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ON A GENERALIZATION OF DUGUNDJI EXTENSION THEOREM

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For every pair of spaces Y and Z, Y being a topological vector space, denote by $C_Y(Z)$ the vector space of all continuous maps $f\colon Z \to Y$.

Let $A \subset X$ be a closed subspace of a space X.

A pair (X, A) has *property* D if, for each locally convex vector space Y, there exists a linear operator $l\colon C_Y(A) \to C_Y(X)$ such that, for every $f \in C_Y(A)$, $l(f)(x) = f(x)$ for $x \in A$, $l(f)(X) = \operatorname{conv}\{f(x)\colon x \in A\}$ and $\|l\| = 1$. If the above is valid for each A closed in X, then the conclusion of the Dugundji Extension Theorem is valid for X and we say that the space X has *property* D.

A space X has a *completely adequate system* (C.A. system; see O'Reilly [7]) if there exists a family $\mathscr{V} = \{V(x)\colon x \in X\}$ of neighbourhood bases $V(x)$ at x with the property that, for every $x \in X$ and every open set $U \subset X$ containing x, there exists an open set $N[x; U]$ with $x \in N[x; U] \subset U$ and such that, for every $y \in N[x; U]$, there exists a neighbourhood $V(y) \in \mathscr{V}(y)$ with $N[x; U] \subset V(y) \subset U$.

A C.A. system $\mathscr{V}$ is called *well ordered* if each $\mathscr{V}(x)$ is well ordered by inclusion, and $\mathscr{V}$ is called *countable* if each $\mathscr{V}(x)$ has only countably many elements.

In [7] it is proved that each space which has a well-ordered C.A. system is hereditarily paracompact and that metrizable spaces and spaces with topologies induced by uniformities having linearly ordered bases (with respect to refinements) have well-ordered C.A. systems (a characterization of these spaces is given by Kucia and this author in [6]).

It is known that the Dugundji Extension Theorem is valid for metrizable spaces (Dugundji [4]), stratifiable spaces (Borges [2]), and subspaces of linearly ordered spaces (Banilower [1]). An example of a compact space for which the Dugundji Extension Theorem is not valid is βN (Gęba and Semadeni [5]). It seems that the class of spaces with property D is rather small.

THEOREM. *If a space X has a well-ordered C.A. system and A is a closed subset of X with compact boundary, then the pair (X, A) has property* D; *if a C.A. system is ordered as natural numbers, then this is valid without any assumption on the boundary of A, i.e. spaces with C.A. systems ordered as natural numbers have property* D.

Proof. Since a C.A. system is well ordered, it follows from the paracompactness of the topology that in the definition of a C.A. system the condition $N[x; U] \subset U$ can be replaced by the condition $\mathrm{cl}\, N[x; U] \subset U$. We can also assume, without loss of generality, that the C.A. system is such that $X \in \mathscr{V}(x)$ for each $x \in X$.

Let A be a closed subset of X. Assume that the boundary $\mathrm{Fr}A = \mathrm{cl}A \cap \mathrm{cl}X - A$ is not empty, for otherwise, if $\mathrm{Fr}A = \emptyset$, then the linear operator $l\colon C_Y(A) \to C_Y(X)$ can be defined by $l(f)(x) = f(x)$ if $x \in A$ and by $l(f)(x) = f(x_0)$ for each $f \in C_Y(A)$ and some fixed $x_0 \in A$.

For each $x \in X - A$, let W_x be the least element of $\mathscr{V}(x)$ such that $\mathrm{cl}\, W_x \cap A = \emptyset$. There is a partition of unity $\{\varphi_s\colon s \in S\}$, $\varphi_s\colon X - A \to [0, 1]$, which is subordinate to the covering $P = \{N[x; W_x]\colon x \in X - A\}$ of $X - A$, i.e., for each $x \in X - A$, we have

$$\sum_{s \in S} \varphi_s(x) = 1,$$

and $\{\varphi_s^{-1}(0, 1]\colon s \in S\}$ is a locally finite refinement of P. For each $s \in S$, let us choose a point c_s,

$$c_s \in \bigcap \{\mathrm{cl}\, V(c(s)) \cap \mathrm{Fr}A\colon \mathrm{cl}\, V(c(s)) \cap \mathrm{Fr}A \neq \emptyset,\ V(c(s)) \in \mathscr{V}(c(s))\},$$

where $c(s)$ is an arbitrary point of $X - A$ such that

$$\varphi_s^{-1}(0, 1] \subset N[c(s); W_{c(s)}]$$

(such a point c_s does exist since we assume that either $\mathrm{Fr}A$ is compact or $\mathscr{V}(c(s))$ is countable). For each $f \in C_Y(A)$, let us put

$$l(f)(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \sum_{s \in S} f(c_s)\varphi_s(x) & \text{if } x \in X - A. \end{cases}$$

It is obvious that $l(f)$ is continuous at each point belonging to $(X - A) \cup \mathrm{int}A$. We show that $l(f)$ is continuous for each $a \in \mathrm{Fr}A$.

Let V be a convex neighbourhood of the point $f(a) = l(f)(a)$. There is an open neighbourhood U_1 of a such that $f(U_1 \cap A) \subset V$. Put $H = N[a; N[a; U]]$, where U is an open neighbourhood of a such that $\mathrm{cl}\, U \subset U_1$. We show that $l(f)(H) \subset V$. It is clear that $l(f)(H \cap A) \subset V$. Now, let $x \in H \cap (X - A)$ and let $s \in S$ be such that $\varphi_s(x) > 0$. Since

$$x \in \varphi_s^{-1}(0, 1] \subset N[c(s); W_{c(s)}],$$

there is a $V(x) \in \mathscr{V}(x)$ such that

$$N[c(s); W_{c(s)}] \subset V(x) \subset W_{c(s)}.$$

Since $x \in H$, there exists a $V'(x) \in \mathscr{V}(x)$ with $H \subset V'(x) \subset N[a; U]$. Notice that $V(x) \cap A = \emptyset$ and $V'(x) \cap A \neq 0$. Hence $V(x) \subset V'(x)$. This implies that

$$c(s) \in N[c(s); W_{c(s)}] \subset N[a; U].$$

There exists a $V\big(c(s)\big) \in \mathscr{V}\big(c(s)\big)$ such that $N[a; U] \subset V\big(c(s)\big) \subset U$. But $c_s \in \operatorname{cl} V\big(c(s)\big)$, since $\operatorname{cl} V\big(c(s)\big) \cap \operatorname{Fr} A \neq \emptyset$. Hence $c_s \in \operatorname{cl} U \subset U_1$.

Thus, if $x \in H \cap X - A$, then $f(c_s) \in V$ for each $s \in S$ such that $\varphi_s(x) > 0$. Hence, for each $x \in H \cap (X - A)$,

$$l(f)(x) = \sum_{s \in S} f(c_s) \varphi_s(x),$$

and thus $l(f)(x)$, being a convex combination of elements belonging to V, belongs itself to V.

It follows immediately from the definition of $l(f)$ that l is a linear operator with $\|l\| = 1$ and $(f)(X) \subset \operatorname{conv} f(A)$ for each $f \in C_Y(A)$.

Remark. There exist non-metrizable spaces with countable C.A. systems, which are not subspaces of linearly ordered spaces. The following example was communicated to the author by Dr. K. Alster:

Example. Let $X = I^2 \times I_Q$, where $I = [0, 1]$ with the usual topology, and I_Q is the interval I with the topology such that irrational points are isolated and rational points have usual bases of neighbourhoods. The space X is non-metrizable, has a countable C.A. system, but it is not a subspace of any linearly ordered space.

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