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# ON THICK SPACES

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According to Arhangelskiĭ [1], a Hausdorff space $X$ is called *thick* if, for each cardinal number $\mathfrak{m}$, there exists a dense subset $D$ of $X$ such that

(*) if $M \subset D$ and $\operatorname{card} M \leqslant \mathfrak{m}$, then $\mathrm{cl}_D M$ is compact and the weight of $\mathrm{cl}_D M$ is not greater than $\mathfrak{m}$.

It was proved in [1] that dyadic spaces are thick (an immediate consequence of the properties: products of thick spaces are thick, a continuous image of a thick space is thick, the two-point discrete space is thick, and thick spaces are compact). The thickness and another property, the flatteness, are components of the property to be Dantean, introduced in the same paper to characterize dyadic spaces. It has been shown there that Dantean spaces have many properties which were originally proved for dyadic spaces, and that dyadic spaces are Dantean. However, whether Dantean spaces are dyadic is a still open question.

Denote by $wX$ the weight of a space $X$, and by $\chi(x, X)$ the minimum cardinality of bases at a point $x$ in $X$. Let

$$\chi(X) = \sup\{\chi(x, X):\ x \in X\}, \qquad ssX = \sup\{sY:\ Y \subset X\},$$

where $sY$ denotes density of $Y$, i.e. the minimum cardinality of dense subsets of $Y$.

It is the aim of this paper to prove the following two results:

THEOREM 1. *If $X$ is thick, then $wX = ssX$.*

THEOREM 2. *If $\beta X$ is thick, then $X$ is pseudocompact.*

These properties of thickness were known to hold for dyadic spaces: for Theorem 1 see Efimov [2] (the case of Dantean spaces is essentially contained in Arhangelskiĭ's paper [1]), and for Theorem 2 see Engelking and Pełczyński [3].

Let us observe that there exist thick spaces which are not dyadic, e.g. one-point compactification of an uncountable discrete space (the Souslin property does not hold) and the long line with $\omega_1$ added. However, $\beta N$ is not a thick space.

LEMMA 1. *If $X$ is thick, then each subset of $X$ realizing $(*)$ for $\mathfrak{m} = ssX$ is $X$ itself.*

Proof. Let $D$ be a dense subset of $X$ satisfying $(*)$ for $\mathfrak{m} = ssX$. Take $x$ from $X\setminus D$. Let $A$ be the family of all open neighbourhoods of $x$. For $W = \bigcup \{D\cap U\colon U\in A\}$ we see that $x\notin W$ and $x\in \mathrm{cl}W$. Since $sX\leqslant ssX$, there exists a dense subset $W'$ of $W$ of cardinality not greater than $ssX$. Hence $x\in \mathrm{cl}W'$. Since $W'\subset D$ and $\mathrm{card}W\leqslant ssX$, $\mathrm{cl}W'\subset D$; a contradiction with $x\in X\setminus D$.

Proof of Theorem 1. By Lemma 1, for each $M\subset X$ such that $\mathrm{card}M\leqslant ssX$, $\mathrm{cl}M$ is compact and $w(\mathrm{cl}M)\leqslant ssX$. Since $sX\leqslant ssX$, there exists a dense subset $M$ of $X$ of cardinality $sX$. We have $wX = w(\mathrm{cl}M) \leqslant ssX$. The converse inequality $ssX\leqslant wX$ being true generally, we have $wX = ssX$.

COROLLARY 1. *If a thick space $X$ is hereditarily separable, then $X$ is metrizable.*

LEMMA 2. *If $\chi(x, X)\leqslant \mathfrak{n}$, then $x$ belongs to each $D$ satisfying $(*)$ for $\mathfrak{m}\geqslant\mathfrak{n}$.*

Proof. Let $\mathfrak{B}$ be a base in $x$ such that $\mathrm{card}\mathfrak{B}\leqslant\mathfrak{n}$. Let $\mathfrak{m}\geqslant\mathfrak{n}$ and let $D$ be a dense subset of $X$ satisfying $(*)$ for this $\mathfrak{m}$. Take $U\in\mathfrak{B}$ and choose one point $x_U$ in each $U\cap D$. Denote by $M$ the set of all these points. Then $\mathrm{card}\mathfrak{B}\leqslant\mathfrak{n}$ implies $\mathrm{card}M\leqslant\mathfrak{n}\leqslant\mathfrak{m}$. Hence $\mathrm{cl}M\subset D$ and $x\in\mathrm{cl}M$.

COROLLARY 2. *Let $X$ be a thick space. If $\mathfrak{m}\geqslant\chi(X)$, then each subset of $X$ satisfying $(*)$ for this $\mathfrak{m}$ is $X$ itself.*

The following is a corollary of the Esenin-Volpin type:

COROLLARY 3. *If $X$ is thick and $sX\leqslant\chi(X)$, then $wX = \chi(X)$.*

For $X$ Dantean, the equality $wX = \chi(X)$ holds without any additional assumption (Arhangelskiĭ [1]).

COROLLARY 4. *If $X$ is thick and $\chi(X)\leqslant sX$, then $wX = sX$.*

The following lemma, provided here with a direct proof, follows also from a theorem in [1].

LEMMA 3. *Let $X$ be a metric space with a countable base. A compactification $rX$ of $X$ is thick if and only if $rX$ is metrizable.*

Proof. Let $X$ be a metric space, $\mathfrak{B}$ a given countable base of the topology on $X$, and $rX$ a compactification of $X$.

I. If $rX$ is thick, take a dense subset $D$ of $rX$ satisfying $(*)$ for $\mathfrak{m} = \aleph_0$. From the well-known fact that $\chi(x, X)$ is equal to $\chi(x, Y)$ if $X$ is dense in $Y$, we infer that $\chi(x, rX)\leqslant\aleph_0$, and by Lemma 2 we have $X\subset D$. In each $U\in\mathfrak{B}$ take one point $x_U\in D\cap U$. Let $A = \{x_U\colon U\in\mathfrak{B}\}$. Since $rX$ is thick, $\mathrm{card}A\leqslant\aleph_0$, and $A\subset D$, we infer from $(*)$ that $\mathrm{cl}A$ is compact with the weight not greater than $\aleph_0$. Since $A$ is dense in $X$, we have

$X \subset \mathrm{cl}_D A \subset rX$ and, in consequence, $\mathrm{cl}_D A = rX$. Hence $rX$ is metric, since so is $\mathrm{cl}_D A$.

II. If $rX$ is metric, it must be thick by having a countable base.

Let us observe that the Čech-Stone compactification of a metric non-compact space with a countable base is not thick.

Proof of Theorem 2. Assume to the contrary that $\beta X$ is thick and that $X$ is not pseudocompact. Then there exists a real-valued function $f$ such that $f(X)$ is not compact. Hence, by the remark preceding the proof, $\beta f(X)$ is not a thick space. Let $f_*\colon \beta X \to \beta f(X)$ be the induced map. Hence $\beta f(X)$ is thick as an image of a thick space. A contradiction.

Since the long line with $\omega_1$ added is thick but not dyadic, Theorem 2 is an essential generalization of a theorem by Engelking and Pełczyński. The converse of Theorem 2 is not true, since there exist pseudocompact spaces having not thick Čech-Stone compactifications, e.g. $\beta N \setminus \{x\}$, where $x \in \beta N \setminus N$.

## REFERENCES

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