(8.1) *For* $1 < p < +\infty$, *all norm continuous products on* $L^p(\mu)$ (μ *purely atomic*) *satisfy* [L] *and* [R] *for the* w^* *and the* lw^* *topologies.*

On the other hand, Rosenthal's generalization of the classical result of Pitt (see [16], Theorem A2.3 and Remark 2 on p. 211) implies that all operators $T\colon L^r(\mu)\to L^s(\mu)$ are compact when μ is purely atomic and $1\leqslant s < r < +\infty$ or $1\leqslant s < 2$, $r = +\infty$. Then, using (2.2), (8.1) and (4.1.i), we conclude that

(8.2) *If* $2 < p < +\infty$, *all norm continuous products on* $L^p(\mu)$, μ *purely atomic, are jointly* lw^**-continuous everywhere.*

It is easy to see that the conclusion is not generally true for $1\leqslant p\leqslant 2$ or for $p = \infty$. In the case $p = \infty$, the space $L^p(\mu)$ being no longer reflexive, the products are not automatically $\vdash$ - products. We can only conclude that:

(8.3) *All* $\vdash$ *- products on* $L^\infty(\mu)$, μ *purely atomic, satisfy* [J, $(0, 0)$] (*and of course* [R]) *for the bounded weak** *convergence.*

In order to interpret the pointwise product in $L^\infty(\mu)$ as a particular case of pointwise products in $L^p(\mu)$ for each p, we first observe that (7.2) can be used to identify $L^p(\mu)$, μ purely atomic, with the space $l^p(S, W)$ of all $(x_a)_{a\in S}$ such that $\sum_{a\in S} |x_a|^p W_a < +\infty$, where $W = (W_a)_{a\in S}$ is a "weight" satisfying $W_a > 0$. The norm in $l^p(S, W)$ is $\big(\sum_{a\in S} |x_a|^p W_a\big)^{1/p}$, $S = \{a\}$ is the set of atoms of μ and $W_a = \mu(a)$. We can now define the *pointwise product* on $L^p(\mu) = l^p(S, W)$ by $(x_a)\cdot(y_a) = (z)_a$, where $z_a = x_a y_a W_a^{(1/p)}$. Clearly this is a $\vdash$ - product with $\vdash$ defined by the same formula. It is now easy to verify that (cf. 1.3.1 and 1.3.5 of [6]):

(8.4) *The pointwise product in* $L^p(\mu)$, μ *purely atomic*, $1\leqslant p\leqslant +\infty$, *is jointly* lw^**-continuous everywhere.*

Routine arguments and (3.1) also yield:

(8.5) *The pointwise product in* $L^p(\mu)$, μ *purely atomic*, $1\leqslant p\leqslant +\infty$, *satisfies* [$J$, $(0, 0)$] *for the* w^* *topology if and only if* $\dim L^p(\mu) < +\infty$ (*i.e.*, μ *has finitely many atoms*).

In the last two statements, $L^1(\mu) = l^1(S, W)$ is to be interpreted as the dual of $c_0(S, W)$. We leave the details to the reader.

9. The Arens product. Let X, Y, Z be Banach spaces and let $m\colon X\times Y\to Z$ be a bounded bilinear map. Define $m^t\colon Y\times X\to Z$ and $m^*\colon Z^*\times X\to Y^*$ by:

$$m^t(x, y) = m(y, x),$$
$$\langle y, m^*(z^*, x)\rangle = \langle m(x, y), z^*\rangle.$$

Iteration of this procedure yields the maps

$$m\colon X\times Y\to Z,$$
$$m^*\colon Z^*\times X\to Y^*,$$
$$m^{**}\colon Y^{**}\times Z^*\to X^*,$$
$$m^{***}\colon X^{**}\times Y^{**}\to Z^{**}.$$

These definitions were introduced by Arens who studied several properties of them in [1], [2].

In the sequel only the case $X = Y = Z$ will be considered and we shall use the symbols $A = X^*$, $B = X^{**}$. Thus we have

$$m\colon X\times X\to X,$$
$$m^*\colon A\times X\to A,$$
$$m^{**}\colon B\times A\to A,$$
$$m^{***}\colon B\times B\to B.$$

Clearly, m^{**} can be considered to be a $\vdash$ - product. From the definition follows that:

(9.1) *The* $\vdash$ *- product associated to* m^{**} *is* m^{***t}.

Thus, by (3.1.i):

(9.2) m^{***} *always has property* [L] *for* w^* (*and then for* lw^*).

We borrow from [2] the following

(9.3) Definition. m is *regular* when $m^{***} = m^{t***t}$. Arens proved that ([2], Theorem 3.3):

(9.4) *m is regular if and only if* m^{***} *has property* [R] *for* lw^*.

In view of (9.2) and (9.4) it is quite natural to take up the question of joint lw^*-continuity. This has been considered by Pym ([15]; cf. also Theorem 2.1 of [11]) and by McKilligan and White ([11], Theorem 2.2). Pym proved that m is regular iff all $a\in A$ are weakly almost periodic and McKilligan and White proved that m^{***} has property [J] for lw^* iff all $a\in A$ are almost periodic. Also for the "scalar case" (i.e., $X = Y$, Z = reals), this is considered in [8].

Our goal here is to give some other conditions closely related to [J] or [J, $(0, 0)$] for the $\vdash$ - product m^{***t}, but in the spirit of our general setup.

We begin by defining an auxiliary space $X^\#$ as follows: $X^\#$ is the vector space $L(X, B)$ of all linear bounded maps $u\colon X\to B$ made into a locally convex space by means of the seminorms

$$p(u) = \sup\{|\langle a, u(x)\rangle|;\ \|x\|\leqslant 1\}, \tag{9.5}$$

where a varies in A. It is clear that this topology is weaker than the norm topology of $L(X,B)$ and therefore the natural inclusion $L(X)\to X^{\#}$ is continuous.

Let us use the notation

$$m_s\colon X\to X^{\#},\qquad m_d\colon X\to X^{\#}$$

for the maps defined by $m_s\colon x\mapsto u$, $m_d\colon x\mapsto v$ with $u(y)=m(x,y)$, $v(y)=m(y,x)$.

(9.6) DEFINITION. We shall say that m is *s-compact* (resp., *d-compact*) when $m_s\colon X\to X^{\#}$ (resp., $m_d\colon X\to X^{\#}$) is a compact operator.

Then we have:

(9.7) THEOREM. *Let X be a Banach space and $m\colon X\times X\to X$ a bounded bilinear map. Consider the following conditions:*

(1) *m is s-compact;*

(2) *m is d-compact;*

(3) *m^{***} has property $[J,(0,0)]$ for lw^*;*

(4) *m^{***} has property $[J]$ for lw^*.*

Then:

(9.7.i) (3) *and* (4) *are equivalent;*

(9.7.ii) (1) *implies* (3) *and* (4);

(9.7.iii) (2) *implies* (3) *and* (4).

Moreover, if X^* is separable, then (1), (2), (3) *and* (4) *are equivalent.*

Proof. We still use the notation $A=X^*$, $B=X^{**}$.

Clearly, (4) implies (3). In order to prove the converse we first observe that for fixed $a_0\in A$, if the operator $T\colon b\mapsto b\vdash a_0$ is compact from B to A, then it is also lw^*-norm continuous. In fact, let $\{b_\alpha\}\subset B$ with $\|b_\alpha\|\leqslant 1$ and $b_\alpha\to 0$ for w^*. Let $\{b_\beta\}$ be an arbitrary subnet of $\{b_\alpha\}$. By compactness, there is a subnet $\{b_\gamma\}$ of $\{b_\beta\}$ with $\{Tb_\gamma\}$ convergent, so that $Tb_\gamma\to a$ in norm for some $a\in A$. However, $\langle x, m^{**}(b_\gamma,a_0)\rangle=\langle m^*(a_0,x),b_\gamma\rangle$ implies that $Tb_\gamma=m^{**}(b_\gamma,a_0)\to 0$ in the norm, as claimed. Now, using (4.1) we conclude that (3) implies (4).

Next we shall establish (1) $\Rightarrow$ (3). It will suffice to show that if m is s-compact and $a_0\in A$ is fixed, then $T\colon B\to A$, $Tb=m^{**}(b,a_0)$ is compact. Since T is the adjoint of $S\colon X\to A$, $Sx=m^*(a_0,x)$, this amounts to proving that S is compact. Let then $\{x_n\}$ be a bounded sequence in X. From (1) follows that there is a subnet $\{y_\alpha\}$ of $\{x_n\}$ such that $m_s(y_\alpha)$ is convergent in $X^{\#}$. This means that there is a bounded linear operator $u\colon X\to B$ such that for each seminorm p, $p\big(m_s(y_\alpha)-u\big)\to 0$. Then, if a is the element of X^* that defines p as in (9.5):

$$\sup\{|\langle a, m_s(y_\alpha)(x)\rangle-\langle a,u(x)\rangle|;\ \|x\|\leqslant 1\}\to 0$$

and so, by definition

$$\sup\{|\langle x, m^*(a,y_\alpha)-u^*(a)\rangle|;\ \|x\|\leqslant 1\}\to 0,$$

where $u^*\colon B^*\to A$ is the adjoint of u. This means that for each $a\in A$, $\|m^*(a,y_\alpha)-u^*(a)\|\to 0$, and so, in particular, $S(y_\alpha)\to u^*(a_0)$. Hence S is compact.

In order to show that (2) $\Rightarrow$ (4), we use implication (1) $\Rightarrow$ (4) applied to m^t to conclude that m^{t***} has $[J]$ for lw^*. But then (9.4) implies that m^t is regular. Now it follows from Theorem 3.3 in [2] (implication "(3.34) $\Rightarrow$ (3.32)") applied to m^{t***} that $(m^t)^t=m$ is regular, i.e., $m^{t***}=m^{***t}$, so that m^{***t} (and therefore m^{***} also) has $[J]$ for lw^* as observed above.

We assume now that A is separable and pick a sequence $D=\{a_n\}_{n=1}^\infty$ dense in A. Assume further that (4) holds. By the argument used in the proof of (1) $\Rightarrow$ (4), we conclude that $S_a\colon x\mapsto m^*(a,x)$ is a compact operator $S\colon X\to A$ for each $a\in A$. We shall abbreviate $S_n=S_a$ when $a=a_n$.

Let now $\{x_n\}$ be a bounded sequence in X. Since S_1 is compact, there is a subsequence $\{x_n'\}$ with $\{S_1x_n'\}$ convergent. Similarly, there is a subsequence $\{x_n''\}$ of $\{x_n'\}$ with $\{S_2x_n'\}$ convergent, and so on. Denote by $\{y_n\}$ the diagonal sequence produced by the iteration. Then $\{S_ky_n\}_{n=1}^\infty$ is convergent for each $k=1,2,\ldots$; let $d_k\in A$ be defined as $d_k=\lim_n S_ky_n$. We define $T\colon D\to A$ by $Ta_k=d_k$. Since $\|S_ky_n-S_jy_n\|=\|m^*(a_k-a_j,y_n)\|\leqslant K\|a_k-a_j\|$, where $K=\|m^*\|\operatorname{Sup}\|y_n\|$, then $\|d_k-d_j\|\leqslant K\|a_k-a_j\|$ and therefore T can be extended to a continuous map (with the same name) $T\colon A\to A$. We claim that T is linear. In fact, let k,j,i be positive integers. Then

$$\begin{aligned}\|T(a_k+a_j)-Ta_k-Ta_j\| &\leqslant \|T(a_k+a_j)-Ta_i\|+\|Ta_i-Ta_k-Ta_j\|\\ &= \|T(a_k+a_j)-Ta_i\|+\lim_n\|(S_i-S_j-S_k)y_n\|\\ &\leqslant \|T(a_k+a_j)-Ta_i\|+\lim_n\|m^*(a_i-a_k-a_j,y_n)\|\\ &\leqslant \|T(a_k+a_j)-Ta_i\|+K\|a_i-(a_k+a_j)\|.\end{aligned}$$

Picking the indices i so that $a_i\to a_k+a_j$, we get $\|T(a_k+a_j)-Ta_k-Ta_j\|=0$ and then, by continuity, $\|T(a'+a'')-Ta'-Ta''\|=0$ for all $a',a''\in A$. Therefore, $T\colon A\to A$ is a continuous linear operator. It is now easy to see that $m^*(a,y_n)\to Ta$ for each a. Let $u\colon X\to B$ be the restriction $u=T^*|X$ to X of the adjoint of T. Our final claim is that $m_s(y_n)\to u$ in $X^{\#}$. In fact, for $a\in A$, $x\in X$,

$$\begin{aligned}\langle a, m_s(y_n,x)\rangle-\langle a,u(x)\rangle &= \langle m(y_n,x),a\rangle-\langle a,Ta\rangle\\ &= \langle x, m^*(a,y_n)-Ta\rangle\end{aligned}$$

so that, with p the seminorm defined by $x^*=a$ in (9.5),

$$p\big(m_s(y_n)-u\big)=\|m^*(a,y_n)-Ta\|\to 0$$

and we are done.

We have therefore proven that (2)$\Rightarrow$(4) and (when A is separable), (4)$\Rightarrow$(1). Thus (2)$\Rightarrow$(1) and so, by symmetry, when A is separable, (1) and (2) are equivalent, and equivalent to (3) and (4), which completes the proof of (9.7).

It is clear from (9.4) and (9.7) that

(9.8) *If* (1) *or* (2) *is satisfied, then* m *is regular.*

One can see that the implications in (9.8) cannot be reversed in general by examining Example (5.2).

We close this section with the observation that s-compactness or d-compactness of m are properties that generalize weak compactness in the scalar case, as in [8], p. 5. In fact, the Grothendieck canonical extensions and the Arens products are closely related, as follows. For X, Y, Z Banach spaces and $m: X\times Y\to Z$ bounded and bilinear, define the *right canonical extension* $m^{\S}: X\times Y^{**}\to Z^{**}$ by first interpreting m as a map $X\to L(Y,Z)$ and then following it by the double-adjoint map $L(Y,Z)\to L(Y^{**},Z^{**})$. The resulting map $X\to L(Y^{**},Z^{**})$ determines $m^{\S}$ in the obvious way. In a similar way we can define ${}^{\S}m: X^{**}\times Y\to Z^{**}$ as the composition $Y\to L(X,Z)\to L(X^{**},Z^{**})$. The two iterated maps ${}^{\S}(m^{\S})$ and $({}^{\S}m)^{\S}$ have the same domain $(X^{**}\times Y^{**})$ and the same range (Z^{****}). It is (tedious but) routine matter to determine the following identities involving Arens products and Grothendieck extensions:

$$m^{***} = P\circ{}^{\S}(m^{\S}),\qquad m^{t***t} = P\circ({}^{\S}m)^{\S},$$

where $P: Z^{***}\to Z^{**}$ is the canonical projection.

10. The Hardy space H^∞. Let Ω be a plane domain. Denote by $B_H(\Omega)$ the vector space of all bounded holomorphic functions on Ω. We assume that Ω supports non-constant bounded holomorphic functions.

For $b\in B_H(\Omega)$ and μ a regular complex measure on Ω, write (with $z = x+yi$):

$$\langle\mu, b\rangle = \iint_\Omega b(z)\,\mu(dx\,dy). \tag{10.1}$$

This defines a duality between the space $M^1(\Omega)$ of all regular complex measures on Ω and $B_H(\Omega)$, with degeneracy. Thus, we can take $N = \{\mu\in M^1(\Omega);\ \langle\mu, b\rangle = 0,\ \forall b\in B_H(\Omega)\}$ and define $A = M^1(\Omega)/N$ as the quotient Banach space. It follows from [19] (4.3, 4.4 and 4.5) that the dual of A can be isometrically identified via the duality (10.1) with $B_H(\Omega)$ under the sup norm (that is, the Hardy space traditionally denoted by $H^\infty(\Omega)$, cf. [17], where the notations used here are introduced).

Also we can define, for $b\in B_H(\Omega)$, $a = \mu+N\in M^1(\Omega)/N$, the element $b\vdash a = b\mu+N$, where $b\mu\in M^1(\Omega)$ is the ordinary product of a bounded function and a measure. The product in $B_H(G)$ associated to $\vdash$ is again

the ordinary product $bc(z) = b(z)c(z)$ of holomorphic functions. Thus we find again the situation in Section 2 with $A = M^1(\Omega)/N$, $B = B_H(\Omega)$ and the ordinary product in $B_H(\Omega)$ as $\vdash$ - product.

The lw^*-continuity properties of this product are known:

(10.2) THEOREM (Rubel and Shields). *The product of $B_H(\Omega)$ is jointly lw^*-continuous.*

This result, however, depends on the fact, due to Rubel and Ryff, that lw^* is the strict topology of $B_H(\Omega)$, i.e., lw^* is the topology defined by the seminorms $\sup_{z\in\Omega}|b(z)\varphi(z)|$, where $\varphi\in C(\Omega)$ with $\varphi = 0$ on $\partial\Omega$. This result is hard to get (see [18]). We can, however, get Theorem (10.2) from Theorem (4.1) and the following easy consequence of the normal families Theorem.

(10.3) PROPOSITION. *Let $b_a(z)$ be holomorphic functions on Ω with $|b_a(z)|\leqslant M$ and assume that $b_a(z)\to 0$ for each $z\in\Omega$, as a increases. Then $\iint_\Omega |b_a(z)|\,|\mu|(dx\,dy)\to 0$ for each $\mu\in M^1(\Omega)$.*

Proof. Write

$$\iint_\Omega |b_a(z)|\,|\mu|(dx\,dy)\leqslant \iint_K |b_a(z)|\,|\mu|(dx\,dy)+M\iint_{\Omega-K}|\mu|(dx\,dy),$$

where $K\subset\Omega$ is a compact subset. Given $\varepsilon>0$, we can pick K such that the last integral does not exceed $\varepsilon/2M$. Now, from the normal families theorem follows that the integral in the middle is also small for a large, and we are done.

This proposition just means that $b\mapsto b\vdash a_0$ is lw^*-norm continuous from $B_H(\Omega)$ into $A = M^1(\Omega)/N$, and therefore from (4.1.ii) follows that the product is jointly lw^*-continuous, i.e., the statement in (10.2).

Similarly, we can use Theorem (3.1) to prove that:

(10.4) THEOREM. *The product in $B_H(\Omega)$ is not jointly w^*-continuous.*

Proof. We first give an elementary proof for Ω bounded. Pick $z_0\in\Omega$ and choose $a>0$ with $z_0+t\in\Omega$ for all $0\leqslant t\leqslant a$. Define the measure μ_n, $n = 0, 1, \ldots$, by

$$\iint_\Omega f(z)\,\mu_n(dx\,dy) = \int_0^a t^n f(z_0+t)\,dt.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be complex numbers and suppose that $\sum\lambda_k\mu_k\in N$, the annihilator of $B_H(\Omega)$ in $M^1(\Omega)$. Then, for $b\in B_H(\Omega)$,

$$0 = \iint_\Omega b(z)\Big(\sum\lambda_k\mu_k\Big)(dx\,dy) = \int_0^a b(z_0+t)P(t)\,dt,$$

where $P(z) = \sum\lambda_k z^k$. Thus, if $Q(z) = \sum\bar\lambda_k(z-z_0)^k$, then

$$0 = \int_0^a Q(z_0+t)P(t)\,dt = \int_0^a |P(t)|^2\,dt,$$

and therefore the polynomial $P(z)$ is identically zero, whence $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. But then $\mu_1+N, \mu_2+N, \ldots, \mu_n+N$ are linearly independent in $M^1(\Omega)/N$. Since $z^n \vdash (\mu_0+N) = \mu_n+N$, it follows that the range of $b \mapsto b \vdash a_0$ is infinite dimensional when $a_0 = \mu_0+N$. Then (10.4) follows from (3.1.ii).

The following proof that (3.1.iii) implies (10.4) for general Ω is due to L. A. Rubel, to whom we wish to express our appreciation.

First, take $f \in B_H(\Omega)$ non-constant and with $\|f\|_\infty = 1$. Let $\omega_n \in \Omega$ be all different for $n = 1, 2, \ldots$ with $|f(\omega_n)| \to 1$ and denote $z_n = f(\omega_n)$. It follows from [9], Corollary, p. 204, that a suitable subsequence (also denoted $\{z_n\}$) is an interpolating sequence for $B_H(\Omega)$. Hence, for each bounded sequence $\{w_n\}$ there is a $g \in B_H(D)$ (D = open unit disc) with $g(z_n) = \omega_n$. The same is true, of course, for $\{\omega_n\}$: there is $b \in B_H(\Omega)$ with $b(w_n) = w_n$ (take $b = g(f)$). Let now μ be the measure $\mu = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k$, where ε_k is the unit point mass at w_k. Clearly, if $b \in B_H(\Omega)$ with $b(\omega_k) = w_k$, then for $a_0 = \mu+N$ we have

$$b \vdash a_0 = \left(\sum 2^{-k} w_k \varepsilon_k\right) + N$$

and this shows that the range of $b \mapsto b \vdash a_0$ contains all the (finite) linear combinations of the measures ε_k and therefore it is not of finite rank.

Theorems (10.2) and (10.4) obviously imply that $w^* \neq lw^*$ (see Theorem 3.14 of [19]); in fact this can be obtained directly from the first and second propositions on p. 180 of [18] (cf. remark of last two lines of [18], loc. cit.).

We remark that the statements above extend with no changes to domains in C^n. In fact, in this case it is also true that the lw^*-topology coincides with the strict topology ([22], p. 476) and that $lw^* \neq w^*$.

11. Operator algebras. The supporting reference for this section is [5]. Let H be a Hilbert space, $L(H)$ the Banach algebra of all bounded linear operators in H under the operator norm and $L_*(H)$ the trace class, that is to say, the space consisting of all $T \in L(H)$ with $\sum (|T| e_a, e_a) < +\infty$, where $\{e_a\}_a$ is an arbitrary complete orthonormal system and $|T| = (T^*T)^{1/2}$. For $T \in L_*(H)$ the trace of T is the number $\mathrm{tr}(T) = \sum (Te_a, e_a)$. It is known that $L_*(H)$ and $\mathrm{tr}(T)$ do not depend on the complete orthonormal system $\{e_a\}_a$; also, $L_*(H)$ is a two sided ideal in $L(H)$, $\|T\|_* = \mathrm{tr}(|T|)$ is a norm on $L_*(H)$ that makes it a Banach space, and

$$\langle T, S\rangle = \mathrm{tr}(TS), \qquad T \in L_*(H),\ S \in L(H),$$

defines a pairing of $L_*(H)$ and $L(H)$ that identifies $L(H)$ with the dual of $L_*(H)$.

With this background, we can consider $A = L_*(H)$ and $B = L(H)$ from the point of view of Section 2, and a natural $\vdash$ - map suggests itself: $S \vdash T = TS$, the composition of operators. It is easy to see that the $\vdash$ - product is again composition of operators: $\langle T, S_1 \cdot S_2\rangle = \langle S_1 \vdash T, S_2\rangle = \langle TS_1, S_2\rangle = \mathrm{tr}(TS_1S_2) = \langle T, S_1S_2\rangle$. The w^* and lw^* continuity properties of the product in $L(H)$ are not hard to get (see [5], I. 3.5, Example 2, or [6], 2-nd paragraph on p. 293) and can be summed up as follows:

11.1. *The product in $L(H)$ has always properties* [L] *and* [R] *for w^* and lw^*; it has property* [J, (0, 0)] *for w^* or lw^* if and only if* $\dim(H) < +\infty$.

We aim to show here that our previous theorems are quite related to 11.1. A further property of the trace will be used (see [5], I. 6, Proposition 1), namely: if $T \in L_*(H)$, $S \in L(H)$, then $\mathrm{tr}(TS) = \mathrm{tr}(ST)$. It follows that if $U \in L(H)$, than also $\langle TU, S\rangle = \langle ST, U\rangle = \langle S \vdash T, U\rangle$. But then it is clear from this identity that if $S_a \xrightarrow{w^*} 0$, then $S_a \vdash T \to 0$ in the weak topology of $L_*(H)$. Consequently, (3.1.ii) implies that the product in $L(H)$ has property [L] for w^* (and hence also for lw^*); since it always has [R] by (3.1.ii), the first half of 11.1 follows.

The w^*- and lw^*-discontinuity when $\dim(H) = \infty$ is not hard to prove directly (see again [6], 2-nd paragraph on p. 293). It can be also obtained from (4.1.i) as follows: pick a countable orthonormal system $\{e_n\}_{n=1}^{\infty}$ and define $T_1 \in L(H)$, $S_n \in L_*(H)$ by $T_1 x = (x, e_1)e_1$, $S_n e_1 = e_n$, $S_n e_n = e_1$, $S_n e_j = e_j$ for $j \neq 1$, $j \neq n$, and $S_n = 0$ on the orthogonal complement to the span of $\{e_n\}$. One has $\|S_n\| = 1$ and $\|S_n \vdash T_1 - S_m \vdash T_1\|_* \geq 2$ when $n \neq m$, so that $S \mapsto S \vdash T_1$ from $L(H)$ into $L_*(H)$ is not compact. Now apply (4.1.i).

References

[1] R. Arens, *Operations induced in function classes*, Monatsh. Math. 55 (1951), pp. 1–19.

[2] — *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), pp. 839–848.

[3] N. Bourbaki, *Integration*, Chapters VII–VIII, Hermann, Paris.

[4] M. M. Day, *Normed linear spaces*, 3-rd Ed., Springer Verlag, 1972.

[5] J. Dixmier, *Les algebres d'opérateurs dans l'espace hilbertien*, 2-nd Ed., Gauthier-Villars, Paris 1969.

[6] E. Dubuc and H. Porta, *Convenient categories of topological algebras, and their duality theory*, J. Pure Appl. Algebra 1 (1971), pp. 281–316.

[7] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York 1958.

[8] A. Grothendieck, *Résumé de la théorie metrique des produits tennsoriels topologiques*, Bol. Soc. Mat. São Paulo 8 (1953), pp. 1–79.

[9] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, 1962.

[10] C. E. Kenig and H. Porta, *Weak continuity of Banach algebra products*, Bull. Amer. Math. Soc. 81 (1975), pp. 605–608.

[11] S. McKilligan and A. J. White, *Representation of L-algebras*, Proc. London Math. Soc., 3-rd Series, 25 (1972), pp. 655–674.
[12] D. Maharan, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. 28 (1942), pp. 108–111.
[13] Z. Phelps, *A Banach space characterization of purely atomic measure spaces*, Proc. Amer. Math. Soc. 12 (1961), pp. 447–452.
[14] H. Porta, *Compactly determined locally convex topologies*, Math. Ann. 196 (1972), pp. 91–100.
[15] J. S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math. Soc., 3-rd Series, 15 (1965), pp. 84–104.
[16] H. Rosenthal, *On quasicomplemented subspaces of Banach spaces with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$*, J. Funct. Analysis 4 (1969), pp. 176–214.
[17] L. A. Rubel, *Bounded convergence of analytic functions*, Bull. Amer. Math. Soc. 77 (1971), pp. 13–23.
[18] — and Z. Ryff, *The bounded weak-star topology and the bounded analytic functions*, J. Funct. Analysis 5 (1970), pp. 167–183.
[19] — and A. L. Shields, *Bounded approximation by polynomials*, Acta Math. 112 (1964), pp. 145–162.
[20] S. Sakai, *C*-algebras and W*-algebras*, Springer–Verlag, Berlin–New York 1971.
[21] J. H. Shapiro, *The bounded weak star topology and the general strict topology*, J. Funct. Analysis 8 (1971), pp. 275–286.
[22] — *Weak topologies on subspaces of $C(S)$*, Trans. Amer. Math. Soc. 157 (1971), pp. 471–479.
[23] A. C. Zaanen, *Integration*, 2-nd Ed., John Wilers, New York 1967.

UNIVERSITY OF CHICAGO
UNIVERSITY OF ILLINOIS, URBANA

Received April 18, 1975 (1002)

STUDIA MATHEMATICA, T. LIX. (1976)

The structure of L-ideals of measure algebras

by

KEIJI IZUCHI (Yokohama)

Abstract. This paper shows that there are L-ideals I_1 and I_2 of the measure algebra on a L. C. A. group such that $Z(I_1) = Z(I_2)$ and there are no L-ideals I such that $I_1 \subsetneqq I \subsetneqq I_2$.

1. Introduction. Let G be a non-discrete L.C.A. group and let $\hat{G}$ be the dual group of G. Let $L^1(G)$ and $M(G)$ be the group algebra on G and the measure algebra on G, respectively. We denote by $\mathrm{Rad}\,L^1(G)$ the radical of $L^1(G)$, that is, $\mathrm{Rad}\,L^1(G)$ is the intersection of all maximal ideals of $M(G)$ which contain $L^1(G)$. For $\mu, \nu \in M(G)$ $\nu \ll \mu$ means that ν is absolutely continuous with respect to μ, and $\nu \perp \mu$ means that ν and μ are mutually singular. For $\mu \in M(G)$, we put $L^1(\mu) = \{\lambda \in M(G);\ \lambda \ll \mu\}$. A closed subspace (ideal, subalgebra) N is called an *L-subspace* (*L-ideal*, *L-subalgebra*) if $L^1(\mu) \subset N$ for every $\mu \in N$.

Taylor [8] showed a characterization of the maximal ideal space of $M(G)$ as follows: There exist compact topological abelian semigroup S and an isometry isomorphism θ of $M(G)$ into $M(S)$ such that the maximal ideal space of $M(G)$ is identified with $\hat{S}$, the set of all continuous semi-characters of S, and the Gelfand transform of $\mu \in M(G)$ is given by $\hat{\mu}(f) = \int_S f d\theta\mu$ for $f \in \hat{S}$.

For a closed ideal I of $M(G)$, we put

$$Z(I) = \{f \in \hat{S};\ \hat{\mu}(f) = 0 \text{ for every } \mu \in I\}.$$

H. Helson ([2]) showed that: If I_1 and I_2 are closed ideals of $L^1(G)$ with $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2)$, then there is a closed ideal I such that $I_1 \subsetneqq I \subsetneqq I_2$.

In this paper, we show that Helson's theorem is not true in the category of L-ideals of $M(G)$ or in the category of closed ideals of $M(G)$. Our results are the following.

THEOREM 1. *There are two L-ideals I_1, I_2 of $M(G)$ such that $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2)$, but there are no L-ideals I so that $I_1 \subsetneqq I \subsetneqq I_2$.*