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A CONNECTION IN A DIFFERENTIAL MODULE

BY

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1. Preliminary definitions and results. Let $(M, \mathscr{C})$ be a differential space ([6], p. 47). We shall denote by $\tau_{\mathscr{C}}$ the weakest topology for M such that all functions in $\mathscr{C}$ are continuous. We set

$$\mathscr{C}(p) = \bigcup_{p \in A \in \tau_{\mathscr{C}}} \mathscr{C}_A \quad \text{for } p \in M.$$

Then the set $\mathscr{C}(p)$ with the addition

$$(\alpha+\beta)(q) = \alpha(q)+\beta(q) \quad \text{for } q \in A \cap B$$

and the multiplication

$$(\alpha\beta)(q) = \alpha(q)\beta(q) \quad \text{for } q \in A \cap B,$$

where α, $\beta \in \mathscr{C}(p)$, $\alpha \in \mathscr{C}_A$ and $\beta \in \mathscr{C}_B$, is a linear algebraic ring over the field E of real numbers. Moreover, a set $T_p(M, \mathscr{C})$ of all linear mappings $v\colon \mathscr{C}(p) \to E$ such that

$$v(\alpha\beta) = \alpha(p)v(\beta)+\beta(p)v(\alpha) \quad \text{for } \alpha, \beta \in \mathscr{C}(p)$$

is a linear space isomorphic (in a natural way) to a linear space $(M, \mathscr{C})_p$ of all linear mappings $v\colon \mathscr{C} \to E$ such that

$$v(\alpha\beta) = \alpha(p)v(\beta)+\beta(p)v(\alpha) \quad \text{for } \alpha, \beta \in \mathscr{C}.$$

We observe that if the topological space $(M, \tau_{\mathscr{C}})$ is a T_2-space, then

$$T_p(M, \mathscr{C}) \cap T_q(M, \mathscr{C}) = \emptyset \quad \text{for } p \neq q \ (p, q \in M).$$

In the sequel we shall identify $T_p(M, \mathscr{C})$ with $(M, \mathscr{C})_p$ and we shall denote it, simply, by M_p for $p \in M$. Thus M_p is a linear space of all vectors tangent to $(M, \mathscr{C})$ at a point $p \in M$ ([6], p. 47).

We shall denote an n-dimensional Euclidean space by E^n. The symbol $\mathscr{E}_n$ denotes the set of all real-valued infinitely differentiable functions defined on E^n. We use $D_i(x)$ to denote a vector in $(E^n)_x$ $(x=(x^1, \ldots, x^n) \in E^n)$

given by

$$D_i(x)(a) = \frac{\partial a}{\partial x^i}(x) \quad \text{for } a \in \mathscr{E}_n,$$

where $\partial a(x)/\partial x^i$ denotes the partial derivative of a with respect to x^i. A symbol $\mathbf{d}f_p$ denotes the differential of a smooth mapping f at a point p ([6], p. 48).

Throughout this paper we assume that $(M, \mathscr{C})$ is a differential space of the differential dimension m ([6], p. 56); as the topological space $(M, \tau_{\mathscr{C}})$ it is a T_2-space. We assume also that $M_p \cap M_q = \emptyset$ for $p \neq q$ $(p, q \in M)$.

Let $\mathscr{W}$ be an n-dimensional differential $\mathscr{C}$-module of Φ-fields on $(M, \mathscr{C})$ ([6], p. 52). We shall denote by $(Q, \mathscr{F})$ the differential space of the differential module $\mathscr{W}$ ([6], p. 67). This differential space has differential dimension $m+n$ [2]. We shall denote by π a projection of Q onto M ([6], p. 67).

For every Φ-field W on M, we set

$$\overline{W}(p) = \big(p, W(p)\big) \quad \text{for } p \in M. \tag{1.1}$$

Then $W \in \mathscr{W}$ iff the mapping $\overline{W}$: $M \to Q$ is smooth ([6], p. 67). If $W_1, \ldots, W_n$ is a vector basis of a $\mathscr{C}$-module $\mathscr{W}$ on an open set A ([6], p. 51), then a mapping f: $A \times E^n \to Q$, defined by

$$f\big(p, (x^1, \ldots, x^n)\big) = \big(p, x^i W_i(p)\big) \quad \text{for } p \in A \text{ and } (x^1, \ldots, x^n) \in E^n, \tag{1.2}$$

will be called *fundamental with respect to this vector basis* or, simply, *fundamental* ([6], p. 67). We shall denote a $\mathscr{C}$-module of all smooth vector fields tangent on $(M, \mathscr{C})$ by $\mathfrak{M}(M)$, and the differential space of $\mathfrak{M}(M)$ by $(TM, \mathscr{T}\mathscr{C})$. Let $V_1, \ldots, V_m$ be a vector basis of a $\mathscr{C}$-module $\mathfrak{M}(M)$ on the set A, and let f be given by (1.2). Then the sequence $Z_1, \ldots, Z_{m+n}$ given by

$$Z_i\big(f(p, x)\big) = \mathbf{d}f(\cdot, x)_p V_i(p) \quad \text{for } (p, x) \in A \times E^n \text{ and } i = 1, \ldots, m, \tag{1.3}$$

$$Z_{m+j}\big(f(p, x)\big) = \mathbf{d}f(p, \cdot)_x D_j(x) \quad \text{for } (p, x) \in A \times E^n \text{ and } j = 1, \ldots, n \tag{1.4}$$

is a vector basis of an $\mathscr{F}$-module $\mathfrak{M}(Q)$ on $\pi^{-1}(A)$ [2]. The vector basis $Z_1, \ldots, Z_{m+n}$ given by (1.3) and (1.4) will be called *associated* with the vector bases $V_1, \ldots, V_m$ and $W_1, \ldots, W_n$. If $Z'_{1'}, \ldots, Z'_{m'+n'}$ is another vector basis of $\mathfrak{M}(Q)$ associated with a vector basis $V'_{1'}, \ldots, V'_{m'}$ of $\mathfrak{M}(M)$ on A' and with a vector basis $W'_{1'}, \ldots, W'_{n'}$ of $\mathscr{W}$ on A', then

$$Z'_{i'}|B = a^i_{i'} \circ \pi \cdot Z_i|B + a^i_{i'} \circ \pi \cdot \beta^{j'}_j \circ \pi \cdot (a^j|B) \cdot Z_i(\beta^k_{j'} \circ \pi) \cdot Z_{m+k}|B \quad \text{for } i' = 1', \ldots, m', \tag{1.5}$$

$$Z'_{m'+j'}|B = \beta^j_{j'} \circ \pi \cdot Z_{m+j}|B \quad \text{for } j' = 1', \ldots, n', \tag{1.6}$$

where

$$V'_{i'}|A\cap A' = a^i_{i'} V_i|A\cap A' \quad \text{for } i' = 1', \ldots, m',$$
$$W'_{j'}|A\cap A' = \beta^j_{j'} W_j|A\cap A', \quad W_j|A\cap A' = \beta^{j'}_j W_{j'}|A\cap A'$$
$$\text{for } j' = 1', \ldots, n',\ j = 1, \ldots, n,$$

$B = \pi^{-1}(A\cap A')$ and $a^1, \ldots, a^n \in \mathscr{F}_{\pi^{-1}(A)}$ are given by

(1.7) $$w = a^j(p, w)W_j(p) \quad \text{for } (p, w)\in \pi^{-1}(A)$$

(see [2]).

Let $Z_1, \ldots, Z_{m+n}$ be the vector basis of $\mathfrak{M}(Q)$ associated with the vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A and with the vector basis $W_1, \ldots, W_n$ of $\mathscr{W}$ on A. It can be proved (see [2]) that

(1.8) $\mathbf{d}\overline{W}_p v = a^i Z_i(\overline{W}(p)) + v(\varphi^j) Z_{m+j}(\overline{W}(p))$ for $v\in M_p$, $W\in\mathscr{W}$ and $p\in A$,

where

(1.9) $W|A = \varphi^j W_j$, $v = a^i V_i(p)$ and $\overline{W}$ is given by (1.1);

(1.10) $\mathbf{d}\pi_{(p,w)}\Big(\sum_{i=1}^{m+n} z^i Z_i(p, w)\Big) = z^i V_i(p)$ for $(p, w)\in \pi^{-1}(A)$;

(1.11) $Z_i(a^j) = 0$ and $Z_{m+k}(a^j) = \delta^j_k$ (the Kronecker symbol) for $i = 1, \ldots, m$, $j, k = 1, \ldots, n$, and $a^1, \ldots, a^n$ are given by (1.7);

(1.12) $Z_i(p, w)(a\circ\pi) = V_i(p)(a)$, $Z_{m+j}(a\circ\pi) = 0$ for $a\in\mathscr{C}$, $(p, w)\in\pi^{-1}(A)$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$;

(1.13) $[Z_i, Z_j] = \gamma^k_{ij}\circ\pi Z_k$, $[Z_l, Z_{m+j}] = 0$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $l = 1, \ldots, m+n$, where $[V_i, V_j] = \gamma^k_{ij} V_k$.

Since every fundamental mapping is a diffeomorphism ([7], p. 148), the topological space $(Q, \tau_{\mathscr{F}})$ is also a T_2-space. Hence

(1.14) $$Q_{(p,w)}\cap Q_{(q,z)} = \emptyset \quad \text{for } (p, w) \neq (q, z)\ \ ((p, w), (q, z)\in Q).$$

We shall denote by π_2 the mapping

$$\pi_2\colon Q \to \bigcup_{p\in M} \Phi(p)$$

given by

(1.15) $$\pi_2(p, w) = w \quad \text{for } (p, w)\in Q.$$

In the case of the linear covariant derivative on a smooth manifold all theorems in the sequel were proved by Dombrowski in [3].

As proved in [1], in a differential space $(M, \mathscr{C})$ of finite differential dimension, the topological space $(M, \tau_{\mathscr{C}})$ of which is locally compact and paracompact, there is a smooth scalar product. Hence and from [5] we infer that on each such a differential space there is a covariant derivative.

2. Determination of a horizontal vector by a covariant derivative. Let ∇ be a covariant derivative in an n-dimensional differential $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields on the differential space $(M, \mathscr{C})$ ([6], p. 68). In the sequel we shall always denote by the same symbol ∇ a covariant derivative in $\mathscr{W}$, its global interpretation ([6], p. 69) and the restriction of ∇ to an open set ([6], p. 71).

The elements of $\mathscr{W}$, interpreted as smooth mappings from $(M, \mathscr{C})$ into $(Q, \mathscr{F})$ given by (1.1), have the following property:

2.1. *Let $W, W' \in \mathscr{W}$ and $p, p' \in M$. If $v \in M_p$, $v' \in M_{p'}$ and $\mathbf{d}\overline{W}_p v = \mathbf{d}\overline{W}'_{p'} v'$, then $p = p'$ and $v = v'$.*

Proof. Since $\mathbf{d}\overline{W}_p v \in Q_{\overline{W}(p)}$ and $\mathbf{d}\overline{W}'_{p'} v' \in Q_{\overline{W}'(p')}$, we infer from (1.14) that $(p, W(p)) = (p', W'(p'))$. Hence $p = p'$. It is clear from the definition of the differential of a smooth mapping that $v(\alpha \circ \pi \circ \overline{W}) = v'(\alpha \circ \pi \circ \overline{W}')$ for $\alpha \in \mathscr{C}$, that is, $v(\alpha) = v'(\alpha)$ for $\alpha \in \mathscr{C}$, which implies $v = v'$.

The covariant derivative ∇ in $\mathscr{W}$ has the following property:

2.2. *If $\mathbf{d}\overline{W}_p v = \mathbf{d}\overline{W}'_p v$, then $\nabla_v W = \nabla_v W'$ for every $W, W' \in \mathscr{W}$, $v \in M_p$ and $p \in M$.*

Proof. Let $V_1, \ldots, V_m$ be a vector basis of $\mathfrak{M}(M)$ on A, and let $W_1, \ldots, W_n$ be a vector basis of $\mathscr{W}$ on A. Let $\Gamma^k_{ij} \in \mathscr{C}_A$ ($i = 1, \ldots, m$, and $j, k = 1, \ldots, n$) be coordinates of ∇ with respect to the vector bases $V_1, \ldots, V_m$ and $W_1, \ldots, W_n$ [5], i.e.,

$$\nabla_{V_i} W_j = \Gamma^k_{ij} W_k \quad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

Then, for $W|A = \varphi^j W_j$, $W'|A = \psi^j W_j$ and $v = a^i V_i(p)$ ($p \in A$, $\varphi^j, \psi^k \in \mathscr{C}_A$ for $j, k = 1, \ldots, n$), we have

$$\nabla_v W = \big(v(\varphi^k) + a^i \varphi^j(p) \Gamma^k_{ij}(p)\big) W_k(p), \tag{2.1}$$

$$\nabla_v W' = \big(v(\psi^k) + a^i \psi^j(p) \Gamma^k_{ij}(p)\big) W_k(p). \tag{2.1'}$$

It follows from the equality $\mathbf{d}\overline{W}_p v = \mathbf{d}\overline{W}'_p v$ that $\overline{W}(p) = \overline{W}'(p)$, that is $(p, W(p)) = (p, W'(p))$. Hence $\varphi^j(p) = \psi^j(p)$ for $j = 1, \ldots, n$. Moreover, it follows from (1.8) that $v(\varphi^j) = v(\psi^j)$ for $j = 1, \ldots, n$. Thus the right-hand sides of (2.1) and (2.1′) are equal.

We denote by h_a for $a \in E$ a mapping from the differential space $(Q, \mathscr{F})$ into itself given by

$$h_a(p, w) = (p, aw) \quad \text{for } (p, w) \in Q.$$

It is obvious that h_a is a smooth mapping.

2.3. *Let ∇ be a covariant derivative in a $\mathscr{C}$-module $\mathscr{W}$. Then there is exactly one smooth mapping $\bar{K}$: $(TQ, \mathscr{T}\mathscr{F}) \to (Q, \mathscr{F})$, $\bar{K}\big((p, w), z\big) = \big(p, K(z)\big)$ for $(p, w) \in Q$ and $z \in Q_{(p,w)}$, such that*

(i) $K(\mathbf{d}\overline{W}_p v) = \nabla_v W$ *for* $W \in \mathscr{W}$, $v \in M_p$ *and* $p \in M$;

(ii) $K|Q_{(p,w)}$ *is a linear mapping from* $Q_{(p,w)}$ *onto* $\Phi(p)$ *for* $(p, w) \in Q$.

Moreover, if we set $K(p, w) = K|Q_{(p,w)}$, $H_{(p,w)} = \ker K(p, w)$ *and* $V_{(p,w)} = \ker \mathbf{d}\pi_{(p,w)}$, *then*

(iii) $Q_{(p,w)} = V_{(p,w)} \oplus H_{(p,w)}$ *for* $(p, w) \in Q$ *and the distribution*

$$H: Q \to \bigcup_{(p,w)\in Q} Q_{(p,w)}$$

given by $H(p, w) = H_{(p,w)}$ *for* $(p, w) \in Q$ *is smooth*;

(iv) *the mapping* $K(p, w)|V_{(p,w)}$ *is a linear isomorphism from the linear space* $V_{(p,w)}$ *onto the linear space* $\Phi(p)$, *and the mapping* $\mathbf{d}\pi_{(p,w)}|H_{(p,w)}$ *is a linear isomorphism from the linear space* $H_{(p,w)}$ *onto the linear space* M_p *for* $(p, w) \in Q$;

(v) $(\mathbf{d}h_a)_{(p,w)} H_{(p,w)} = H_{(p,aw)}$ *for* $a \in E - \{0\}$ *and* $(p, w) \in Q$.

Proof. Condition (i) is well defined by virtue of 2.2. Since in the set $\{\mathbf{d}\overline{W}_p v\colon W \in \mathscr{W},\ v \in M_p\}$ there is a basis of the linear space $Q_{(p,w)}$ for $(p, w) \in Q$ (see [2]), the uniqueness of the mapping K follows from 2.2, (i) and (ii).

We set

$$\mathfrak{M} = \{\mathbf{d}\overline{W}_p v\colon W \in \mathscr{W},\ v \in M_p \text{ and } p \in M\}.$$

Let a mapping $K|\mathfrak{M}\colon \mathfrak{M} \to Q$ be given by

$$(K|\mathfrak{M})(\mathbf{d}\overline{W}_p v) = \nabla_v W \quad \text{for } \mathbf{d}\overline{W}_p v \in \mathfrak{M}. \tag{2.2}$$

Let $p \in M$ and let A be an open neighbourhood of p in M such that there exist a vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A and a vector basis $W_1, \ldots, W_n$ of $\mathscr{W}$ on A. Let $Z_1, \ldots, Z_{m+n}$ be the vector basis of $\mathfrak{M}(Q)$ associated with the two vector bases. Then, for every $W \in \mathscr{W}$ and $v \in M_p$, we have $\varphi^1, \ldots, \varphi^n \in \mathscr{C}_A$, and $a^1, \ldots, a^m \in E$ such that $W|A = \varphi^j W_j$ and $v = a^i V_i(p)$. If $\alpha^1, \ldots, \alpha^n$ are the functions given by (1.7), then $\varphi^j(p) = \alpha^j\big(\overline{W}(p)\big)$ for $j = 1, \ldots, n$. Let Γ_{ij}^k be the coordinates of ∇ with respect to the vector bases $V_1, \ldots, V_m$ and $W_1, \ldots, W_n$. Then we can write (2.1) in the form

$$\nabla_v W = \big(v(\varphi^k) + a^i \alpha^j\big(\overline{W}(p)\big)\Gamma_{ij}^k(p)\big)W_k(p), \tag{2.1''}$$

and (2.2) in the form

$$(K|\mathfrak{M})\Big(\sum_{i=1}^{m+n} z^i Z_i\big(\overline{W}(p)\big)\Big) = \big(z^{m+k} + z^i \alpha^j\big(\overline{W}(p)\big)\Gamma_{ij}^k(p)\big)W_k(p), \tag{2.2'}$$

where $z^i = a^i$ for $i = 1, \ldots, m$, and $z^{m+j} = v(\varphi^j)$ for $j = 1, \ldots, n$.

It is immediate from (2.2′) that

(2.3) If $z, z', az+a'z' \in Q_{(p,w)} \cap \mathfrak{M}$, then $(K|\mathfrak{M})(az+a'z')$
$= a(K|\mathfrak{M})(z)+a'(K|\mathfrak{M})(z')$ for $a, a' \in E$ and $(p, w) \in Q$.

It is easy to verify that a mapping $\bar{K}\colon TQ \to Q$ given by

$$K\Big(\sum_{i=1}^{m+n} z^i Z_i(p, w)\Big) = \big(z^{m+k}+z^i a^j(p, w)\cdot\Gamma^k_{ij}(p)\big) W_k(p) \tag{2.4}$$
$$\text{for } (p, w) \in \pi^{-1}(A),\ z^1, \ldots, z^{m+n} \in E$$

is an extension of $K|\mathfrak{M}$, since the sets of the form $\pi^{-1}(A)$, where $A \in \tau_{\mathscr{C}}$ is such that there is a vector basis of $\mathfrak{M}(M)$ and of $\mathscr{W}$ on A, make an open covering of TQ. It follows from (2.4) that $\bar{K}\,|\,T\pi^{-1}(A)$ is a smooth mapping from the differential space $\big(T\pi^{-1}(A),\ \mathscr{T}\mathscr{F}_{\pi^{-1}(A)}\big)$ onto the differential space $\big(\pi^{-1}(A), \mathscr{F}_{\pi^{-1}(A)}\big)$. Thus $\bar{K}$ is smooth.

It follows from (1.10) that $Z_{m+1}(p, w), \ldots, Z_{m+n}(p, w)$ is a basis of $V_{(p,w)}$ for $(p, w) \in \pi^{-1}(A)$. It is immediate from (1.10) and (2.4) that $V_{(p,w)} \cap H_{(p,w)} = 0$ (zero vector of $Q_{(p,w)}$). Moreover, it is clear from (2.4) that a vector $\sum_{i=1}^{m+n} z^i Z_i(p, w)$ belongs to $H_{(p,w)}$ iff

$$z^{m+k}+z^i a^j(p, w)\Gamma^k_{ij}(p) = 0 \quad \text{for } k = 1, \ldots, n. \tag{2.5}$$

Since the solutions of (2.5) form an m-dimensional linear space, we infer that

$$\dim H_{(p,w)} = m \quad \text{for } (p, w) \in Q. \tag{2.6}$$

Hence

$$Q_{(p,w)} = V_{(p,w)} \oplus H_{(p,w)} \quad \text{for } (p, w) \in Q.$$

Next, by (2.5), it is clear that the sequence

(2.7) $A_1(p, w) = Z_1(p, w) - a^j(p, w)\Gamma^k_{1j}(p) Z_{m+k}(p, w), \quad \ldots,$
$A_m(p, w) = Z_m(p, w) - a^j(p, w)\Gamma^k_{mj}(p) Z_{m+k}(p, w) \quad$ for $(p, w) \in \pi^{-1}(A)$

is a basis of $H_{(p,w)}$ for $(p, w) \in \pi^{-1}(A)$. It is obvious that vector fields $A_1, \ldots, A_m$ are smooth, i.e. they belong to $\mathfrak{M}(Q)_{\pi^{-1}(A)}$. Thus the distribution H is smooth.

From (1.10) and (2.7) it is clear that the mapping $d\pi_{(p,w)}|H_{(p,w)}$ is a linear isomorphism from the linear space $H_{(p,w)}$ onto the linear space M_p for $(p, w) \in Q$. Next, it follows from (2.4) that the mapping $K(p, w)\,|\,V_{(p,w)}$ is a linear isomorphism from the linear space $V_{(p,w)}$ onto the linear space $\Phi(p)$ for $(p, w) \in Q$. This proves (iv).

In order to prove (v) it is sufficient to show that

$$(2.8)\qquad (\mathbf{d}h_a)_{(p,w)}A_i(p, w) = A_i(p, aw) \quad \text{for } i = 1, \dots, m,\ a \in E-\{0\} \text{ and } (p, w) \in \pi^{-1}(A).$$

From (1.7) it is obvious that

$$(2.9)\qquad \alpha^j(h_a(p, w)) = a\alpha^j(p, w) \quad \text{for } j = 1, \dots, n,\ a \in E \text{ and } (p, w) \in \pi^{-1}(A).$$

Next, it follows from (1.2) that

$$f(p, ax) = h_a(f(p, x)) \quad \text{for } (p, x) \in A \times E^n \text{ and } a \in E.$$

Therefore, by (1.3) we have

$$\begin{aligned}(\mathbf{d}h_a)_{f(p,x)}Z_i(f(p, x))(\gamma) &= Z_i(f(p, x))(\gamma\circ h_a) = V_i(p)(\gamma\circ h_a\circ f(\cdot, x))\\ &= V_i(p)(\gamma\circ f(\cdot, ax)) = Z_i(f(p, ax))(\gamma)\\ &= Z_i(h_a(f(p, x)))(\gamma)\end{aligned}$$

for $i = 1, \dots, m,\ a \in E-\{0\},\ (p, x) \in A \times E^n$ and $\gamma \in \mathscr{F}$.

Thus we proved

$$(2.10)\qquad (\mathbf{d}h_a)_{(p,w)}Z_i(p, w) = Z_i(h_a(p, w)) \quad \text{for } i = 1, \dots, m,\ a \in E-\{0\} \text{ and } (p, w) \in \pi^{-1}(A).$$

Using the chain rule and (1.4), we have

$$\begin{aligned}(\mathbf{d}h_a)_{f(p,x)}Z_{m+j}(f(p, x))(\gamma) &= Z_{m+j}(f(p, x))(\gamma\circ h_a) = D_j(x)(\gamma\circ h_a\circ f(p, \cdot))\\ &= D_j(x)(\gamma\circ f(p, a\cdot)) = aD_j(ax)(\gamma\circ f(p, \cdot))\\ &= aZ_{m+j}(h_a(f(p, x)))(\gamma)\end{aligned}$$

for $j = 1, \dots, n,\ a \in E-\{0\},\ (p, x) \in A \times E^n$ and $\gamma \in \mathscr{F}$.

This implies

$$(2.11)\qquad (\mathbf{d}h_a)_{(p,w)}Z_{m+j}(p, w) = aZ_{m+j}(h_a(p, w)) \quad \text{for } j = 1, \dots, n,\ a \in E-\{0\} \text{ and } (p, w) \in \pi^{-1}(A).$$

By (2.9)-(2.11) we have (2.8) which proves (v). Thus the proof of 2.3 is completed.

The only K satisfying 2.3 will be called the *mapping of the covariant derivative* ∇ and the vectors belonging to the distribution H given in 2.3 will be called *horizontal with respect to the covariant derivative* ∇.

3. Determination of the covariant derivative by the smooth distribution. Recall that π is the projection of Q onto M and $V_{(p,w)} = \ker \mathbf{d}\pi_{(p,w)}$ for $(p, w) \in Q$.

3.1. *Let $\{HQ_{(p,w)}\}_{(p,w)\in Q}$ be a smooth distribution on the differential space $(Q, \mathscr{F})$ such that*

(a) $Q_{(p,w)} = V_{(p,w)} \oplus HQ_{(p,w)}$ *for* $(p, w) \in Q$;

(b) $(\mathrm{d}h_a)_{(p,w)} HQ_{(p,w)} = HQ_{(p,aw)}$ *for* $(p, w) \in Q$ *and* $a \in E - \{0\}$.

Then there is exactly one covariant derivative ∇ in a $\mathscr{C}$-module $\mathscr{W}$ such that the horizontal vectors with respect to ∇ belong to $\{HQ_{(p,w)}\}_{(p,w)\in Q}$.

Proof. First we determine $\bar{K}$: $(TQ, \mathscr{T}\mathscr{F}) \to (Q, \mathscr{F})$ which is the mapping of the covariant derivative ∇. In order to define $\bar{K}$ it is sufficient to determine K in the vector bases. Let $(p', w') \in Q$. Then there exist an open neighbourhood B' of (p', w') in $(Q, \mathscr{F})$ and the vector fields $A_1, \ldots, A_m \in \mathfrak{M}(Q)_{B'}$ such that a sequence $A_1(p, w), \ldots, A_m(p, w)$ is a basis of $HQ_{(p,w)}$ for $(p, w) \in B'$. Let A be an open neighbourhood of p' in $(M, \tau_{\mathscr{C}})$ such that there exist a vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A and a vector basis $W_1, \ldots, W_n$ of $\mathscr{W}$ on A. Let $Z_1, \ldots, Z_{m+n}$ be the vector basis of $\mathfrak{M}(Q)$ associated with those two vector bases. Then it is clear from (1.10) and (a) that the vector fields $A_1, \ldots, A_m, Z_{m+1}, \ldots, Z_{m+n}$ form a vector basis of $\mathfrak{M}(Q)$ on $B = B' \cap \pi^{-1}(A)$. Hence there is the unique sequence of functions $a_i^j \in \mathscr{F}_B$ $(i = 1, \ldots, m,\ j = 1, \ldots, m+n)$ such that

$$A_i | B = a_i^j Z_j | B + a_i^{m+k} Z_{m+k} | B \quad \text{for } i = 1, \ldots, m. \tag{3.1}$$

It follows from (a) that

$$\det[a_i^j(p, w):\ i, j = 1, \ldots, m] \neq 0 \quad \text{for } (p, w) \in B.$$

We set

$$[a_i^j(p, w):\ i, j = 1, \ldots, m]^{-1} = [\beta_i^j(p, w):\ i, j = 1, \ldots, m].$$

Now it is easy to verify that

$$\text{if } z = \sum_{i=1}^{m+n} z^i Z_i(p, w), \text{ then } z = b^i A_i(p, w) + b^{m+j} Z_{m+j}(p, w) \quad \text{for } (p, w) \in B, \tag{3.2}$$

where

$$b^i = \beta_j^i(p, w) z^j \quad \text{for } i = 1, \ldots, m, \tag{3.3}$$

$$b^{m+k} = z^{m+k} - a_j^{m+k}(p, w) \cdot \beta_i^j(p, w) z^i \quad \text{for } k = 1, \ldots, n. \tag{3.4}$$

We set

$$K_B\Big(\sum_{i=1}^{m+n} z^i Z_i(p, w)\Big) = \big(z^{m+k} - a_j^{m+k}(p, w) \cdot \beta_i^j(p, w) z^i\big) W_k(p) \quad \text{for } (p, w) \in B \tag{3.5}$$

or, by virtue of (3.3) and (3.4),

$$K_B\big(b^i A_i(p, w) + b^{m+k} Z_{m+k}(p, w)\big) = b^{m+k} W_k(p) \quad \text{for } (p, w) \in B,\ b^1, \ldots, b^{m+n} \in E. \tag{3.5'}$$

Let $A'_{1'}, \ldots, A'_{m'}, Z'_{m'+1'}, \ldots, Z'_{m'+n'}$ be another vector basis of $\mathfrak{M}(Q)$ on B' given in the analogical way as $A_1, \ldots, A_m, Z_{m+1}, \ldots, Z_{m+n}$ above and let $K_{B'}$ be an analogon to K_B above. Then we infer from (1.5) and (1.6) by a simple verification that

$$K_B | B \cap B' = K_{B'} | B \cap B'. \tag{3.6}$$

Let $\mathscr{B}$ be a class of all open sets B in $(Q, \tau_{\mathscr{F}})$ on which there is a vector basis $A_1, \ldots, A_m, Z_{m+1}, \ldots, Z_{m+n}$ of $\mathfrak{M}(Q)$ such that a sequence $A_1(p, w), \ldots, A_m(p, w)$ is a basis of $HQ_{(p,w)}$ and a sequence $Z_{m+1}(p, w), \ldots \ldots, Z_{m+n}(p, w)$ is a basis of $V_{(p,w)}$ for $(p, w) \in B$. Then $\mathscr{B}$ is an open covering of Q, and a mapping $\bar{K}\colon TQ \to Q$ given by

$$K | B = K_B \quad \text{for } B \in \mathscr{B} \tag{3.7}$$

is well defined by virtue of (3.6). It follows directly from (3.5) that $\bar{K}$ is a smooth mapping. As in 2.3, we denote by $K(p, w)$ the mapping $K | Q_{(p,w)}$. Then it is obvious that

$$K(p, w)\colon Q_{(p,w)} \to \Phi(p) \text{ is a linear mapping for } (p, w) \in Q, \tag{3.8}$$

$$\ker K(p, w) = HQ_{(p,w)} \text{ for } (p, w) \in Q. \tag{3.9}$$

We set

$$\nabla_v W = K(\mathbf{d}\overline{W}_p v) \quad \text{for } W \in \mathscr{W},\ v \in M_p \text{ and } p \in M. \tag{3.10}$$

From (3.8) and (3.10) it is obvious that

$$\nabla_{v+v'} W = \nabla_v W + \nabla_{v'} W \quad \text{for } W \in \mathscr{W},\ v, v' \in M_p \text{ and } p \in M, \tag{3.11}$$

$$\nabla_{a(p)v} W = a(p) \nabla_v W \quad \text{for } W \in \mathscr{W},\ a \in \mathscr{C},\ v \in M_p \text{ and } p \in M, \tag{3.12}$$

$$\nabla_v (W + W') = \nabla_v W + \nabla_v W' \quad \text{for } W, W' \in \mathscr{W},\ v \in M_p \text{ and } p \in M. \tag{3.13}$$

By virtue of (3.2)-(3.5) we can write (3.10) in the vector basis $A_1, \ldots, A_m, Z_{m+1}, \ldots, Z_{m+n}$ as

$$\nabla_v W = \big(v(\varphi^k) - a^i \cdot \beta_i^j(\overline{W}(p)) \cdot a_j^{m+k}(\overline{W}(p))\big) W_k(p) \quad \text{for } v \in M_p,\ W \in \mathscr{W}, \tag{3.14}$$

where $v = a^i V_i(p)$, $W | A = \varphi^j W_j$, $\overline{W}(p) \in B$, and $Z_1, \ldots, Z_{m+n}$ is the vector basis of $\mathfrak{M}(Q)$ associated with the vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A and with the vector basis $W_1, \ldots, W_n$ of $\mathscr{W}$ on A. Next, it follows from (2.10) and (2.11) that

$$(\mathbf{d}h_a)_{(p,w)}\Big(\sum_{i=1}^{m+n} z^i Z_i(p, w)\Big) = z^i Z_i(p, aw) + a z^{m+k} Z_{m+k}(p, aw) \tag{3.15}$$
$$\text{for } (p, w) \in \pi^{-1}(A) \text{ and } z^1, \ldots, z^{m+n} \in E.$$

By (3.5′) and (3.4) we get

$$(3.16)\qquad \sum_{i=1}^{m+n} z^i Z_i(p,w)\in HQ_{(p,w)} \text{ iff } z^{m+k}-z^i\cdot\beta_i^j(p,w)\cdot a_j^{m+k}(p,w)=0$$
$$\text{for } k=1,\dots,n \text{ and } (p,w)\in B.$$

Therefore, by (b) and (3.15), we have

$$(3.17)\qquad \text{if } \sum_{i=1}^{m+n} z^i Z_i(p,w)\in HQ_{(p,w)},$$
$$\text{then } az^{m+k}-z^i\cdot\beta_i^j(p,aw)\cdot a_j^{m+k}(p,aw)=0$$
$$\text{for } k=1,\dots,n \text{ and } a\in E-\{0\}.$$

On the other hand, by (3.16) we have

$$(3.18)\qquad \text{if } \sum_{i=1}^{m+n} z^i Z_i(p,w)\in HQ_{(p,w)}, \text{ then } az^{m+k}-az^i\cdot\beta_i^j(p,w)\cdot a_j^{m+k}(p,w)=0$$
$$\text{for } k=1,\dots,n \text{ and } a\in E-\{0\}.$$

Therefore, by (3.17) and (3.18) we have

$$(3.19)\qquad \beta_i^j(p,aw)\cdot a_j^{m+k}(p,aw)=a\cdot\beta_i^j(p,w)\cdot a_j^{m+k}(p,w)$$
$$\text{for } i=1,\dots,m,\ k=1,\dots,n,\ (p,w)\in B \text{ and } a\in E-\{0\}.$$

From the continuity of a_i^j ($i=1,\dots,m$, $j=1,\dots,m+n$) it follows that (3.19) is true also for $a=0$. Hence, for $v\in M_p$, $W\in\mathscr{W}$ and $a\in\mathscr{C}$, if $v=a^iV_i(p)$ and $W|A=\varphi^jW_j$, then

$$\begin{aligned}\nabla_v aW &= \big(v(\varphi^k a)-a^i\cdot\beta_i^j(p,a(p)W(p))\cdot a_j^{m+k}(p,a(p)W(p))\big)W_k(p)\\ &= v(a)\varphi^k(p)W_k(p)+a(p)\big(v(\varphi^k)-a^i\cdot\beta_i^j(p,W(p))\cdot a_j^{m+k}(p,W(p))\big)W_k(p)\\ &= v(a)W(p)+a(p)\nabla_vW.\end{aligned}$$

Thus

$$(3.20)\qquad \nabla_v aW=v(a)W(p)+a(p)\nabla_vW$$
$$\text{for } a\in\mathscr{C},\ W\in\mathscr{W},\ v\in M_p \text{ and } p\in M.$$

Setting

$$(3.21)\qquad (\nabla_VW)(p)=\nabla_{V(p)}W \quad \text{for } V\in\mathfrak{M}(M),\ W\in\mathscr{W} \text{ and } p\in M,$$

by (3.14) we obtain $\nabla_VW\in\mathscr{W}$ for $V\in\mathfrak{M}(M)$ and $W\in\mathscr{W}$.

Therefore, it follows from (3.11)-(3.13) and (3.20) that ∇ given by (3.21) is a covariant derivative in $\mathscr{W}$. The mapping of this covariant derivative is K given by (3.7) in virtue of (3.9) and (3.10). Therefore, the covariant derivative ∇ satisfies 3.1. Moreover, by virtue of (3.8)-(3.10) and 2.3, there is exactly one covariant derivative which satisfies 3.1. This completes the proof of 3.1.

As a corollary to 2.3 and 3.1 we obtain

3.2. *There is a one-to-one correspondence between the covariant derivatives in a differential module and the smooth distributions H on the differential space of $\mathscr{W}$ satisfying conditions* (a) *and* (b).

4. Vertical and horizontal lifts. An almost complex structure. In this section, let ∇ be a covariant derivative on the differential space $(M, \mathscr{C})$. We shall denote by K the mapping of the covariant derivative from the differential space $(TTM, \mathscr{T}\mathscr{T}\mathscr{C})$ onto the differential space $(TM, \mathscr{T}\mathscr{C})$ given in 2.3. By π we denote the projection of TM onto M. We define $H_{(p,w)}$, $V_{(p,w)}$ and $K(p, w)$ as in 2.3. As a corollary to 2.3 we obtain

4.1. *For any vectors $v, u, w \in M_p$ $(p \in M)$ there is exactly one vector $z \in (TM)_{(p,w)}$ such that*

$$\mathbf{d}\pi_{(p,w)} z = v \quad \text{and} \quad K(p, w)(z) = u. \tag{4.1}$$

Proof. If we set

$$z^- = (d\pi_{(p,w)} | H_{(p,w)})^{-1}(v) \quad \text{and} \quad z' = \big(K(p, w) | V_{(p,w)}\big)^{-1}(u),$$

then the vector $z = z^- + z'$ satisfies 4.1.

It follows from 4.1 that if X is a tangent vector field on $(M, \mathscr{C})$, then there is exactly one tangent vector field X^{h} on $(TM, \mathscr{T}\mathscr{C})$ and there is exactly one tangent vector field X^{v} on $(TM, \mathscr{T}\mathscr{C})$ such that

$$\mathbf{d}\pi_{(p,w)} X^{\mathrm{h}}(p, w) = X(p) \quad \text{and} \quad K(p, w)\big(X^{\mathrm{h}}(p, w)\big) = 0 \quad \text{for } (p, w) \in TM, \tag{4.2}$$

$$\mathbf{d}\pi_{(p,w)} X^{\mathrm{v}}(p, w) = 0 \quad \text{and} \quad K(p, w)\big(X^{\mathrm{v}}(p, w)\big) = X(p) \quad \text{for } (p, w) \in TM, \tag{4.3}$$

where 0 denotes zero vector of M_p. The tangent vector fields X^{h} and X^{v} satisfying 4.2 and 4.3 will be called the *horizontal* and *vertical lifts* of the vector field X, respectively.

If $Z_1, \ldots, Z_{2m}$ is the vector basis of $\mathfrak{M}(TM)$ associated with a vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A, then by (1.10) and (2.4) we obtain the local expressions of X^{h} and X^{v}, i.e.,

$$X^{\mathrm{h}} | \pi^{-1}(A) = \varphi^i \circ \pi \cdot Z_i = \varphi^i \circ \pi \cdot a^j \cdot \Gamma^k_{ij} \circ \pi \cdot Z_{m+k}, \tag{4.4}$$

$$X^{\mathrm{v}} | \pi^{-1}(A) = \varphi^i \circ \pi \cdot Z_{m+i}, \tag{4.5}$$

where $X | A = \varphi^i V_i$ and $a^1, \ldots, a^m \in \mathscr{T}\mathscr{C}_{\pi^{-1}(A)}$ are given by (1.7). Therefore, by (4.4) and (4.5), we have

$$X^{\mathrm{h}}, X^{\mathrm{v}} \in \mathfrak{M}(TM) \quad \text{for } X \in \mathfrak{M}(M). \tag{4.6}$$

Moreover, by (1.10), (2.7), (4.4) and (4.5), we obtain

(4.7) vectors $(V_1)^{\mathrm{h}}(p, w), \ldots, (V_m)^{\mathrm{h}}(p, w)$ form a basis of $H_{(p,w)}$ for $(p, w) \in \pi^{-1}(A)$,

(4.8) vectors $(V_1)^{\mathrm{v}}(p, w), \ldots, (V_m)^{\mathrm{v}}(p, w)$ form a basis of $V_{(p,w)}$ for $(p, w) \in \pi^{-1}(A)$.

Thus we have

4.2. *Let ∇ be a covariant derivative on $(M, \mathscr{C})$. Then*

(i) *horizontal and vertical lifts of a smooth tangent vector field are also smooth;*

(ii) *if $V_1, \ldots, V_m$ is a vector basis of $\mathfrak{M}(M)$ on A, then the vector fields $(V_1)^{\mathrm{h}}, \ldots, (V_m)^{\mathrm{h}}, (V_1)^{\mathrm{v}}, \ldots, (V_m)^{\mathrm{v}}$ form a vector basis of $\mathfrak{M}(TM)$ on $\pi^{-1}(A)$ satisfying* (4.7) *and* (4.8).

Next, it follows immediately from 4.1 that if $(p, w) \in TM$, then for every $z \in (TM)_{(p,w)}$ there is exactly one element $Jz \in (TM)_{(p,w)}$ such that

$$\mathbf{d}\pi_{(p,w)}Jz = -K(p, w)(z) \quad \text{and} \quad K(Jz) = \mathbf{d}\pi_{(p,w)}z. \tag{4.9}$$

For any vector field $Z \in \mathfrak{M}(TM)$, let $J(Z)$ denote a tangent vector field on $(TM, \mathscr{T}\mathscr{C})$ given by

$$\big(J(Z)\big)(p, w) = JZ(p, w) \quad \text{for } (p, w) \in TM, \tag{4.10}$$

where the vector $JZ(p, w)$ satisfies (4.9). It is obvious from (4.9) and (4.10) that

$$J^2 = -id_{\mathfrak{M}(TM)}. \tag{4.11}$$

Let $Z_1, \ldots, Z_{2m}$ be the vector basis of $\mathfrak{M}(TM)$ on $\pi^{-1}(A)$ associated with a vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A. Let $A_1, \ldots, A_m$ be defined by (2.7). Then, by (1.10), (2.4) and (4.9), we have the local expression of $J(Z)$, i.e.,

$$J(Z) \,|\, \pi^{-1}(A) = \gamma^i Z_{m+i} - \gamma^{m+i} A_i, \tag{4.12}$$

where

$$Z \,|\, \pi^{-1}(A) = \gamma^i A_i + \gamma^{m+i} Z_{m+i}. \tag{4.13}$$

Thus

$$J(Z) \in \mathfrak{M}(TM) \quad \text{for } Z \in \mathfrak{M}(TM). \tag{4.14}$$

Moreover, if $\gamma, \gamma' \in \mathscr{T}\mathscr{C}$ and $Z, Z' \in \mathfrak{M}(TM)$, then by the linearity of mappings $\mathbf{d}\pi_{(p,w)}$ and $K(p, w)$ we have

$$\begin{aligned} \big(J(\gamma Z + \gamma' Z')\big)(p, w) &= J\big(\gamma(p, w)Z(p, w) + \gamma'(p, w)Z'(p, w)\big) \\ &= \gamma(p, w)JZ(p, w) + \gamma'(p, w)JZ'(p, w) \\ &= \big(\gamma J(Z) + \gamma' J(Z')\big)(p, w) \quad \text{for } (p, w) \in TM. \end{aligned}$$

Thus the mapping $J: \mathfrak{M}(TM) \to \mathfrak{M}(TM)$ given by (4.10) is $\mathcal{T}\mathcal{C}$-linear. Since J satisfies (4.11), it is an almost complex structure of $(TM, \mathcal{T}\mathcal{C})$. Next, it follows from (4.4), (4.5) and (4.12) that

$$J(X^{\mathrm{h}}) = X^{\mathrm{v}} \quad \text{and} \quad J(X^{\mathrm{v}}) = -X^{\mathrm{h}} \quad \text{for } X \in \mathfrak{M}(M). \tag{4.15}$$

Thus we have

4.3. *Any covariant derivative ∇ on the differential space $(M, \mathcal{C})$ determines an almost complex structure J of the differential space $(TM, \mathcal{T}\mathcal{C})$ satisfying* (4.9), *where K is the mapping of ∇.*

Let $N(W, Z)$ be a tangent vector field on the differential space $(TM, \mathcal{T}\mathcal{C})$ given by

$$\begin{aligned} N(W, Z) = [W, Z] + J\big([W, J(Z)]\big) - J\big([J(W), Z]\big) - \\ - [J(W), J(Z)] \quad \text{for } W, Z \in \mathfrak{M}(TM), \end{aligned} \tag{4.16}$$

where J is the almost complex structure defined by (4.10). By a straightforward calculation, we see that

$$N: \mathfrak{M}(TM) \times \mathfrak{M}(TM) \to \mathfrak{M}(TM)$$

is a skew-symmetric tensor on $(TM, \mathcal{T}\mathcal{C})$. This tensor will be called the *torsion of J* (cf. [4]).

4.4. *Let T and R be the torsion and curvature tensors of the covariant derivative ∇ on $(M, \mathcal{C})$, respectively. Let J be the almost complex structure of $(TM, \mathcal{T}\mathcal{C})$ defined in* 4.3, *and let N be the torsion tensor of J. Then*

$$\mathbf{d}\pi_{(p,z)}\big(N(X^{\mathrm{v}}, Y^{\mathrm{v}})\big)(p, z) = T(X, Y)(p) \quad \textit{for } X, Y \in \mathfrak{M}(M) \textit{ and } (p, z) \in TM, \tag{4.17}$$

$$K(p, z)\big(N(X^{\mathrm{v}}, Y^{\mathrm{v}})\big)(p, z) = R_{X(p), Y(p)} z \quad \textit{for } X, Y \in \mathfrak{M}(M) \textit{ and } (p, z) \in TM. \tag{4.18}$$

We first prove

4.5. *Under the hypothesis and notation of* 4.4, *for any $X, Y \in \mathfrak{M}(M)$ and $(p, w) \in TM$, we have*

$$[X^{\mathrm{v}}, Y^{\mathrm{v}}] = 0, \tag{4.19}$$

$$[X^{\mathrm{h}}, Y^{\mathrm{v}}] = (\nabla_X Y)^{\mathrm{v}}, \tag{4.20}$$

$$\mathbf{d}\pi_{(p,w)}([X^{\mathrm{h}}, Y^{\mathrm{h}}])(p, w) = [X, Y](p), \tag{4.21}$$

$$K(p, w)([X^{\mathrm{h}}, Y^{\mathrm{h}}])(p, w) = -R_{X(p), Y(p)} w. \tag{4.22}$$

Proof. Let $Z_1, \ldots, Z_{m+n}$ be a vector basis of $\mathfrak{M}(TM)$ on $\pi^{-1}(A)$ associated with a vector basis $V_1, \ldots, V_m$ of $\mathfrak{M}(M)$ on A. Let

$$X|A = \varphi^i V_i \quad \text{and} \quad Y|A = \psi^i V_i \quad (\varphi^1, \ldots, \varphi^m, \psi^1, \ldots, \psi^m \in \mathscr{C}_A).$$

Then, by (1.12), (1.13) and (4.5), we obtain

$$\begin{aligned}[X^{\mathrm{v}}, Y^{\mathrm{v}}]|\pi^{-1}(A) &= [\varphi^i\circ\pi\cdot Z_{m+i}, \psi^j\circ\pi\cdot Z_{m+j}] \\ &= \varphi^i\circ\pi\cdot\psi^j\circ\pi\cdot[Z_{m+i}, Z_{m+j}] + \varphi^i\circ\pi\cdot Z_{m+i}(\psi^j\circ\pi)Z_{m+j} - \\ &\quad - \psi^j\circ\pi\cdot Z_{m+j}(\varphi^i\circ\pi)Z_{m+i} = 0,\end{aligned}$$

which proves (4.19).

Next, from (1.11)-(1.13), (4.4) and (4.5) we get

$$\begin{aligned}[X^{\mathrm{h}}, Y^{\mathrm{v}}]|\pi^{-1}(A) &= [\varphi^i\circ\pi\cdot Z_i - \varphi^i\circ\pi\cdot a^j\cdot\Gamma^k_{ij}\circ\pi\cdot Z_{m+k}, \psi^i\circ\pi\cdot Z_{m+i}] \\ &= \varphi^i\circ\pi\cdot\psi^j\circ\pi\cdot[Z_i, Z_{m+j}] + \varphi^i\circ\pi\cdot Z_i(\psi^j\circ\pi)Z_{m+j} - \\ &\quad - \psi^i\circ\pi\cdot Z_{m+i}(\varphi^j\circ\pi)Z_{m+j} + \\ &\quad + \psi^l\circ\pi\cdot Z_{m+l}(\varphi^i\circ\pi\cdot a^j\cdot\Gamma^k_{ij}\circ\pi)Z_{m+k} - \\ &\quad - \varphi^i\circ\pi\cdot a^j\cdot\Gamma^k_{ij}\circ\pi\cdot\psi^l\circ\pi\cdot[Z_{m+k}, Z_{m+l}] - \\ &\quad - \varphi^i\circ\pi\cdot a^j\cdot\Gamma^k_{ij}\circ\pi\cdot Z_{m+k}(\psi^l\circ\pi)Z_{m+l} \\ &= \big(\varphi^i\circ\pi\cdot V_i(\psi^k)\circ\pi + \varphi^i\circ\pi\cdot\psi^j\circ\pi\cdot\Gamma^k_{ij}\circ\pi\big)Z_{m+k}\end{aligned}$$

and, on the other hand,

$$\begin{aligned}(\nabla_X Y)^{\mathrm{v}}|\pi^{-1}(A) &= \big((\varphi^i\cdot V_i(\psi^k) + \varphi^i\cdot\psi^j\cdot\Gamma^k_{ij})V_k\big)^{\mathrm{v}} \\ &= \big(\varphi^i\circ\pi\cdot V_i(\psi^k)\circ\pi + \varphi^i\circ\pi\cdot\psi^j\circ\pi\cdot\Gamma^k_{ij}\circ\pi\big)Z_{m+k},\end{aligned}$$

which proves (4.20).

We assume that $[V_i, V_j] = \gamma^k_{ij}V_k$ ($\gamma^k_{ij} \in \mathscr{C}_A$ for $i, j, k = 1, \ldots, m$). Then, by (1.11)-(1.13) and (4.4), we have

$$\begin{aligned}&[X^{\mathrm{h}}, Y^{\mathrm{h}}]|\pi^{-1}(A) \\ &= [\varphi^i\circ\pi\cdot Z_i - \varphi^i\circ\pi\cdot a^j\cdot\Gamma^k_{ij}\circ\pi\cdot Z_{m+k}, \psi^i\circ\pi\cdot Z_i - \psi^l\circ\pi\cdot a^j\cdot\Gamma^k_{lj}\circ\pi\cdot Z_{m+k}] \\ &= \big(\varphi^i\cdot\psi^j\cdot\gamma^k_{ij} + \varphi^i\cdot V_i(\psi^k) - \psi^i\cdot V_i(\varphi^k)\big)\circ\pi\cdot Z_k + \\ &\quad + \big(\varphi^i\circ\pi\cdot\psi^l\circ\pi\cdot a^j\cdot V_l(\Gamma^k_{ij})\circ\pi + \psi^l\circ\pi\cdot a^j\cdot\Gamma^k_{ij}\circ\pi\cdot V_l(\varphi^i)\circ\pi - \\ &\quad - \varphi^i\circ\pi\cdot\Gamma^k_{lj}\circ\pi\cdot a^j\cdot V_i(\psi^l)\circ\pi + \varphi^i\circ\pi\cdot\psi^l\circ\pi\cdot a^j\cdot\big((\Gamma^r_{ij}\cdot\Gamma^k_{lr})\circ\pi\big) - \\ &\quad - \varphi^i\circ\pi\cdot\psi^l\circ\pi\cdot a^j\cdot V_i(\Gamma^k_{lj})\circ\pi - \psi^l\circ\pi\cdot\varphi^r\circ\pi\cdot a^i\cdot\big((\Gamma^j_{li}\cdot\Gamma^k_{rj})\circ\pi\big)\big)Z_{m+k}.\end{aligned}$$

Hence and from (1.10) we have

$$\begin{aligned}\mathbf{d}\pi_{(p,w)}\big([X^{\mathrm{h}}, Y^{\mathrm{h}}](p, w)\big) &= \big(\varphi^i\psi^j\gamma^k_{ij} + \varphi^i V_i(\psi^k) - \psi^i V_i(\varphi^k)\big)V_k(p) \\ &= [X, Y](p) \quad \text{for } (p, w) \in \pi^{-1}(A),\end{aligned}$$

which proves (4.21).

Next, by (2.4) we have

$$K(p,w)\big([X^{\mathrm{h}}, Y^{\mathrm{h}}](p,w)\big) = \Gamma^{r}_{ik}(p)\Gamma^{l}_{jr}(p) - \Gamma^{r}_{jk}(p)\Gamma^{l}_{ir}(p) +$$
$$+ V_j(p)(\Gamma^{l}_{ik}) - V_i(p)(\Gamma^{l}_{ik}) + \Gamma^{l}_{rk}(p)\gamma^{r}_{ij}(p)\cdot\varphi^{i}(p)\cdot\psi^{j}(p)\cdot a^{k}(p,w)\, V_l(p)$$
$$= -R_{X(p),Y(p)}w \quad \text{for } (p,w)\in\pi^{-1}(A),$$

which proves (4.22).

Proof of 4.4. By virtue of (4.15) and (4.16), we observe that

$$(4.23) \qquad N(X^{\mathrm{v}}, Y^{\mathrm{v}}) = [X^{\mathrm{v}}, Y^{\mathrm{v}}] - J([X^{\mathrm{v}}, Y^{\mathrm{h}}] + [X^{\mathrm{h}}, Y^{\mathrm{v}}]) - [X^{\mathrm{h}}, Y^{\mathrm{h}}] \quad \text{for } X, Y\in\mathfrak{M}(M).$$

Hence, using (1.10) and 4.5, we have

$$\mathbf{d}\pi_{(p,w)}\big(N(X^{\mathrm{v}}, Y^{\mathrm{v}})(p,w)\big)$$
$$= \mathbf{d}\pi_{(p,w)}\Big(J\big((\nabla_Y X)^{\mathrm{v}}(p,w) - (\nabla_X Y)^{\mathrm{v}}(p,w)\big) - [X^{\mathrm{h}}, Y^{\mathrm{h}}](p,w)\Big)$$
$$= \mathbf{d}\pi_{(p,w)}\big((\nabla_X Y)^{\mathrm{h}}(p,w)\big) - \mathbf{d}\pi_{(p,w)}\big((\nabla_Y X)^{\mathrm{h}}(p,w)\big) - \mathbf{d}\pi_{(p,w)}\big([X^{\mathrm{h}}, Y^{\mathrm{h}}](p,w)\big)$$
$$= (\nabla_X Y - \nabla_Y X - [X,Y])(p) = T(X,Y)(p)$$

and by (2.4) and 4.5 we obtain

$$K(p,w)\big(N(X^{\mathrm{v}}, Y^{\mathrm{v}})(p,w)\big)$$
$$= K(p,w)\big((\nabla_X Y)^{\mathrm{h}}(p,w)\big) - K(p,w)\big((\nabla_Y X)^{\mathrm{h}}(p,w)\big) -$$
$$- K(p,w)\big([X^{\mathrm{h}}, Y^{\mathrm{h}}](p,w)\big)$$
$$= R_{X(p),Y(p)}w \quad \text{for } X, Y\in\mathfrak{M}(M) \text{ and } (p,w)\in TM.$$

As a corollary to 4.4 we obtain

4.6. *Under the hypothesis and notation of* 4.4, $N = 0$ *iff* $T = 0$ *and* $R = 0$.

Proof. If $N = 0$, then (4.17) and (4.18) imply $T = 0$ and $R = 0$. If $T = 0$ and $R = 0$, then from condition (iv) of 2.3, (4.17) and (4.18) it follows that $N(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0$ for $X, Y\in\mathfrak{M}(M)$. We observe that

$$N(W,Z) = J\big(N(J(W),Z)\big) = J\big(N(W,J(Z))\big) = -N\big(J(W),J(Z)\big) \quad \text{for } W, Z\in\mathfrak{M}(TM).$$

Hence, by virtue of (4.15),

$$N(X^{\mathrm{v}}, Y^{\mathrm{v}}) = N(X^{\mathrm{v}}, Y^{\mathrm{h}}) = N(X^{\mathrm{h}}, Y^{\mathrm{v}}) = N(X^{\mathrm{h}}, Y^{\mathrm{h}}) = 0 \quad \text{for } X, Y\in\mathfrak{M}(M).$$

Thus, by 4.2 and 4.4, $N = 0$.

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