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The structure of L-ideals of measure algebras

by

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Abstract. This paper shows that there are L-ideals I_1 and I_2 of the measure algebra on a L. C. A. group such that $Z(I_1) = Z(I_2)$ and there are no L-ideals I such that $I_1 \subsetneqq I \subsetneqq I_2$.

1. Introduction. Let G be a non-discrete L.C.A. group and let $\hat{G}$ be the dual group of G. Let $L^1(G)$ and $M(G)$ be the group algebra on G and the measure algebra on G, respectively. We denote by $\operatorname{Rad} L^1(G)$ the radical of $L^1(G)$, that is, $\operatorname{Rad} L^1(G)$ is the intersection of all maximal ideals of $M(G)$ which contain $L^1(G)$. For $\mu, \nu \in M(G)$ $\nu \ll \mu$ means that ν is absolutely continuous with respect to μ, and $\nu \perp \mu$ means that ν and μ are mutually singular. For $\mu \in M(G)$, we put $L^1(\mu) = \{\lambda \in M(G);\ \lambda \ll \mu\}$. A closed subspace (ideal, subalgebra) N is called an *L-subspace* (*L-ideal*, *L-subalgebra*) if $L^1(\mu) \subset N$ for every $\mu \in N$.

Taylor [8] showed a characterization of the maximal ideal space of $M(G)$ as follows: There exist compact topological abelian semigroup S and an isometry isomorphism θ of $M(G)$ into $M(S)$ such that the maximal ideal space of $M(G)$ is identified with $\hat{S}$, the set of all continuous semi-characters of S, and the Gelfand transform of $\mu \in M(G)$ is given by $\hat{\mu}(f) = \int_S f d\theta\mu$ for $f \in \hat{S}$.

For a closed ideal I of $M(G)$, we put

$$Z(I) = \{f \in \hat{S};\ \hat{\mu}(f) = 0 \text{ for every } \mu \in I\}.$$

H. Helson ([2]) showed that: If I_1 and I_2 are closed ideals of $L^1(G)$ with $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2)$, then there is a closed ideal I such that $I_1 \subsetneqq I \subsetneqq I_2$.

In this paper, we show that Helson's theorem is not true in the category of L-ideals of $M(G)$ or in the category of closed ideals of $M(G)$. Our results are the following.

THEOREM 1. *There are two L-ideals I_1, I_2 of $M(G)$ such that $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2)$, but there are no L-ideals I so that $I_1 \subsetneqq I \subsetneqq I_2$.*

THEOREM 2. *There are closed ideals I_1, I_2 of $M(G)$ and an L-ideal I_3 of $M(G)$ such that $I_1 \subsetneqq I_2 \subsetneqq I_3$ and $Z(I_1) = Z(I_2) = Z(I_3)$, but there are neither closed ideals I with $I_1 \subsetneqq I \subsetneqq I_2$ nor with $I_2 \subsetneqq I \subsetneqq I_3$.*

THEOREM 3. *There exist L-ideals $\{I_k\}_{k=1}^{\infty}$ of $M(G)$ such that $L^1(G) \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \ldots \subsetneqq I_n \subsetneqq \ldots \subsetneqq \operatorname{Rad} L^1(G)$.*

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2. Proof of Theorem 1. After some lemmas, we show Theorem 1. For $\mu \in M(G)$, we denote by $I(\mu)$ the smallest L-ideal of $M(G)$ which contains μ. For L-ideals I_1, I_2 of $M(G)$ such that $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2)$, we say that (I_1, I_2) *has property* $(\#)$ if the following condition is satisfied:

$(\#)$ *There are no L-ideals I of $M(G)$ such that $I_1 \subsetneqq I \subsetneqq I_2$.*

For $x \in G$, $\delta(x)$ is the unit point mass at x. For an L-subspace N, we put $N^{\perp} = \{\mu \in M(G);\ \mu \perp N\}$. An L-ideal I is called *prime* if $I^{\perp}$ is an L-subalgebra.

LEMMA 1. *Let H be a closed subgroup of G. If there are L-ideals I_1, I_2 of $M(H)$ such that (I_1, I_2) has property $(\#)$, then there are L-ideals J_1, J_2 of $M(G)$ such that (J_1, J_2) has property $(\#)$.*

Proof. We change the topology of G by adjointing to the original collection τ of open sets arbitrary unions of sets of the form $(H+x) \cap V$, where $x \in G$ and $V \in \tau$. We denote its L.C.A. group by G_H. Then we have $M(G) = M(G_H) \oplus M(G_H)^{\perp}$, $M(G_H)^{\perp}$ is a prime L-ideal of $M(G)$ and $M(H) \subset M(G_H)$. Let $\tilde{I}_1$ $(\tilde{I}_2)$ be the L-ideal of $M(G_H)$ generated by I_1 (I_2). Then we have $\tilde{I}_1 \subsetneqq \tilde{I}_2$. Next we show $Z(\tilde{I}_1) = Z(\tilde{I}_2)$. Suppose that $Z(\tilde{I}_1) \supsetneqq Z(\tilde{I}_2)$. Then there is a complex homomorphism ψ of $M(G_H)$ such that $\psi(\lambda) = 0$ for every $\lambda \in \tilde{I}_1$, but there is $\nu \in \tilde{I}_2$ with $\psi(\nu) \neq 0$. Since $\tilde{I}_2$ coincides with the L-subspace generated by $\{\delta(x) * \eta;\ x \in G_H,\ \eta \in I_2\}$, there are $x \in G_H$ and $\eta \in I_2$ such that $\psi(\delta(x) * \eta) \neq 0$. Thus we have $\psi(\eta) \neq 0$, but $\psi(\lambda) = 0$ for every $\lambda \in I_1$. This contradicts $Z(I_1) = Z(I_2)$. We put $J_1 = \tilde{I}_1 \oplus M(G_H)^{\perp}$ and $J_2 = \tilde{I}_2 \oplus M(G_H)^{\perp}$. Then J_1 and J_2 are distinct L-ideals of $M(G)$, and $Z(J_1) = Z(J_2)$ is clear. Let J be an L-ideal such that $J_1 \subsetneqq J \subset J_2$. We put $I = \{\mu \in J;\ \mu \in M(G_H)\}$. Since $M(G_H)^{\perp}$ is a prime L-ideal, I is an L-ideal of $M(G_H)$ and $I_1 \subsetneqq I \subset I_2$. By the assumption of this lemma, we have $I = I_2$. Then $J = J_2$. Thus (J_1, J_2) has property $(\#)$.

Let K be a compact subgroup of G and let φ be the canonical homomorphism from G onto G/K. Then φ induces the homomorphism Φ from $M(G)$ onto $M(G/K)$. We denote by m_K the normalized Haar measure on K. We note that if $\mu_1, \mu_2 \in M(G)$ and $\Phi(\mu_1) = \Phi(\mu_2)$, then $\mu_1 * m_K = \mu_2 * m_K$.

LEMMA 2 ([1], Lemma 3.2). *Let $f \in \hat{S}$, the maximal ideal space of $M(G)$, such that $f = 1$ a.e. θm_K. Then there is a non-zero complex homomorphism ψ of $M(G/K)$ such that*

$$\hat{\mu}(f) = \psi(\Phi\mu) \quad \text{for every } \mu \in M(G).$$

Remark. If $f \in \hat{S}$ and $f \neq 1$ a.e. θm_K, then $\hat{m}_K(f) = 0$. Thus if $f \in \hat{S}$ and $\hat{m}_K(f) \neq 0$, then there is a non-zero complex homomorphism ψ of $M(G/K)$ such that $\hat{\mu}(f) = \psi(\Phi\mu)$ for every $\mu \in M(G)$.

LEMMA 3. *Let K be a compact subgroup of G. If there are L-ideals I_1, I_2 of $M(G/K)$ such that (I_1, I_2) has property $(\#)$, then there are L-ideals J_1, J_2 of $M(G)$ such that (J_1, J_2) has property $(\#)$.*

Proof. Let $J_1 = \Phi^{-1}(I_1)$; then J_1 is an L-ideal of $M(G)$. Let J_2' be an L-ideal of $M(G)$ generated by $\Phi^{-1}(I_2) * m_K$. Since $\Phi(\Phi^{-1}(I_2) * m_K) = I_2$, we have $\Phi(J_2') = I_2$. Suppose that J is an L-ideal of $M(G)$ such that $\Phi(J) = I_2$. Since $J \supset J * m_K = \Phi^{-1}(I_2) * m_K$, we have $J \supset J_2'$. We put $J_2 = J_1 + J_2'$; then J_2 is an L-ideal of $M(G)$, $\Phi(J_2) = \Phi(J_1) + \Phi(J_2') = I_1 + I_2 = I_2$ and $J_1 \subsetneqq J_2$. Next we show $Z(J_1) = Z(J_2)$. Suppose that $Z(J_2) \subsetneqq Z(J_1)$. Then there exist $\hat{f} \in S$ and $\mu \in J_2$ such that $\mu \in J_2'$, $\hat{\mu}(f) \neq 0$ and $\hat{\lambda}(f) = 0$ for every $\lambda \in J_1$. Since $\Phi^{-1}(I_2) * m_K$ is dense in J_2', we have $\hat{m}_K(f) \neq 0$. By the remark of Lemma 2, there is a non-zero complex homomorphism ψ of $M(G/K)$ such that $\psi(\Phi(\nu)) = \hat{\nu}(f)$ for every $\nu \in M(G)$. Then we have $\psi \in Z(I_1)$, but $\psi \notin Z(I_2)$. This contradicts $Z(I_1) = Z(I_2)$. Thus we have $Z(J_1) = Z(J_2)$. Let J be an L-ideal of $M(G)$ such that $J_1 \subsetneqq J \subset J_2$. Then we have $I_1 = \Phi(J_1) \subset \Phi(J) \subset \Phi(J_2) = I_2$. Since $J_1 = \Phi^{-1}(I_1)$, we have $I_1 \subsetneqq \Phi(J) \subset I_2$. By the assumption of this lemma, we have $\Phi(J) = I_2$. This implies $J_2' \subset J$. Since $J_2 = J_1 + J_2' \subset J$, we get $J_2 = J$. Thus (J_1, J_2) has property $(\#)$.

LEMMA 4. *Suppose that there is a positive measure μ on G satisfying the following conditions*: (1) *and* (2); *then there are L-ideals I_1, I_2 of $M(G)$ such that (I_1, I_2) has property $(\#)$.*

(1) *$I(\mu) = I(\lambda)$ for every non-zero $\lambda \ll \mu$.*

(2) *There exist $f \in \hat{S}$ and a positive number b $(0 < b < 1)$ such that $|f| = b$ a.e. $\theta\mu$.*

Proof. Let μ be a positive measure satisfying the conditions of this lemma. We put $I_2 = I(\mu)$ and put $I_1 = \{\lambda \in I_2;\ |\lambda| * M(G) \perp \mu\}$; then I_1 is an L-ideal, $I_1 \perp \mu$ and $I_1 \subsetneqq I_2$. By condition (2), there exists $f \in \hat{S}$ and $0 < b < 1$ such that $|f| = b$ a.e. $\theta\mu$. Since $|f| = b^2$ a.e. $\theta(\mu * \mu)$, we have $(\mu * \mu) * M(G) \perp \mu$. Thus we have $\mu * \mu \in I_1$. Since $Z(I(\mu)) = Z(I(\mu * \mu))$ and $I(\mu * \mu) \subset I_1 \subset I_2$, we have $Z(I_1) = Z(I_2)$. Next we show that there are no L-ideals I such that $I_1 \subsetneqq I \subsetneqq I_2$. Suppose that I is an L-ideal such that $I_1 \subsetneqq I \subset I_2$. Then there exists $\nu \in I$ such that $\nu \geqslant 0$ and $\nu \perp I_1$. By the definition of I_1, there exists $\eta \in M(G)$ such that $\eta > 0$ and $\nu * \eta$ non $\perp \mu$. Then there is $\lambda_0 \in L^1(\mu)$ so that $\lambda_0 \neq 0$ and $\lambda_0 \ll \nu * \eta$. By condition (1), we have

$I(\lambda_0) = I(\mu) = I_2$. Since $\lambda_0 \in I$, we have $I(\lambda_0) \subset I$. Thus $I = I_2$. This completes the proof.

Let G_1 be one of the following metrizable L.C.A. groups;

(1) a countable product of finite cyclic groups $\prod_{i=1}^{\infty} Z(p_i)$,

(2) a p-adic integer group Δ_p,

(3) the circle group T,

(4) the additive group of the real line R.

In [5], Johnson shows that there is a measure μ on G_1 which satisfies condition (2) of Lemma 4. We will show that such μ satisfies condition (1) of Lemma 4.

Let $V_0, V_1, \ldots$ be a basic system of compact neighbourhoods of 0 in G_1 with $V_i \supset V_{i+1} + V_{i+1}$; $m_1, m_2, \ldots$, positive integers and x_{ij}, $i = 1, 2, \ldots, j = 0, 1, \ldots, m_i - 1$ such that $x_{i0} = 0$, $x_{ij} \in V_{i-1}$, and V_{i-1} is equal to the union of the disjoint sets $x_{ij} + V_i$, $j = 0, 1, \ldots, m_i - 1$. Let

$$X_i = \{x_{ij};\ j = 0, 1, \ldots, m_i - 1\}, \qquad Y_i = X_1 + X_2 + \ldots + X_i,$$

$$M_i = \frac{2}{m_i(m_i+1)}, \qquad \delta_i = M_i \sum_{j=0}^{m_i-1} (m_i - j)\,\delta(x_{ij}),$$

$\mu = \mathop{\ast}_{i=1}^{\infty} \delta_i$, and $\mu_n = \mathop{\ast}_{i=n+1}^{\infty} \delta_i$ (convergence in the weak-* topology). Then μ_n is supported in V_n, $\mu_n \geqslant 0$ and $\|\mu_n\| = 1$. We note that

$$\mu = \sum_{y \in Y_i} \mu(y + V_i)\,\delta(y) * \mu_i$$

and $\delta(y) * \mu_i \perp \delta(y') * \mu_i$ for distinct $y, y' \in Y_i$.

Lemma 5. *$L^1(\mu)$ is the closed linear span of $\{\delta(y) * \mu_i;\ y \in Y_i$, $i = 1, 2, \ldots\}$.*

Proof. Since $\bigcap_{i=0}^{\infty} V_i = \{0\}$ and V_{i-1} is the union of the disjoint sets $x_{ij} + V_i$, Lemma 5 is clear.

Remark. For $\varepsilon > 0$ and for $\nu \in L^1(\mu)$, there are a positive integer m and complex numbers $\{a(y)\}_{y \in Y_m}$ such that

$$\Big\|\nu - \sum_{y \in Y_m} a(y)\,\delta(y) * \mu_m\Big\| < \varepsilon.$$

Lemma 6. *Let N be an L-subspace and $\lambda \in M(G_1)$. If for every $\nu \ll \lambda$ there is $\eta \in N$ such that λ non $\perp \eta$, then $\lambda \in N$.*

Proof. It is clear by Lebesgue's decomposition theorem.

Lemma 7. *Such a measure μ as above satisfies condition* (1) *of Lemma* 4.

Proof. Let $\lambda \in L^1(\mu)$ and $\lambda \neq 0$. Then we have $I(\lambda) \subset I(\mu)$. Next we show that $\lambda \in L^1(\mu)$ implies $\mu \in I(\lambda)$. To show this fact, it is sufficient to show that for every $\nu \ll \mu$ there is $x \in G_1$ such that $\delta(x) * \lambda$ non $\perp \nu$ by Lemma 6. Here we may assume that $\lambda \geqslant 0$, $\nu \geqslant 0$ and $\|\lambda\| = \|\nu\| = 1$. By Lemma 5, there are a positive integer n and complex numbers $\{a(y), b(y)\}_{y \in Y_n}$ such that

$$\Big\|\lambda - \sum_{y \in Y_n} a(y)\,\delta(y) * \mu_n\Big\| < \tfrac{1}{4} \tag{1}$$

and

$$\Big\|\nu - \sum_{y \in Y_n} b(y)\,\delta(y) * \mu_n\Big\| < \tfrac{1}{4}. \tag{2}$$

For $y \in Y_n$, we can decompose

$$\mu_n = \mu_{n,y,\lambda} + \mu'_{n,y,\lambda}, \tag{3}$$

where $\delta(y) * \mu_{n,y,\lambda} \ll \lambda$ and $\delta(y) * \mu'_{n,y,\lambda} \perp \lambda$, and

$$\mu_n = \mu_{n,y,\nu} + \mu'_{n,y,\nu}, \tag{4}$$

where $\delta(y) * \mu_{n,y,\nu} \ll \nu$ and $\delta(y) * \mu'_{n,y,\nu} \perp \nu$. Since

$$\Big\|\lambda - \sum_{y \in Y_n} a(y)\,\delta(y) * \mu_n\Big\| = \Big\|\lambda - \sum_{y \in Y_n} a(y)\,\delta(y) * \mu_{n,y,\lambda}\Big\| + \Big\|\sum_{y \in Y_n} a(y)\,\delta(y) * \mu'_{n,y,\lambda}\Big\| < \tfrac{1}{4},$$

we have

$$\Big\|\sum_{y \in Y_n} a(y)\,\delta(y) * \mu'_{n,y,\lambda}\Big\| < \tfrac{1}{4}. \tag{5}$$

On the other hand, (1) implies that

$$\tfrac{3}{4} < \sum_{y \in Y_n} |a(y)| < \tfrac{5}{4}. \tag{6}$$

Here we show that there is $y_0 \in Y_n$ such that $a(y_0) \neq 0$ and $\|\mu_{n,y_0,\lambda}\| > \frac{1}{2}$. Suppose that $\|\mu_{n,y,\lambda}\| \leqslant \frac{1}{2}$ for every $y \in Y_n$ with $a(y) \neq 0$. Then we have $\|\mu'_{n,y,\lambda}\| > \frac{1}{2}$ for every $y \in Y_n$ with $a(y) \neq 0$.

By (5), we have

$$\tfrac{1}{4} > \Big\|\sum_{y \in Y_n} a(y)\,\delta(y) * \mu'_{n,y,\lambda}\Big\| = \sum_{y \in Y_n} |a(y)|\,\big\|\mu'_{n,y,\lambda}\big\| > \tfrac{1}{2} \sum_{y \in Y_n} |a(y)|.$$

Hence we have $\frac{1}{2} > \sum_{y \in Y_n} |a(y)|$. This contradicts (6). Thus there is $y_0 \in Y_n$ such that

$$a(y_0) \neq 0 \quad \text{and} \quad \|\mu_{n,y_0,\lambda}\| > \tfrac{1}{2}. \tag{7}$$

By the same argument as above, there is $y_1 \in Y_n$ such that

(8) $$b(y_1) \neq 0 \quad \text{and} \quad \|\mu_{n,y_1,\nu}\| > \tfrac{1}{2}.$$

By (3) and (4), there are characteristic functions χ_1 and χ_2 such that $\mu_{n,y_0,\lambda} = \chi_1 \mu_n$ and $\mu_{n,y_1,\nu} = \chi_2 \mu_n$. By (7) and (8), we have $\mu_{n,y_0,\lambda}$ non $\perp \mu_{n,y_1,\nu}$. This implies that

$$\delta(y_1 - y_0) * \big(\delta(y_0) * \mu_{n,y_0,\lambda}\big) \text{ non} \perp \delta(y_1) * \mu_{n,y_1,\nu},$$

thus we have $\delta(y_1 - y_0) * \lambda * \nu$. This completes the proof.

LEMMA 8 ([5], 3.1, 4.1, 5.1, 5.4). *There exist $\{V_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ such that μ satisfies condition* (2) *of Lemma* 4.

By Lemmas 1, 2, 3, 7 and 8, we have Theorem 1 as usual.

3. Proofs of another theorems. We put $M_c = \{\mu \in M(G);\ \mu$ is a continuous measure$\}$, then M_c is an L-ideal of $M(G)$. For a subset N of $M(G)$, $[N]$ means the closed subalgebra generated by N. By Varopoulos [9], we have $[M_c * M_c] \subsetneqq M_c$ and $[M_c * M_c]$ is an L-ideal of $M(G)$.

Proof of Theorem 2. Let $I_3 = M_c$ and $I_0 = [M_c * M_c]$; then I_0 is an L-ideal, $I_0 \subsetneqq I_3$ and $Z(I_0) = Z(I_3)$. We can decompose $I_3 = I_0 \oplus N$ such that $N \subset M_c$ and $N \perp I_0$. For $\gamma_1, \gamma_2 \in \hat{G}$ $(\gamma_1 \neq \gamma_2)$, we put $I_4 = \{\mu \in N;\ \hat{\mu}(\gamma_1) = 0\}$ and $I_5 = \{\mu \in N;\ \hat{\mu}(\gamma_1) = \hat{\mu}(\gamma_2) = 0\}$. Then I_4 and I_5 are closed subspaces of N and $I_5 \subsetneqq I_4$. We put $I_1 = I_0 \oplus I_5$ and $I_2 = I_0 \oplus I_4$; then I_1 and I_2 are closed ideals of $M(G)$, $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2) = Z(I_3)$. At first, we show that there are no closed ideals I such that $I_2 \subsetneqq I \subsetneqq I_3$. Suppose that I is a closed ideal with $I_2 \subsetneqq I \subset I_3$. Then there exists $\lambda \in I$ such that $\lambda \notin I_2$ and $\lambda \in N$, so we have $\hat{\lambda}(\gamma_1) \neq 0$. For $\nu \in N$, we have

$$\left(\nu - \frac{\hat{\nu}(\gamma_1)}{\hat{\lambda}(\gamma_1)}\lambda\right)^{\wedge}(\gamma_1) = 0.$$

Thus

$$\nu - \frac{\hat{\nu}(\gamma_1)}{\hat{\lambda}(\gamma_1)}\lambda \in I_4 \quad \text{and} \quad \nu \in \frac{\hat{\nu}(\gamma_1)}{\hat{\lambda}(\gamma_1)}\lambda + I_4 \subset I.$$

This implies that $I_3 = I_0 \oplus N \subset I$ and $I = I_3$. As the same way as the above, we can prove the fact that there are no closed ideals I such that $I_1 \subsetneqq I \subsetneqq I_2$.

As the same way as Theorem 2, we have the following:

THEOREM 2'. *Let M be an L-ideal of $M(G)$ such that $M \neq [M * M]$. Then there are closed ideals I_1 and I_2 such that $I_1 \subsetneqq I_2 \subsetneqq M$ and $Z(I_1) = Z(I_2) = Z(M)$, but there are neither closed ideals I with $I_1 \subsetneqq I \subsetneqq I_2$ nor with $I_2 \subsetneqq I \subsetneqq M$.*

For subsets N, M of $M(G)$, we put $N * M = \{\mu_1 * \mu_2;\ \mu_1 \in N,\ \mu_2 \in M\}$ and we put $N^n = N * N^{n-1}$ for a positive integer n.

To prove Theorem 3, we use the following lemma.

LEMMA 9 ([3], P. 419). *For each integer $k > 1$, there is a non-zero positive measure μ in* $\operatorname{Rad} L^1(G)$ *such that $\mu^n \perp L^1(G)$ $(n = 1, 2, \ldots, k-1)$, where $\mu^n = \mu * \mu^{n-1}$ and $\mu^k \in L^1(G)$.*

Proof of Theorem 3. We construct such L-ideals $\{I_k\}_{k=1}^{\infty}$ inductively. The first step, there is a measure $\mu_1 > 0$ such that $\mu_1 \perp L^1(G)$ and $\mu_1^2 \in L^1(G)$ by Lemma 9. We put $I_1 = I(\mu_1)$. Since $L^1(G)$ is the smallest L-ideal of $M(G)$, we have $L^1(G) \subsetneqq I_1 \subsetneqq \operatorname{Rad} L^1(G)$ and $I_1 * I_1 \subset L^1(G)$. Suppose that for a positive integer k there are L-ideals $I_1, I_2, \ldots, I_k$ and a positive integer $s(k)$ such that $L^1(G) \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \ldots \subsetneqq I_k \subsetneqq \operatorname{Rad} L^1(G)$ and $I_k^{s(k)} \subset L^1(G)$. By Lemma 9, there is a measure $\mu_{k+1} > 0$ such that $\mu_{k+1}^n \perp L^1(G)$ $\big(n = 1, 2, \ldots, s(k)\big)$ and $\mu_{k+1}^{s(k)+1} \in L^1(G)$. We put $I_{k+1} = I_k + I(\mu_{k+1})$; then I_{k+1} is an L-ideal. Since $\mu_{k+1}^{s(k)} \perp L^1(G)$, we have $\mu_{k+1} \notin I_k$ and $\mu_{k+1} \in I_{k+1}$. Thus $I_k \subsetneqq I_{k+1}$. We put $s(k+1) = 2s(k)$; then we have $I_{k+1}^{s(k+1)} \subset L^1(G)$. This implies that $I_{k+1} \subsetneqq \operatorname{Rad} L^1(G)$. This completes the proof.

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