87 (1922), pp. 39–65). This work was based on his refined techniques for treating sums of exponentials and was certainly one of the deepest pieces of analytic number theory ever made. Some further improvements led him a few years later to the exponent 27/82. The method could be applied to the circle problem by his student L.W. Nieland, with the same error estimate $O(x^{27/28})$. Another problem attacked by the method was the order of growth of the Riemann zeta function, on which Van der Corput collaborated with his student J. F. Koksma.

Until about 1940 Van der Corput was extremely active in analytic number theory, with series of papers on Diophantine approximation, Vinogradov's method, Goldbach's conjecture, and Geometry of numbers. After 1940 Van der Corput added to his work on number theory an active interest in many other branches of mathematics. His earlier work on the asymptotic method of stationary phase initialized extensive work in his later theory of neutrices. He wrote on a wide variety of subjects, like the study of functional equations for the elementary functions, and a new proof for the fundamental theorem of algebra. Nevertheless he kept working in number theory too. Special attention might be given to what was the first complete account on the Erdös–Selberg elementary proof of the prime number theorem (Math. Centrum Amsterdam, Scriptum no. 1 (1948)).

Van der Corput was very stimulating as a teacher, and made his students collaborate with him on his best ideas. In particular four of his best students might be mentioned who all died before him: L. W. Nieland, J. F. Koksma, J. Popken, C. S. Meijer.

A very remarkable episode in Van der Corput's life was his initiative in 1946 to start a national institution for the promotion of both pure and applied mathematics, in order to give the mathematical background for the post-war industrial development in the Netherlands. This Mathematical Centre did indeed provide such a background in various areas, especially in statistics and computer science. Van der Corput was its first director (1946–1953).

The Royal Netherlands Academy of Sciences and Letters made Van der Corput a member in 1929. Furthermore he was honoured by doctorates honoris causa from the University of Bordeaux and from the Technological University at Delft, and by the membership of the Royal Academy for Sciences and Letters of Belgium.

Van der Corput was an editor of Acta Arithmetica from its start in 1936, and he had an article in its first volume.

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Superprimes and a generalized Frobenius symbol

by

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Dedicated to Karl Dörge on occasion of his 75th birthday

This paper can be considered from three points of view: 1. ultrafilter invariants, 2. J. Ax's theory of ultraproducts of finite fields [2], and 3. Jarden's "translation principle" ([10], [12]) which connects the Dirichlet density with the Haar measure in the Galois group. We shall give an extension of ordinary arithmetic of global fields k, which — in a certain sense — can be interpreted as arithmetic of special non-standard models of k. The results will be applied to the first two cases mentioned above.

According to the general philosophy, non-principal ultrafilters are so highly unconstructive that they cannot be distinguished from one another (at least in the case of a countable index set, Bell–Slomson [5]). However, we shall define number theoretic invariants, related to class field theory, which allow us to divide all non-principal ultrafilters on the set of all primes into $2^{\aleph_0}$ different classes (Section 2).

More generally, we consider the space Ω_k of all non-principal ultrafilters U on the set P_k of all prime divisors of a global field k. Since they are related to prime divisors of a non-standard model $^*k$ of k we call the ultrafilters $U \in \Omega_k$ the "superprimes" of k. As is well known ([4] or [7]) Ω_k is a compact subspace of the Stone–Čech compactification $\hat{P}_k$ of the discrete countable set P_k. For those superprimes we can define the usual notions, such as "lying over", ramification, residue class fields, etc., and obtain analogous elementary properties, e.g. $n = \sum e_i f_i$ (Section 1). It turns out that superprimes are always unramified.

To each superprime of k we can attach a generalized Frobenius- and Artin symbol. Our main result states that for a (not necessary finite) Galois extension $K|k$ of a global field k the generalized Artin symbol defines a continuous surjective mapping

$$\left(\frac{K|k}{}\right): \Omega_k \to \varkappa G(K|k)$$

of the compact superprime space Ω_k onto the compact space $\varkappa G(K|k)$ of all conjugacy classes of the Galois group $G(K|k)$. This mapping extends the ordinary reciprocity homomorphism of class field theory via the invariant mapping inv_k: $\Omega_k \to \bar{C}_k$ into the reduced idele class group $\bar{C}_k$ of k (theorem (3.7)).

Since for a Galois extension $K|k$ the Galois group $G(K|k)$ acts in a natural way on Ω_K, we can define the decomposition group of a superprime to be its fixed group, as usual. It turns out to be a procyclic group generated by the corresponding generalized Frobenius symbol (theorem (4.2)). The decomposition fields can be described as the field of k-algebraic elements in certain ultraproducts of finite fields.

In Section 5 we give the applications to Ax's theory [2]. In his paper Ax characterizes the field of absolute algebraic elements of certain ultraproducts of finite fields. Our refinement of Ax's result is as follows: the absolute algebraic subfield $\mathrm{Abs}(\bar{k}_U)$ of an ultraproduct $\bar{k}_U = \prod_{\mathfrak{p}\in P_k} \bar{k}_\mathfrak{p}/U$ of the (finite) residue class fields $\bar{k}_\mathfrak{p}$ of a number field k is just the decomposition field of the superprime U (theorem (5.2)). Moreover two ultraproducts $\prod_{\mathfrak{p}\in P_k} k_\mathfrak{p}/U_i$ of the $\mathfrak{p}$-adic hulls $k_\mathfrak{p}$ are k-isomorphic if and only if the generalized Artin symbols of U_1 and U_2 lie in the same division ("Abteilung") of the total Galois group G_k over k (theorem (5.4)).

Parts of this paper were the subject of a lecture the first author gave at King's College, London, about two years ago. A small problem posed in this talk has been solved in the meantime by M. Jarden [11].

1. The space of superprimes and U-adic hulls. For a global field k let P_k denote the set of all prime divisors of k and Ω_k the space of all non-principal ultrafilters on P_k. Hence $\Omega_k = \hat{P}_k \setminus P_k$, where $\hat{P}_k$ denotes the Stone–Čech-compactification of the discrete space P_k. Ω_k is a compact space with respect to the Zariski topology: The system of closed subsets of Ω_k is given by all $\Omega_\mathscr{F} := \{U \in \Omega_k \mid \mathscr{F} \subseteq U\}$, $\mathscr{F}$ a filter on P_k (inclusively the power set $\mathscr{P}(P_k)$ of P_k). The mapping $P_k \supseteq D \mapsto \Omega^D := \{U \in \Omega_k \mid D \in U\}$ is a lattice morphism from the power set $\mathscr{P}(P_k)$ onto a basis of the open and of the closed subsets of Ω_k. (For facts on Stone–Čech-compactification and its connection with ultrafilters see [4] and [7].)

Let $k_\mathfrak{p}$ denote the $\mathfrak{p}$-adic completion of k with respect to a prime-divisor $\mathfrak{p} \in P_k$. We define for every superprime $U \in \Omega_k$ the ultraproduct $\prod_{\mathfrak{p}\in P_k} k_\mathfrak{p}/U =: k_U$ as the *U-adic hull* of k. (For basic facts concerning ultraproducts see Bell–Slomson [5], ch. 5.)

According to the theorem of Łoś (see Bell–Slomson [5], loc. cit.) these are henselian valued fields with residue class field $\prod_{\mathfrak{p}\in P_k} \bar{k}_\mathfrak{p}/U$, where $\bar{k}_\mathfrak{p}$ denotes the residue class field of k with respect to $\mathfrak{p}$. The valuation is given by v_U: $k_U^\times \to \boldsymbol{P}^{P_k}/U$, $(a_\mathfrak{p})^U_{\mathfrak{p}\in P_k} \mapsto (v_\mathfrak{p}(a_\mathfrak{p}))^U_{\mathfrak{p}\in P_k}$. (Here $v_\mathfrak{p}$ denotes the normed valuation $v_\mathfrak{p}$: $k_\mathfrak{p}^\times \to \boldsymbol{Z}$ corresponding to $\mathfrak{p}$. The elements in an ultraproduct $\prod_{i\in I} M_i/U$ are equivalence classes of $(m_i)_{i\in I} \in \prod_{i\in I} M_i$ modulo the ultrafilter U, they will be denoted by $(m_i)^U_{i\in I}$ or $(m_i)^U$.)

We remark explicitly that k_U is not complete with respect to v_U, but only henselian. Nevertheless we call k_U the U-adic hull of k in order to stress their analogy with the usual $\mathfrak{p}$-adic hulls $k_\mathfrak{p}$.

Every U-adic hull k_U of k contains a non-standard model of k, namely the ultrapower ${}^*k \simeq k^{P_k}/U$. Since for different ultrafilters U these ultrapowers are saturated, of same cardinality and elementarily equivalent, they are isomorphic to each other (Bell–Slomson [5], ch. 11), and therefore ${}^*k$ is uniquely determined up to isomorphism. Thus a superprime U of k can be regarded as a prime divisor v_U on a non-standard model ${}^*k$ of k.

In order to define notions like "lying over", ramification, etc. one has to look for the possibility of embedding k_U into K_V for global fields $k \subseteq K$ and superprimes U, V of these fields. Firstly, let us make the following remark whose proof is straightforward:

(1.1) Remark. (a) Every mapping φ: $I \to J$ of sets I, J induces a continuous mapping $\hat{\varphi}$ of the ultrafilter spaces by $U \mapsto \hat{\varphi} U := \{M \subseteq J \mid \varphi^{-1}M \in U\}$. ($\hat{\varphi}$ is exactly the Stone–Čech-extension of φ.)

(b) $\hat{\varphi}$ is surjective if and only if φ is, and in that case $\hat{\varphi}U = \{\varphi M \mid M \in U\}$ holds.

(c) $\hat{\varphi}U$ is a principal ultrafilter if and only if there is a set $M \in U$ on which φ is constant. In particular, $\hat{\varphi}U$ is principal, if U is principal.

(d) Hence for the spaces Ω_I, Ω_J of all non-principal ultrafilters on I resp. J the following holds:

$\hat{\varphi}$ maps Ω_I into Ω_J if and only if all fibres $\varphi^{-1}\{j\}$ of φ are finite.

Let $K|k$ be a finite extension of global fields and j: $P_K \to P_k$ the restriction of primes.

(1.2) Lemma. *The restriction j induces a finite covering $j_{K|k}$: $\Omega_K \to \Omega_k$, $V \mapsto V_k$ of compact spaces, where V_k denotes the superprime*

$$\hat{j}V = \{M \subseteq P_k \mid j^{-1}M \in U\} = \{jD \mid D \in V\}.$$

Instead of $V_k = U \in \Omega_k$ we write $V|U$ and say: V divides U.

Proof. According to (1.1) j induces a continuous surjective mapping $\hat{j} =: j_{K|k}$: $\Omega_K \to \Omega_k$. Since the number of elements in each fibre $j^{-1}\{\mathfrak{p}\}$ is bounded by $n = (K:k)$, P_K is a union $P_K \doteq \bigcup_{i=1}^n D_i$ of sets D_i for which j: $D_i \simeq P_k$ is bijective. (For D_1 take in each fibre "the first" element, for D_2 "the second" and so on; in general the union is not disjoint because not all fibres contain n elements.) From $P_K = \bigcup_{i=1}^n D_i$ it follows that

$\Omega_K = \Omega^{P_K} = \bigcup_{i=1}^{n} \Omega^{D_i}$ with open subsets Ω^{D_i}. Finally, these Ω^{D_i} are homeomorphic to Ω_k under $j_{K|k}$, since ultrafilters $U \in \Omega^{D_i}$ are uniquely determined by their restrictions $U|_{D_i} := \{M \cap D_i \mid M \in U\}$ to D_i. Hence the proof of lemma (1.2) is complete.

(1.3) Remark. Each k-isomorphism $\sigma\colon K_1 \to K_2$ of global fields determines naturally a homeomorphism (again called σ) $\sigma\colon \Omega_{K_1} \to \Omega_{K_2}$, for which the diagram

$$\begin{tikzcd} \Omega_{K_1} \arrow[rr, "\sigma"] \arrow[dr, "j_{K_1|k}"'] & & \Omega_{K_2} \arrow[dl, "j_{K_2|k}"] \\ & \Omega_k & \end{tikzcd}$$

commutes.

In particular, if $K|k$ is a finite Galois extension then the Galois group $G(K|k)$ acts faithfully on Ω_K.

Proof. $\sigma\colon K_1 \to K_2$ induces a bijection $\sigma'\colon P_{K_1} \to P_{K_2}$ for which the diagram

$$\begin{tikzcd} P_{K_1} \arrow[rr, "\sigma'"] \arrow[dr] & & P_{K_2} \arrow[dl] \\ & P_k & \end{tikzcd}$$

commutes. Hence, according to (1.1) σ' determines a homeomorphism $\tilde{\sigma}\colon \Omega_{K_1} \to \Omega_{K_2}$, defined by $\tilde{\sigma} U = \{\sigma' M \mid M \in U\}$. This homeomorphism may also be called σ, because different isomorphisms $\tau, \sigma\colon K_1 \to K_2$ induce different mappings $\tilde{\tau}, \tilde{\sigma}$. For, let $\tilde{\tau} = \tilde{\sigma}$. Then $\tilde{\varrho} = \mathrm{id}$ with the k-automorphism $\varrho = \sigma^{-1} \circ \tau$ of K_1. Since $\tilde{\varrho} U = U$ is equivalent to $\mathrm{Fix}(\varrho') \in U$ (Klingen [13], Prop. 5.2) $\tilde{\varrho} = \mathrm{id}$ means that $\mathrm{Fix}(\varrho')$ is a cofinite set in P_{K_1}. Therefore almost all primes of the fixed field L of ϱ are undecomposed in K_1. Hence $L = K_1$ and ϱ is the identity. This shows that in the case of a Galois extension the action of the Galois group is faithful, and the proof of (1.3) is complete.

(1.4) Lemma. (a) *The natural (diagonal) embedding* $\prod_{\mathfrak{p} \in P_k} k_{\mathfrak{p}} \to \prod_{\mathfrak{P} \in P_K} K_{\mathfrak{P}}$ *induces an embedding of valued fields* $k_U := \prod_{\mathfrak{p} \in P_k} k_{\mathfrak{p}}/U \to \prod_{\mathfrak{P} \in P_K} K_{\mathfrak{P}}/V =: K_V$ $(U \in \Omega_k,\ V \in \Omega_K)$ *if and only if* V *divides* U.

(b) *For ultrapowers one gets a still stronger result: If* $V \in \Omega_K$ *divides* $U \in \Omega_k$, *the induced embedding* $\boldsymbol{Z}^{P_k}/U \to \boldsymbol{Z}^{P_K}/V$ *is an isomorphism.*

Since for all non-principal U resp. V the ultrapowers $\boldsymbol{Z}^{P_k}/U$ and $\boldsymbol{Z}^{P_K}/V$ are isomorphic — use the same argument concerning ${}^*k$ — the emphasis of (b) lies in fact that there is a natural isomorphism if V divides U.

Proof of (1.4). (a) Let $\varphi\colon \prod_{\mathfrak{p} \in P_k} k_{\mathfrak{p}} \to \prod_{\mathfrak{P} \in P_K} K_{\mathfrak{P}}$ denote the natural embedding, that means $\varphi\big((a_{\mathfrak{p}})_{\mathfrak{p} \in P_k}\big) = (a_{j(\mathfrak{P})})_{\mathfrak{P} \in P_K}$. Because of the definition of the valuations v_U and v_V on k_U resp. K_V this φ induces an embedding $\tilde{\varphi}\colon k_U \to K_V$ of valued fields if and only if the following equivalence holds for every $(a_{\mathfrak{p}}), (\beta_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in P_k} k_{\mathfrak{p}}$:

$$\{\mathfrak{p} \in P_k \mid a_{\mathfrak{p}} = \beta_{\mathfrak{p}}\} \in U \Leftrightarrow \{\mathfrak{P} \in P_K \mid a_{j(\mathfrak{P})} = \beta_{j(\mathfrak{P})}\} \in V.$$

But

$$\{\mathfrak{P} \in P_K \mid a_{j(\mathfrak{P})} = \beta_{j(\mathfrak{P})}\} = j^{-1}\{\mathfrak{p} \in P_k \mid a_{\mathfrak{p}} = \beta_{\mathfrak{p}}\},$$

so the validity of this equivalence for all $(a_{\mathfrak{p}}), (\beta_{\mathfrak{p}})$ means: $M \in U \Leftrightarrow j^{-1} M \in V$ for all $M \subseteq P_k$.

(b) In analogy to (a) one has an embedding $\iota\colon \boldsymbol{Z}^{P_k}/U \to \boldsymbol{Z}^{P_K}/V$. Now let $P_K = \bigcup_{i=1}^{n} D_i$ with $j\colon D_i \xrightarrow{\sim} P_k$ bijective (see proof of (1.2)). Then there is a $\nu \in \{1, \ldots, n\}$ with $D_\nu \in V$. Let $\varrho\colon P_k \to D_\nu$ denote the inverse mapping of $j|_{D_\nu}$. Then one finds for every $(m_{\mathfrak{P}})^V_{\mathfrak{P} \in P_K} \in \boldsymbol{Z}^{P_K}/V$ the element $(m_{\varrho(\mathfrak{p})})^U_{\mathfrak{p} \in P_k} \in \boldsymbol{Z}^{P_k}/U$ as an inverse image under ι; for,

$$\iota\big((m_{\varrho(\mathfrak{p})})^U_{\mathfrak{p} \in P_k}\big) = (m_{\varrho(j(\mathfrak{P}))})^V_{\mathfrak{P} \in P_K} = (m_{\mathfrak{P}})^V_{\mathfrak{P} \in P_K}$$

since $\{\mathfrak{P} \in P_K \mid m_{\varrho(j(\mathfrak{P}))} = m_{\mathfrak{P}}\} \supseteq D_\nu \in V$. This proves the lemma.

As a consequence of (1.4) we can make the following definitions: Let $K|k$ be a finite extension of global fields and V resp. U superprimes of K resp. k. If V divides U we define:

$e(V|U) = e_{K|k}(V) = \big(v_V(K_V^\times) : v_V(k_U^\times)\big)$ *ramification index* of $V|U$,

$f(V|U) = f_{K|k}(V) = (\bar{K}_V : \bar{k}_U)$ *residue class degree* of $V|U$,

$g_{K|k}(U)$ = number of $V \in \Omega_K$ dividing U.

Since for relational structures M with finite underlying set the diagonal embedding $d\colon M \to M^I/U$ into an arbitrary ultrapower is an isomorphism, such ultrapowers will be identified with their basis. Thus

$$(m_i)^U_{i \in I} = m \Leftrightarrow (m_i)^U_{i \in I} = d(m) \Leftrightarrow \{i \in I \mid m_i = m\} \in U.$$

Using this identification we get under the assumptions of the preceding definitions:

(1.5) Proposition. *Let* $K|k$ *be a finite extension of global fields and* $V|U$ *superprimes. Then:*

(a) $e_{K|k}(V) = \big(e_{K|k}(\mathfrak{P})\big)^V_{\mathfrak{P} \in P_K}$, $f_{K|k}(V) = \big(f_{K|k}(\mathfrak{P})\big)^V_{\mathfrak{P} \in P_K}$ *and* $g_{K|k}(U) = \big(g_{K|k}(\mathfrak{p})\big)^U_{\mathfrak{p} \in P_k}$ *are natural numbers* $\leqslant (K : k)$. *In any case* $e_{K|k}(V) = 1$ *holds: All superprimes are unramified.*

(b) *The residue class degree equals the "local degree":*

$$f_{K|k}(V) = (K_V : k_U).$$

(c) *The usual relation*

$$(K:k) = \sum_W f(W\,|\,U)$$

holds, the sum taken over all $W \in \Omega_K$ dividing a fixed superprime $U \in \Omega_k$.

(d) *In the case of a Galois extension $K|k$ all superprimes V of K lying over U are conjugate under the action of the Galois group $G(K|k)$. Hence in that case*

$$(K:k) = g_{K|k}(U)\cdot f_{K|k}(U)$$

where $f_{K|k}(U) := f(V\,|\,U)$ for all $V\,|\,U$.

Proof. (a) The presentation of the number $e_{K|k}(V)$ and $f_{K|k}(V)$ as elements of an ultrapower of natural numbers is an immediate consequence of the identification $k_U \to K_V$ and Łoś theorem. For instance one has

$$e_{K|k}(V) = \big(v_V(K_V^\times) : v_V(k_U^\times)\big) = \Big(\prod_{\mathfrak{P}\in P_K} v_{\mathfrak{P}}(K_{\mathfrak{P}}^\times)\big/V : \prod_{\mathfrak{P}\in P_K} v_{\mathfrak{P}}(k_{j(\mathfrak{P})}^\times)\big/V\Big).$$

which, using Łoś theorem and the boundedness of all $e_{K|k}(\mathfrak{P})$ by $(K:k)$, is the following natural number $\leqslant (K:k)$:

$$\big(v_{\mathfrak{P}}(K_{\mathfrak{P}}^\times) : v_{\mathfrak{P}}(k_{j(\mathfrak{P})}^\times)\big)_{\mathfrak{P}\in P_K}^V = \big(e_{K|k}(\mathfrak{P})\big)_{\mathfrak{P}\in P_K}^V.$$

Similarly for the numbers f. Since $e_{K|k}(\mathfrak{P}) = 1$ for almost all $\mathfrak{P} \in P_K$ we conclude $e_{K|k}(V) = 1$. This proves the statements concerning e and f.

For a natural number g the equality $g = \big(g_{K|k}(\mathfrak{p})\big)_{\mathfrak{p}\in P_k}^U$ means that the set

$$M := \{\mathfrak{p}\in P_k \mid \text{there exist exactly } g \text{ primes of } K \text{ over } \mathfrak{p}\}$$

belongs to U, and hence

$$N := \{\mathfrak{P}\in P_K \mid j(\mathfrak{P}) \in M\} \in V \quad \text{for every } V\,|\,U.$$

By definition of M we can write N as a union $N = \bigcup_{i=1}^{g} N_i$ of disjoint sets $N_i \subseteq P_K$ with $j\colon N_i \xrightarrow{\sim} M$ bijective (see proof of (1.2)). Hence the superprimes $V \in \Omega_K$ lying over U contain exactly one N_i and, since $j\colon N_i \xrightarrow{\sim} M$ is bijective, two superprimes $V_1, V_2\,|\,U$ containing the same N_i must coincide. So there are exactly as many superprimes V over U as sets N_i, namely g. This proves $g_{K|k}(U) = \big(g_{K|k}(\mathfrak{p})\big)_{\mathfrak{p}\in P_k}^U$.

(b) Because of Łoś theorem $(K_V : k_U)$ equals

$$(K_{\mathfrak{P}} : k_{j(\mathfrak{P})})_{\mathfrak{P}\in P_K}^V = \big(e_{K|k}(\mathfrak{P}) f_{K|k}(\mathfrak{P})\big)_{\mathfrak{P}\in P_K}^V$$

and, using (a), this proves (b).

(c) The statements (a) and (b) suggest the validity of (c) because of the fundamental equations $(K:k) = \sum_{\mathfrak{P}|\mathfrak{p}} e_{K|k}(\mathfrak{P}) f_{K|k}(\mathfrak{P})$ in global fields. However, one has to pay attention to the fact that different ultrafilters appear in the sum of (c). Let $V_1, \ldots, V_g$ be the superprimes of K dividing U and $e_i := e(V_i\,|\,U)$, $f_i := f(V_i\,|\,U)$. That means

$$E_i := \{\mathfrak{P}\in P_K \mid e_{K|k}(\mathfrak{P}) = e_i\} \in V_i \quad\text{and}\quad F_i := \{\mathfrak{P}\in P_K \mid f_{K|k}(\mathfrak{P}) = f_i\} \in V_i.$$

Moreover, because of (a), we have $M := \{\mathfrak{p}\in P_k \mid g_{K|k}(\mathfrak{p}) = g\} \in U$, and hence $j^{-1}M =: N = \bigcup_{i=1}^{g} N_i$ with $N_i \in V_i$ (see proof of (a)). Since $\bigcap_{i=1}^{g} j(E_i \cap F_i \cap N_i) \in U$ holds, we can choose $\mathfrak{q} \in P_k$ and $\mathfrak{P}_i \in E_i \cap F_i \cap N_i$ such that $\mathfrak{P}_i | \mathfrak{q}$ for all $i \in \{1, \ldots, g\}$. Now $\mathfrak{P}_i \in E_i \cap F_i$ implies $e_i = e_{K|k}(\mathfrak{P}_i)$ and $f_i = f_{K|k}(\mathfrak{P}_i)$, and $\mathfrak{P}_i \in N_i$ $(i = 1, \ldots, g)$ means that $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ are exactly the prime divisors of $\mathfrak{q}$ in K. Therefore

$$\sum_{W|U} e_{K|k}(W) f_{K|k}(W) = \sum_{i=1}^{g} e_i f_i = \sum_{i=1}^{g} e_{K|k}(\mathfrak{P}_i)\, f_{K|k}(\mathfrak{P}_i) = (K:k).$$

(d) Choose $M \subseteq P_K$ with $j\colon M \xrightarrow{\sim} P_k$ bijective. Since $K|k$ is a Galois extension, one has $P_K = \bigcup_{\sigma\in G(K|k)} \sigma M$. Hence for every $V, V' \in \Omega_K$ there exist $\sigma, \sigma' \in G(K|k)$ with $\sigma M \in V$, $\sigma' M \in V'$. If V and V' divide U then also $\sigma^{-1} V$, $\sigma'^{-1} V'$ divide U (see (1.3)), and therefore the restrictions $\sigma^{-1} V|_M$ and $\sigma'^{-1} V'|_M$ must coincide since $j\colon M \xrightarrow{\sim} P_k$ is bijective. From this we conclude $\sigma^{-1} V = \sigma'^{-1} V'$ (see proof of (1.2)), which proves (d) and hence the proposition.

Using inverse resp. direct limits we will extend the definitions of superprimes and U-adic hulls from global fields k to arbitrary separable algebraic extensions K of global fields. For such fields K we define the *space of superprimes* by $\Omega_K := \varprojlim \Omega_k$; here the projective limit is taken over all global subfields k of K and with respect to the mappings in (1.2). For an extension $L|K$ of separable algebraic extensions of global fields let $j_{L|K}\colon \Omega_L \to \Omega_K$ denote the natural projection. For $V \in \Omega_K$ the *V-adic hull* of K is defined by $K_V := \varinjlim k_{V_k}$; here the direct limit is taken over all global subfields k of K with respect to the mappings of (1.4), and V_k denotes the restriction $j_{K|k}(V)$ of V to k. In the case that K and L are global fields all these definitions agree with the former ones.

An immediate generalization of (1.3) is the following fact:

(1.6) Lemma. *Let K_0, K, L be separable algebraic extensions of global fields and $K_0 \subseteq K$, $K_0 \subseteq L$. Then each K_0-isomorphism $\sigma\colon K \xrightarrow{\sim} L$ induces* *a homeomorphism $\sigma\colon \Omega_K \to \Omega_L$ with $j_{L|K_0} \circ \sigma = j_{K|K_0}$. If $L|K$ is a Galois extension, the Galois group $G(L|K)$ acts faithfully on Ω_L and the fibres of $j_{L|K}$ are exactly the orbits of Ω_L under the action of $G(L|K)$.*

Proof. Since $\Omega_L = \varprojlim_l \Omega_l = \varprojlim_k \Omega_{\sigma[k]}$, where l resp. k run over all global subfields of L resp. K, the first part of (1.6) follows from (1.3). The faithful action of $G(L|K)$ on Ω_L in the Galois case is an immediate consequence of (1.3) too.

Because of $j_{L|K} \circ \sigma = j_{L|K}$ each orbit of $G(L|K)$ lies in one fibre of $j_{L|K}$. Now take superprimes V, W of L with $j_{L|K} V = j_{L|K} W$. Since $L|K$ is a Galois extension, the set $\mathscr{L}$ of all global fields $l \subseteq L$ which are Galois extensions of $l \cap K$ is cofinal in the set of all global fields $l \subseteq L$. For, let $a \in L$ be an arbitrary element and k a global subfield of K containing the coefficients of the minimal polynomial f of a over K. Then, if we choose l as the normal hull of $k(a)|k$, we get a global field $l \subseteq L$ which is Galois over $l \cap K$ and contains a. Because of the cofinality of $\mathscr{L}$ we have $\Omega_L = \varprojlim_{l \in \mathscr{L}} \Omega_l$. According to proposition (1.5) (d) the sets $M_l = \{\varrho \in G(l|l \cap K) \mid \varrho(V_l) = W_l\}$ are non-empty for all $l \in \mathscr{L}$ and hence we can choose $\sigma \in \varprojlim_{l \in \mathscr{L}} M_l$. For all $l \in \mathscr{L}$ we have $\sigma|_l(V_l) = W_l$ and hence $\sigma V = W$. Moreover, for all global subfields k of K we have $\sigma|_k = \mathrm{id}$, so that σ is an element in $G(L|K)$ with the desired property. This completes the proof of lemma (1.6).

2. Stone–Čech extension and a number theoretic invariant of ultrafilters. As mentioned in Section 1 the space Ω_k of superprimes of a global field k is a compact subspace of the Stone–Čech compactification $\hat{P}_k$ of the discrete space P_k. Hence arbitrary mappings $\tau\colon P_k \to C$ into compact spaces C extend uniquely to continuous mappings $\hat{\tau}\colon \hat{P}_k \to C$, which especially are defined on Ω_k.

A slightly different situation occurs if the space C is a projective limit of compact spaces C_i $(i \in I)$

$$C = \varprojlim C_i \text{ with defining } \pi_{ij}\colon C_j \to C_i,$$

and (τ_i) a family of compatible maps into the C_i defined on *varying* cofinite subsets D_i of a discrete set P

$$\tau_i\colon D_i \to C_i \text{ with } P \setminus D_i \text{ finite}.$$

It turns out that the family (τ_i) can be extended to a continuous mapping, not on the whole compactification $\hat{P}$ but on the compact subspace $\Omega_P = \hat{P} \setminus P$, with values in C. This is the content of the

(2.1) Extension principle. *By assumption, the family $(\tau_i)_{i \in I}$ of mappings $\tau_i\colon D_i \to C_i$, defined almost everywhere, i.e. on cofinite sets D_i, fulfils the compatibility conditions*

$$\tau_i = \pi_{ij} \circ \tau_j \quad \text{on} \quad D_i \cap D_j \quad \text{for } i \leqslant j.$$

The natural projections $\pi_i\colon C \to C_i$ from the projective limit $C = \varprojlim_{i \in I} C_i$ into C_i are assumed to be surjective.

(a) *Then there is a uniquely determined continuous mapping $\hat{\tau}\colon \Omega_P \to C$ with the property*

$$\pi_i \circ \hat{\tau} = \hat{\tau}_i \quad \text{for all } i \in I.$$

Here $\hat{\tau}_i\colon \Omega_P \to C_i$ denotes the restriction to Ω_P of the Stone–Čech extension of τ_i.

(b) *In the case of finite sets C_i the map $\hat{\tau}$ can be described explicitly as follows:*

For an element $c = (c_i)_{i \in I} \in C$ and a non-principal ultrafilter $U \in \Omega_P$ the following holds:

$$\hat{\tau}(U) = c \Leftrightarrow U \text{ contains the filter basis } \{\tau_i^{-1}\{c_i\} \mid i \in I\}.$$

Hence, $\hat{\tau}$ is surjective if and only if all fibres of all τ_i are infinite.

Proof. In order to prove (a) one observes that the closure $\overline{D_i}$ of D_i in $\hat{P}$ contains Ω^{D_i} (see [7]) and that $\Omega^{D_i} = \Omega_P$ since D_i is cofinite. The Stone–Čech extensions of the τ_i are defined on $\overline{D_i}$ and their restrictions $\hat{\tau}_i$ to Ω_P fulfil the corresponding compatibility conditions:

$$\hat{\tau}_i = \pi_{ij} \circ \hat{\tau}_j \quad \text{for} \quad i \leqslant j.$$

Hence the family $(\hat{\tau}_i)$ determines a unique continuous mapping $\hat{\tau}\colon \Omega_P \to C$ into the projective limit $C = \varprojlim_{i \in I} C_i$ with the asserted property.

For a proof of (b) we assume all C_i to be finite sets. Because of $\overline{D_i} = \{U \in \hat{P} \mid D_i \in U\}$ (see [7]) the equivalence class $(\tau_i(p))_{p \in P}^U \in C_i^P/U$ is well defined for every $U \in \overline{D_i}$ and can be considered as an element of C_i by the identification in Section 1. This defines an extension $\tilde{\tau}_i\colon \overline{D_i} \to C_i$ of τ_i by

$$\tilde{\tau}_i(U) = (\tau_i(p))_{p \in P}^U \in C_i \quad \text{for} \quad U \in \overline{D_i}$$

which in fact is the Stone-Čech extension of τ_i since it is continuous:

According to the definition (and the identification in Section 1) we have

$$\tilde{\tau}_i(U) = c_i \Leftrightarrow U \ni \tau_i^{-1}\{c_i\}$$

and hence the inverse image of $c_i \in C_i$ under $\tilde{\tau}_i$ is

$$\{U \in \overline{D}_i \mid \tau_i^{-1}\{c_i\} \in U\}$$

which clearly is an open set (see Sec. 1) in $\hat{P}$.

So we have shown that the continuous mappings $\hat{\tau}_i$ of (a) are given explicitly by

$$\text{(2.2)} \qquad \begin{aligned} &\hat{\tau}_i(U) = \big(\tau_i(p)\big)^U_{p\in P} \in C_i \quad \text{for} \quad U \in \Omega_P, \text{ or} \\ &\hat{\tau}_i(U) = c_i \Leftrightarrow U \ni \tau_i^{-1}\{c_i\}. \end{aligned}$$

For $\hat{\tau}$ this means:

$$\hat{\tau}(U) = (c_i)_{i\in I} \Leftrightarrow U \ni \tau_i^{-1}\{c_i\} \quad \text{for all } i \in I.$$

This proves the first part of (b).

As to the last part we notice that for each element $c = (c_i)_{i\in I} \in C$ the sets $\tau_i^{-1}\{c_i\}$, $i \in I$, form a filter basis on P. Hence there is a non-principal ultrafilter U containing this filter basis if and only if all $\tau_i^{-1}\{c_i\}$ are infinite. This completes the proof of the extension principle.

As a first application of the extension principle we shall define a number theoretic invariant of ultrafilters as a continuous mapping

$$\text{(2.3)} \qquad \mathrm{inv}_k\colon \Omega_k \to \overline{C_k}$$

which attaches to every superprime U of a global field k an element in the reduced idele class group $\overline{C_k}$ of k. This group is defined as the projective limit

$$\overline{C_k} = \varprojlim_{\mathscr{H}} C_k/\mathscr{H}$$

where $\mathscr{H}$ runs through all open subgroups of C_k of finite index. Hence $\overline{C_k}$ is a profinite group, isomorphic to the Galois group of the maximal abelian extension of k under the reciprocity isomorphism. In the number field case, $\overline{C_k}$ is the idele class group C_k modulo the connected component D_k of the identity:

$$\overline{C_k} = C_k/D_k \quad \text{in the number field case.}$$

We need some notations:

$\mathfrak{m}$ a divisor modulus of k in the sense of class field theory,

$D_k^{\mathfrak{m}}$ the group of (finite) divisors of k relatively prime to $\mathfrak{m}$,

$S_k(\mathfrak{m})$ the "Hauptdivisorstrahl" modulo $\mathfrak{m}$,

$\mathscr{E}_k(\mathfrak{m}) = \{\varepsilon k^\times \mid \varepsilon \text{ idele unit of } k,\ \varepsilon \equiv 1(\mathfrak{m})\}$ the group of idele classes representable by idele units congruent 1 modulo $\mathfrak{m}$,

$\delta(\mathfrak{m})$ set of prime divisors of $\mathfrak{m}$.

Since the ray class groups $D_k^{\mathfrak{m}}/S_k(\mathfrak{m})$ are isomorphic to $C_k/\mathscr{E}_k(\mathfrak{m})$ and since subgroups of C_k of finite index are open if and only if they contain some $\mathscr{E}_k(\mathfrak{m})$, we get

$$\overline{C_k} = \varprojlim_{\mathfrak{m},\mathscr{H}^{(\mathfrak{m})}} C_k/\mathscr{H}^{(\mathfrak{m})} \simeq \varprojlim_{\mathfrak{m},H^{(\mathfrak{m})}} D_k^{\mathfrak{m}}/H^{(\mathfrak{m})}.$$

Here for each divisor modulus $\mathfrak{m}$ $\mathscr{H}^{(\mathfrak{m})}$ resp. $H^{(\mathfrak{m})}$ runs over all subgroups of C_k resp. $D_k^{\mathfrak{m}}$ of finite index which contain $\mathscr{E}_k(\mathfrak{m})$ resp. $S_k(\mathfrak{m})$. Following Hasse [9], we call such subgroups $H^{(\mathfrak{m})}$ *ideal groups* mod $\mathfrak{m}$.

We want to remark that for number fields k we have the following description of $\overline{C_k}$, since in that case the ray class groups $D_k^{\mathfrak{m}}/S_k(\mathfrak{m})$ are finite:

$$\overline{C_k} = \varprojlim_{\mathfrak{m}} D_k^{\mathfrak{m}}/S_k(\mathfrak{m}) \quad \text{if } k \text{ is a number field.}$$

In the function field case, however, the ray class groups are merely finitely generated, and so, in general, we have to use the description of $\overline{C_k}$ mentioned above.

Clearly, the mappings

$$P_k \setminus \delta(\mathfrak{m}) \to D_k^{\mathfrak{m}}/H^{(\mathfrak{m})}, \quad \mathfrak{p} \mapsto \mathfrak{p}H^{(\mathfrak{m})}$$

form a compatible system of mappings defined almost everywhere on P_k. (Here, $\mathfrak{m}$ runs through all divisor moduli of k and $H^{(\mathfrak{m})}$ through all ideal groups mod $\mathfrak{m}$.) Therefore they induce, according to (2.1), a continuous mapping $\mathrm{inv}_k\colon \Omega_k \to \overline{C_k}$. This is the first step towards the following

(2.4) THEOREM. *For all global fields k there is a natural continuous and surjective mapping*

$$\mathrm{inv}_k\colon \Omega_k \to \overline{C_k}$$

attaching to each non-principal ultrafilter U on P_k an element in the reduced idele class group, called the invariant of U.

We have the following explicit description:

$$\mathrm{inv}_k U = (\mathfrak{a}_{H^{(\mathfrak{m})}} H^{(\mathfrak{m})})_{\mathfrak{m},H^{(\mathfrak{m})}} \in \varprojlim_{\mathfrak{m},H^{(\mathfrak{m})}} D_k^{\mathfrak{m}}/H^{(\mathfrak{m})}$$

if and only if U contains the filter basis formed by the arithmetic prime progressions

$$\{\mathfrak{p} \in P_k \mid \mathfrak{p} \equiv \mathfrak{a}_{H^{(\mathfrak{m})}} \bmod H^{(\mathfrak{m})}\}$$

($\mathfrak{m}$ divisor modulus of k, $H^{(\mathfrak{m})}$ ideal group mod $\mathfrak{m}$).

In the number field case this description can be simplified:

$$\operatorname{inv}_k U = (\mathfrak{a}_\mathfrak{m} S_k(\mathfrak{m})) \in \varprojlim_{\mathfrak{m}} D_k^\mathfrak{m}/S_k(\mathfrak{m})$$

if and only if U contains the filter basis of all arithmetic prime progressions

$$D_\mathfrak{m}(\mathfrak{a}_\mathfrak{m}) = \{\mathfrak{p} \in P_k \mid \mathfrak{p} \equiv \mathfrak{a}_\mathfrak{m} \bmod S_k(\mathfrak{m})\}.$$

Proof. The explicit formula for the invariant is based on the extension principle (2.1), (b). The simplified formula in the number field case follows from the finiteness of the ray class groups. Hence, to complete the proof of (2.4), we have only to show the surjectivity of inv_k. According to the explicit description we have to show: For all divisor moduli $\mathfrak{m}$, all ideal groups $H^{(\mathfrak{m})}$ mod $\mathfrak{m}$ and all $\mathfrak{a} \in D_k^\mathfrak{m}$ the sets

$$D_{H^{(\mathfrak{m})}}(\mathfrak{a}) = \{\mathfrak{p} \in P_k \mid \mathfrak{p} \equiv \mathfrak{a} \bmod H^{(\mathfrak{m})}\}$$

are infinite. In the number field case this is true according to the theorem on arithmetic progressions. The proof of this fact (see for instance Hasse [9], I § 5, 8) relies on the behaviour of the L-functions at $s = 1$. Since, according to F. K. Schmidt [16], the L-functions have the same behaviour in the function field case, the sets $D_{H^{(\mathfrak{m})}}(\mathfrak{a})$ are infinite in general.

3. The generalized Frobenius symbol as extension of the global reciprocity isomorphism. Let $k'|k$ be a finite Galois extension of global fields. By the procedure in Section 2 the usual Frobenius symbol

$$\left[\frac{k'|k}{\quad}\right]: P_{k'} \setminus R' \to G(k'|k),$$

where R' is the set of primes of k' ramified over k, has a unique continuous extension

$$(3.1)\qquad \left[\frac{k'|k}{\quad}\right]: \Omega_{k'} \to G(k'|k)$$

to the superprime space $\Omega_{k'}$. The explicit definition is, using (2.1),

$$(3.2)\qquad \begin{aligned} &\text{(i)} \quad \left[\frac{k'|k}{U}\right] = \left(\left[\frac{k'|k}{\mathfrak{p}'}\right]\right)^U_{\mathfrak{p}'\in P_{k'}} \in G(k'|k)^{P_{k'}}/U = G(k'|k), \\ &\text{(ii)} \quad \left[\frac{k'|k}{U}\right] = \sigma \ \text{ iff } \ U \ni \hat{D}_{k'|k}(\sigma) \end{aligned}$$

where for $\sigma \in G(k'|k)$

$$\hat{D}_{k'|k}(\sigma) = \left\{\mathfrak{p}' \in P_{k'} \,\middle|\, \left[\frac{k'|k}{\mathfrak{p}'}\right] = \sigma\right\}$$

denotes the *Čebotarev set* of σ.

Using the ordinary restriction formula for Frobenius symbols of global fields $k''|k'|k$

$$\left[\frac{k''|k}{\mathfrak{p}''}\right]\Bigg|_{k'} = \left[\frac{k'|k}{\mathfrak{p}'}\right] \quad \text{for} \quad \mathfrak{p}''|\mathfrak{p}'|\mathfrak{p},$$

and according to the definition (3.2), the mapping (3.1) can be defined for arbitrary Galois extensions $K|k$ of a global field k: There is a continuous mapping

$$(3.3)\qquad \left[\frac{K|k}{\quad}\right]: \Omega_K \to G(K|k) = G$$

uniquely determined by the conditions

$$(3.3')\qquad \left[\frac{K|k}{V}\right]\Bigg|_{k'} = \left[\frac{k'|k}{V_{k'}}\right] \quad \text{for all finite Galois } k'|k.$$

This means that for all finite Galois subextensions $k'|k$ of $K|k$ the diagrams

$$\begin{array}{ccc} \Omega_K & \xrightarrow{\left[\frac{K|k}{\ }\right]} & G(K|k) \\ {\scriptstyle j_{K|k'}}\downarrow & & \downarrow \\ \Omega_{k'} & \xrightarrow{\left[\frac{k'|k}{\ }\right]} & G(k'|k) \end{array}$$

commute and we have

$$\left[\frac{K|k}{V}\right] = \sigma \quad \text{iff} \quad V_{k'} \ni \hat{D}_{k'|k}(\sigma|_{k'}) \quad \text{for all } k'.$$

The mapping (3.3) will be called the *generalized Frobenius symbol* of $K|k$.

For the definition of the corresponding Artin symbol we can rely directly on the extension principle (2.1). Let $\varkappa(\sigma)$ denote the conjugacy class of $\sigma \in G(K|k)$, $\varkappa(G)$ the compact space of conjugacy classes of the compact group G in the quotient topology. Then, using (2.1), the ordinary Artin symbols

$$\left(\frac{k'|k}{\quad}\right): P_k \setminus R \to \varkappa(G(k'|k)),$$

being a compatible system of maps from cofinite sets into finite sets, can uniquely be extended to a continuous mapping

$$(3.4)\qquad \left(\frac{K|k}{\quad}\right): \Omega_k \to \varkappa(G(K|k));$$

the explicit definition is, according to (2.1), (b)

$$(3.5)\quad \left(\frac{K\,|\,k}{U}\right) = \varkappa(\sigma) \text{ iff } U \supset \{D_{k'|k}(\sigma|_{k'}) \mid k' \subset K,\ k'|k \text{ finite Galois}\};$$

here the Čebotarev set in the ground field k is denoted by

$$D_{k'|k}(\varrho) = \left\{\mathfrak{p} \in P_k \,\middle|\, \left(\frac{k'\,|\,k}{\mathfrak{p}}\right) = \varkappa(\varrho)\right\} \quad \text{for} \quad \varrho \in G(k'\,|\,k).$$

We call the mapping (3.4) the *generalized Artin symbol* of $K\,|\,k$.

We notice firstly some basic properties of the generalized Frobenius symbol:

(3.6) Proposition. *All extension fields of the global field k are supposed to be Galois extensions. Then one has the usual rules*

(i) $\left[\frac{K'\,|\,k}{V'}\right]\Big|_{K(U_1|U)} = \left[\frac{K\,|\,k}{V}\right]$ *for $K'\,|\,K\,|\,k$ and $V'\,|\,V\,|\,U$,*

(ii) $\left[\frac{K\,|\,k}{V}\right]^{f(U_1|U)} = \left[\frac{K\,|\,k_1}{V}\right]$ *for $K\,|\,k_1\,|\,k$ and $V\,|\,U_1\,|\,U$ for a finite subextension $k_1\,|\,k$,*

(iii) $\left[\frac{K\,|\,k}{\sigma V}\right] = \sigma\left[\frac{K\,|\,k}{V}\right]\sigma^{-1}$ *for $\sigma \in G(K\,|\,k)$, $V \in \Omega_K$.*

The last rule shows that the Frobenius mapping (3.3) is a $G = G(K\,|\,k)$-morphism, G acting on itself by conjugation.

For a proof one observes that for finite extensions the properties (i)–(iii) are easily deduced from the corresponding properties of the usual symbols. Once proved in the finite case they follow immediately in the general case according to definition (3.3′).

As main result we can formulate

(3.7) Theorem. *Let $K\,|\,k$ be an arbitrary Galois extension of the global field k and A its maximal abelian subextension over k. Then the following holds:*

(a) *The generalized Frobenius symbol and the generalized Artin symbol*

$$\left[\frac{K\,|\,k}{\ }\right]\colon\ \Omega_K \to G(K\,|\,k) \qquad \textit{resp.} \qquad \left(\frac{K\,|\,k}{\ }\right)\colon\ \Omega_k \to \varkappa G(K\,|\,k)$$

are continuous and surjective mappings.

(b) *The generalized Artin symbol extends, via the invariant mapping (2.3), the reciprocity homomorphism of class field theory: The following diagrams commute*

$$\begin{tikzcd}
\Omega_K \arrow[r, "{\left[\frac{K|k}{\ }\right]}"] \arrow[d, "j_{K|k}"'] & G(K\,|\,k) \arrow[d, "\varkappa"] \\
\Omega_k \arrow[r, "{\left(\frac{K|k}{\ }\right)}"] \arrow[d, "\mathrm{inv}_k"'] & \varkappa G(K\,|\,k) \arrow[d, "\text{restriction}"] \\
\overline{C_k} \arrow[r, "{(\ ,A|k)}"] & G(A\,|\,k)
\end{tikzcd}$$

where $(\ ,A\,|\,k)$ is the Chevalley symbol of $A\,|\,k$.

Proof. (a) For every $\sigma \in G(K\,|\,k)$ the compact sets

$$\Omega(k',\sigma) := \{U' \in \Omega_{k'} \mid \hat{D}_{k'|k}(\sigma|_{k'}) \in \dot{U'}\}$$

form a projective subsystem of the $\Omega_{k'}$ ($k'\,|\,k$ finite Galois). Hence there exists a superprime $V \in \Omega_K$ with $\left[\frac{K\,|\,k}{V}\right] = \sigma$ if and only if the projective limit $\varprojlim_{k'} \Omega(k',\sigma)$ is non-empty, that means, if and only if all $\Omega(k',\sigma)$ are non-empty. Therefore one has to show that all $\hat{D}_{k'|k}(\sigma|_{k'})$, respectively all $D_{k'|k}(\sigma|_{k'}) = j\hat{D}_{k'|k}(\sigma|_{k'})$ are infinite sets. In the number field case, this is true according to the Čebotarev density theorem. The same theorem holds in the function field case, though it seems not to be mentioned explicitly in the literature. Deuring's idea [6], rediscovered by MacCluer [15], of reducing Čebotarev's density theorem to the case of cyclic extensions (see Lang [14] or Goldstein [8]), can also be applied to function fields. In the cyclic case the density theorem is — via the reciprocity mapping — identical with the theorem on arithmetic progressions, which is also true for function fields (see proof of theorem (2.4)). Hence the sets $D_{k'|k}(\sigma|_{k'})$ are infinite. This completes the proof that the generalized Frobenius symbol is surjective. The same is true for the generalized Artin symbol since for all superprimes $V \in \Omega_K$, $U \in \Omega_k$ with $V\,|\,U$ the relation

$$\left(\frac{K\,|\,k}{U}\right) = \varkappa\left[\frac{K\,|\,k}{V}\right]$$

holds.

(b) Because of this last remark one has only to prove that the lower half of the diagram commutes. For an ideal group $H^{(\mathfrak{m})}$ mod $\mathfrak{m}$ let $L_{H(\mathfrak{m})}$ be the class field of k belonging to $H^{(\mathfrak{m})}$ and $K_{H(\mathfrak{m})} := L_{H(\mathfrak{m})} \cap K$. Hence

$$G(A\,|\,k) = \varprojlim_{\mathfrak{m},H^{(\mathfrak{m})}} G(K_{H(\mathfrak{m})}\,|\,k) \quad \text{and} \quad \overline{C_k} = \varprojlim_{\mathfrak{m},H^{(\mathfrak{m})}} D_k^{\mathfrak{m}}/H^{(\mathfrak{m})}.$$

For the Chevalley symbol $(\ ,A\,|\,k)$ the following diagrams commute, where φ denotes the usual Artin symbol:

$$\begin{tikzcd}
\overline{C}_k \arrow[r, "{(\ ,A|k)}"] \arrow[d, "\text{natural projection } \pi_{\mathfrak{m}}"'] & G(A\,|\,k) \arrow[d, "\text{restriction}"] \\
D_k^{\mathfrak{m}}/H^{(\mathfrak{m})} \arrow[r, "\varphi"] & G(K_{H^{(\mathfrak{m})}}\,|\,k)
\end{tikzcd}$$

Because of proposition (3.6), (i) we have

$$\left(\frac{K\,|\,k}{}\right)\Big|_{K_{H^{(\mathfrak{m})}}} = \left(\frac{K_{H^{(\mathfrak{m})}}\,|\,k}{}\right)$$

and therefore one has only to show that for all divisor moduli $\mathfrak{m}$ and all ideal groups $H^{(\mathfrak{m})}$ mod $\mathfrak{m}$ the following diagrams commute:

$$\begin{tikzcd}
\Omega_k \arrow[r, "\left(\frac{K_{H^{(\mathfrak{m})}}|k}{}\right)"] \arrow[d, "\pi_{\mathfrak{m}}\circ \mathrm{inv}_k"'] & \varkappa G(K_{H^{(\mathfrak{m})}}\,|\,k) \arrow[d, equal] \\
D_k^{\mathfrak{m}}/H^{(\mathfrak{m})} \arrow[r, "\varphi"] & G(K_{H^{(\mathfrak{m})}}\,|\,k)
\end{tikzcd}$$

These diagrams commute since they are the continuous extension of the corresponding (obviously commuting) diagram

$$\begin{tikzcd}
\mathfrak{p}\quad P_k\setminus\delta(\mathfrak{m}) \arrow[rr, "\left(\frac{K_{H^{(\mathfrak{m})}}|k}{}\right)"] \arrow[d] \arrow[dr] & & G(K_{H^{(\mathfrak{m})}}\,|\,k) \\
\mathfrak{p}H^{(\mathfrak{m})} & D_k^{\mathfrak{m}}/H^{(\mathfrak{m})} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

and because $P_k\setminus\delta(\mathfrak{m})$ is dense in $\hat{P}\setminus\delta(\mathfrak{m}) \supseteq \Omega_k$. Hence theorem (3.7) is proved.

Finally we want to mention, though in a few words, the connection between the Artin mapping $\left(\frac{\tilde{Q}\,|\,k}{}\right)$: $\Omega_k \to \varkappa G(\tilde{Q}\,|\,k)$ (k a number field, $\tilde{Q}$ the algebraic closure of Q) and Jarden's translation principle ([10]). As shown in [12] the Artin mapping turns out to be measurable with respect to the Haar measure on $\varkappa G(\tilde{Q}\,|\,k)$ and with respect to all, in a certain sense reasonable, measures on Ω_k extending the Dirichlet density on P_k. This fact yields a natural supplement and a new proof of the translation principle.

4. Decomposition fields of superprimes. Let k be a global field, $K\,|\,k$ a Galois extension and V a superprime of K.

The *decomposition group* of V over k is defined, as usual, as fixed group of V:

$$\mathfrak{Z}_{K|k}(V) = \{\sigma\in G(K\,|\,k)\mid \sigma V = V\}.$$

The corresponding fixed field $Z_{K|k}(V) = \mathrm{Fix}_K(\mathfrak{Z}_{K|k}(V))$ is the *decomposition field* of V over k. In the same way as for usual primes one proves

(4.1) (a) Decomposition groups and fields belonging to different $V\in\Omega_K$ dividing $U\in\Omega_k$ are conjugate under $G(K\,|\,k)$.

(b) If $K\,|\,k$ is finite and V divides $U\in\Omega_k$ the following holds:

(i) $(K:Z_{K|k}(V)) = f(V\,|\,U)$;

(ii) The decomposition field $Z_{K|k}(V)$ is the biggest field L between k and K such that $f(V_L\,|\,U) = 1$.

(4.2) THEOREM. *Let $K\,|\,k$ be an arbitrary Galois extension of the global field k and $V\,|\,U$ superprimes in Ω_K, Ω_k respectively. Then the decomposition group $\mathfrak{Z}_{K|k}(V)$ is a procyclic subgroup of $G(K\,|\,k)$, generated topologically by the generalized Frobenius symbol:*

$$\mathfrak{Z}_{K|k}(V) = \overline{\left\langle\left[\frac{K\,|\,k}{V}\right]\right\rangle}.$$

Every procyclic subgroup of $G(K\,|\,k)$ occurs as a decomposition group. The decomposition field $Z_{K|k}(V)$ is the intersection of K with the U-adic field k_U (within K_V):

$$Z_{K|k}(V) = K\cap k_U \quad \textit{in } K_V$$

under the natural embeddings introduced in (1.4).

The proof is based on two lemmas, the first of which is proved in [13].

(4.3) LEMMA (Klingen [13]). *Let I be a set, $\sigma\colon I\to I$ a permutation of I and U an ultrafilter on I. Then the following holds:*

$$\sigma U = U \textit{ if and only if } U \textit{ contains } \mathrm{Fix}(\sigma) = \{i\in I\mid \sigma(i) = i\}.$$

(4.4) LEMMA. *Let $L\,|\,k$ be a finite Galois extension and W a superprime of L. Then the decomposition group $\mathfrak{Z}_{L|k}(W)$ is isomorphic to the ultraproduct $\prod_{\mathfrak{P}\in P_L}\mathfrak{Z}_{L|k}(\mathfrak{P})/W$ of the standard decomposition groups* (under the diagonal mapping $d\colon G(L\,|\,k)\to G(L\,|\,k)^{P_L}/W$). *Hence the following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{Z}_{L|k}(W) \arrow[r] \arrow[d, "\wr"] & G(L\,|\,k) \arrow[d, "\wr\, d"] \\
\prod_{\mathfrak{P}\in P_L}\mathfrak{Z}_{L|k}(\mathfrak{P})/W \arrow[r] & G(L\,|\,k)^{P_L}/W
\end{tikzcd}$$

Proof. For $\sigma\in G(L\,|\,k)$ $d(\sigma)$ belongs to $\prod_{\mathfrak{P}\in P_L}\mathfrak{Z}_{L|k}(\mathfrak{P})/W$ if and only if W contains $\{\mathfrak{P}\in P_L\mid \sigma\in\mathfrak{Z}_{L|k}(\mathfrak{P})\} = \{\mathfrak{P}\in P_L\mid \sigma\mathfrak{P} = \mathfrak{P}\}$. Because of (4.3) this happens if and only if $\sigma W = W$, that means if and only if σ belongs to the decomposition group $\mathfrak{Z}_{L|k}(W)$. This proves lemma (4.4).

Proof of theorem (4.2). According to the action of $G(K|k) = \varprojlim_L G(L|k)$ on $\Omega_K = \varprojlim_L \Omega_L$ (L running over all finite Galois subextensions of $K|k$) we have $\mathfrak{Z}_{K|k}(V) = \varprojlim_L \mathfrak{Z}_{L|k}(V_L)$. Hence, to show $\mathfrak{Z}_{K|k}(V) = \overline{\left\langle\left[\frac{K|k}{V}\right]\right\rangle}$ one has only to prove that every $\mathfrak{Z}_{L|k}(V_L)$ is generated by $\left[\frac{K|k}{V}\right]\Big|_L = \left[\frac{L|k}{V_L}\right]$. By lemma (4.4) $\mathfrak{Z}_{L|k}(V_L)$ is canonically isomorphic to an ultraproduct of the (for almost all $\mathfrak{P}$ cyclic) groups $\mathfrak{Z}_{L|k}(\mathfrak{P})$ of bounded order (namely $\leqslant (K:k)$), and hence it is itself cyclic. Since the $\mathfrak{Z}_{L|k}(\mathfrak{P})$ are generated (for almost all $\mathfrak{P}$) by the usual Frobenius symbol $\left[\frac{L|k}{\mathfrak{P}}\right]$, the ultraproduct is generated by

$$\left(\left[\frac{L|k}{\mathfrak{P}}\right]\right)_{\mathfrak{P}\in P_L}^{V_L} = \left[\frac{L|k}{V_L}\right].$$

Since the generalized Frobenius symbol is a surjective mapping from Ω_K onto $G(K|k)$ (theorem (3.7)), every procyclic subgroup of $G(K|k)$ is a decomposition group.

The statement of (4.2) about the decomposition field has again to be proved only for finite Galois extensions $L|k$, since we have

$$Z_{K|k}(V) = \varinjlim_L Z_{L|k}(V_L) = \bigcup_L Z_{L|k}(V_L)$$

because of $\mathfrak{Z}_{K|k}(V) = \varprojlim_L \mathfrak{Z}_{L|k}(V_L)$.

$$\begin{tikzcd}[column sep=small]
 & L_{\mathfrak{P}} \\
L \arrow[ur, dash] & \\
 & k_{\mathfrak{p}} \arrow[uu, "\iota_{\mathfrak{P}}"'] \\
Z_{L|k}(\mathfrak{P}) \arrow[uu, dash] \arrow[ur, dash] & \\
k \arrow[u, dash] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & \prod\limits_{\mathfrak{P}\in P_L} L_{\mathfrak{P}}/V_L = L_{V_L} \\
L \arrow[ur, "d"] & \\
 & \prod\limits_{\mathfrak{p}\in P_k} k_{\mathfrak{p}}/U = k_U \arrow[uu, "\iota"'] \\
Z_{L|k}(V_L) \arrow[uu, dash] \arrow[ur] & \\
k \arrow[u, dash] &
\end{tikzcd}$$

The diagrams on the left side commute for all $\mathfrak{P}\in P_L$ and induce (by taking the ultraproduct with respect to $V_L \in \Omega_L$) the commuting diagram on the right side. Here U is the superprime of k lying under V and ι denotes the natural embedding of k_U into L_{V_L} (see (1.4)). The only fact left to be proved is:

$$Z_{L|k}(V_L) = L\cap k_U = L\cap \prod_{\mathfrak{p}\in P_k} k_{\mathfrak{p}}/U \quad \text{ in } L_{V_L}.$$

Now, $\alpha\in L_{V_L}$ belongs to $L\cap \prod_{\mathfrak{p}\in P_k} k_{\mathfrak{p}}/U$ if and only if

$$\alpha\in L \quad\text{and}\quad \{\mathfrak{P}\in P_L \mid \alpha\in k_{j(\mathfrak{P})}\}\in V_L,$$

that means

$$V_L \ni \{\mathfrak{P}\in P_L \mid \alpha\in L\cap k_{j(\mathfrak{P})} = Z_{L|k}(\mathfrak{P})\} = \{\mathfrak{P}\in P_L \mid \alpha\in \mathrm{Fix}_L(\mathfrak{Z}_{L|k}(\mathfrak{P}))\}.$$

Because of lemma (4.4) this happens if and only if $\alpha\in \mathrm{Fix}_L(\mathfrak{Z}_{L|k}(V_L)) = Z_{L|k}(V_L)$. This completes the proof of theorem (4.2).

5. Absolute algebraic fields. For an extension $K|k$ of fields let $\mathrm{Alg}(K|k)$ denote the field of all elements of K separable algebraic over k. If k is the prime field of K $\mathrm{Alg}(K|k)$ is the field $\mathrm{Abs}(K)$ of all absolute algebraic elements in K (in the sense of J. Ax [2]). With $\mathrm{Ab}(K|k)$ we denote the corresponding maximal abelian extension of k in $\mathrm{Alg}(K|k)$.

Let k be a global field and U a superprime of k. Since for every $a\in k^\times$ $v_{\mathfrak{p}}(a) = 0$ for almost all $\mathfrak{p}\in P_k$ we have $v_U(a) = 0$ for all $a\in k^\times$. Hence k is contained in the valuation ring R_U of k_U and the mapping $k\to\bar{k}_U$, $a\mapsto\bar{a}$ is a well defined monomorphism.

(5.1) Remark. $\mathrm{Alg}(k_U|k)$ and $\mathrm{Alg}(\bar{k}_U|k)$ are naturally k-isomorphic.

Proof. We have to show that $\mathrm{Alg}(k_U|k)$ is contained in the valuation ring R_U and mapped onto $\mathrm{Alg}(\bar{k}_U|k)$ under the natural mapping $\nu\colon R_U\to\bar{k}_U$. Take $\alpha\in \mathrm{Alg}(k_U|k)$ and let $f\in k[X]$ be its monic irreducible polynomial. Hence $\alpha = (\alpha_{\mathfrak{p}})_{\mathfrak{p}\in P_k}^U$ with $\alpha_{\mathfrak{p}}\in k_{\mathfrak{p}}$ and $f(\alpha_{\mathfrak{p}}) = 0$ for all $\mathfrak{p}\in P_k$. Since for almost all $\mathfrak{p}\in P_k$ f is a polynomial over the valuation ring $R_{\mathfrak{p}}$ of $k_{\mathfrak{p}}$ we have $\alpha_{\mathfrak{p}}\in R_{\mathfrak{p}}$ for almost all $\mathfrak{p}\in P_k$, hence $v_U(\alpha)\geqslant 0$. Obviously ν induces a k-monomorphism from $\mathrm{Alg}(k_U|k)$ into $\mathrm{Alg}(\bar{k}_U|k)$. This k-monomorphism is surjective because k_U is a henselian valued field. This proves (5.1).

If in theorem (4.2) one chooses for K the separable closure k_s or the maximal abelian extension A of k, one gets the following characterization of the fields $\mathrm{Alg}(k_U|k)$ resp. $\mathrm{Ab}(k_U|k)$ as fixed fields. Obviously these fields are realized in k_s only up to conjugacy.

(5.2) Theorem. *Let k be a global field and U a superprime of k. Then:*

(a) *The subfield of k-algebraic elements in k_U is — up to conjugacy — the fixed field of $\sigma\in G(k_s|k)$ with $\varkappa(\sigma) = \left(\frac{k_s|k}{U}\right)$, or in short terms:*

$$\mathrm{Alg}(k_U|k) = \mathrm{Alg}(\bar{k}_U|k) = \mathrm{Fix}_{k_s}\left(\frac{k_s|k}{U}\right)$$

the fixed field of the Artin symbol of U.

(b) *The maximal abelian subextension of $k_U|k$ is the fixed field of the value of the invariant of U under the Chevalley symbol*:

$$\mathrm{Ab}(k_U|k) = \mathrm{Ab}(\bar{k}_U|k) = \mathrm{Fix}_A(\mathrm{inv}_k U, A|k).$$

Proof. (b) follows from (a) and theorem (3.7):

$$\varkappa(\sigma|_A) = \left(\frac{A|k}{U}\right) = (\mathrm{inv}_k U, A|k).$$

According to theorem (4.2) $\mathrm{Alg}(k_U|k)$ is k-isomorphic to $Z_{k_s|k}(V)$ for any $V \in \Omega_{k_s}$ with $V|U$. Because of $\left(\frac{k_s|k}{U}\right) = \varkappa(\sigma)$ there exists a $V \in \Omega_{k_s}$ with $V|U$ and $\left[\frac{k_s|k}{V}\right] = \sigma$. So we get $\mathrm{Alg}(k_U|k) \cong_k Z_{k_s|k}(V) = \mathrm{Fix}_{k_s}(\sigma)$ and complete the proof of (5.2).

As a consequence one can deduce the following result of J. Ax ([2], Prop. 7 and 7′):

(5.3) Corollary. (a) *Let K be an algebraic extension of $\boldsymbol{Q}$ with $G_K = G(\tilde{K}|K)$ procyclic* ($\tilde{K}$ the algebraic closure of K). *Then there exists an ultrafilter U on the set P of all prime numbers such that*

$$K \simeq \mathrm{Abs}\Big(\prod_{p\in P} \boldsymbol{F}_p\big/U\Big).$$

(b) *Let K be an algebraic extension of the prime field $\boldsymbol{F}_p$ of characteristic p. Then there exists an ultrafilter U on $\boldsymbol{N}$ such that $K \simeq \mathrm{Abs}\big(\prod_{n\in \boldsymbol{N}} \boldsymbol{F}_{p^n}/U\big)$.* ($\boldsymbol{F}_{p^n}$ the finite field of p^n elements.)

Proof. (a) Let $\sigma \in G(\tilde{\boldsymbol{Q}}|\boldsymbol{Q})$ be a generator of $G_K = G(\tilde{\boldsymbol{Q}}|K)$. Since the generalized Artin symbol is surjective (theorem (3.7)) there exists a superprime $U \in \Omega_{\boldsymbol{Q}}$ with $\left(\frac{\tilde{\boldsymbol{Q}}|\boldsymbol{Q}}{U}\right) = \varkappa(\sigma)$. Hence, according to theorem (5.2), we have

$$K = \mathrm{Fix}_{\tilde{\boldsymbol{Q}}}(\sigma) \simeq \mathrm{Alg}(\bar{\boldsymbol{Q}}_U|\boldsymbol{Q}) = \mathrm{Abs}\Big(\prod_{p\in P} \boldsymbol{F}_p\big/U\Big).$$

(b) Since the absolute Galois group $G_{\boldsymbol{F}_p} = G(\tilde{\boldsymbol{F}}_p|\boldsymbol{F}_p)$ is procyclic, there is an automorphism σ of $\tilde{\boldsymbol{F}}_p$ with $K = \mathrm{Fix}_{\tilde{\boldsymbol{F}}_p}(\sigma)$. Let σ' be the unique extension of σ to a k-automorphism of $\tilde{\boldsymbol{F}}_p(X)$, where k denotes the rational function field $\boldsymbol{F}_p(X)$ in one variable over $\boldsymbol{F}_p$. Theorem (5.2) can now be applied to the global field k and an arbitrary $\bar{\sigma} \in G_k = G(k_s|k)$ with $\bar{\sigma}|_{\tilde{\boldsymbol{F}}_p(X)} = \sigma'$. With the same argument as in the proof of (a) we get $L := \mathrm{Fix}_{k_s}(\bar{\sigma}) \cong_k \mathrm{Alg}(\bar{k}_U|k)$ for some superprime U of k. Since $K = \mathrm{Fix}_{\tilde{\boldsymbol{F}}_p}(\sigma) = \tilde{\boldsymbol{F}}_p \cap L = \mathrm{Abs}(L)$, we have:

$$K \simeq \mathrm{Abs}(\bar{k}_U) = \mathrm{Abs}\Big(\prod_{\mathfrak{p}\in P_k} \bar{k}_{\mathfrak{p}}\big/U\Big).$$

To complete the proof of (5.3) (b) we have only to show that $\mathrm{Abs}\big(\prod_{\mathfrak{p}\in P_k} \bar{k}_{\mathfrak{p}}/U\big)$ is isomorphic to $\mathrm{Abs}\big(\prod_{n\in\boldsymbol{N}} \boldsymbol{F}_{p^n}/U'\big)$ for some ultrafilter U' on $\boldsymbol{N}$. The valuations of $\boldsymbol{F}_p(X)$ correspond (with one exception) to the set $\mathscr{F}$ of all irreducible monic polynomials of $\boldsymbol{F}_p[X]$. Their residue class fields are the finite fields $\boldsymbol{F}_{p^n}$, n being the degree $d(f)$ of the corresponding polynomial f. Hence

$$\prod_{\mathfrak{p}\in P_k} \bar{k}_{\mathfrak{p}}\big/U = \prod_{f\in\mathscr{F}} \boldsymbol{F}_{p^{d(f)}}\big/U.$$

Since the mapping $d\colon \mathscr{F}\to\boldsymbol{N}$, $f\mapsto d(f)$ has finite fibres, using (1.1), the ultrafilter dU on $\boldsymbol{N}$ is non-principal. It suffices to prove

$$\mathrm{Abs}\Big(\prod_{n\in\boldsymbol{N}} \boldsymbol{F}_{p^n}\big/dU\Big) \cong \mathrm{Abs}\Big(\prod_{f\in\mathscr{F}} \boldsymbol{F}_{p^{d(f)}}\big/U\Big).$$

The isomorphism is given by the mapping

$$D\colon \prod_{n\in\boldsymbol{N}} \boldsymbol{F}_{p^n}\big/dU \to \prod_{f\in\mathscr{F}} \boldsymbol{F}_{p^{d(f)}}\big/U, \quad (a_n)_{n\in\boldsymbol{N}}^{dU} \mapsto (a_{d(f)})_{f\in\mathscr{F}}^{U}.$$

D is a well defined monomorphism as one can easily verify (definition of dU!). Take

$$\beta = (\beta_f)^U \in \mathrm{Abs}\Big(\prod_{f\in\mathscr{F}} \boldsymbol{F}_{p^{d(f)}}\big/U\Big)$$

and let $\varphi \in \boldsymbol{F}_p[X]$ be the monic irreducible polynomial belonging to β. Hence U must contain the set $E := \{f\in\mathscr{F} \mid \varphi(\beta_f) = 0\}$. Since $\{\beta_f \mid f\in E\}$ is finite, we can find $m\in\boldsymbol{N}$ and $\gamma\in\boldsymbol{F}_{p^m}$ with $U \ni \{f\in\mathscr{F} \mid \beta_f = \gamma\}$. Defining

$$a_n := \begin{cases} \gamma & \text{if } n \geqslant m,\\ \in \boldsymbol{F}_{p^n} \text{ arbitrary} & \text{if } n < m,\end{cases}$$

we have found

$$a = (a_n)_{n\in\boldsymbol{N}}^{dU} \in \prod_{n\in\boldsymbol{N}} \boldsymbol{F}_{p^n}\big/dU \quad \text{with} \quad D(a) = \beta.$$

Thus the proof of (5.3) is complete.

It seems worthwhile to notice the special case of the rational ground field in theorem (5.2), (b) separately. For this purpose let $\mathsf{U} = \prod_p \mathsf{U}_p$

be the ("finite part" of the) rational idele unit group, i.e. the product of all rational-p-adic unit groups U_p. This group U can be identified with the Galois group $G(\mathsf{P}|\boldsymbol{Q})$ of the field P of all roots of unity ξ by the ordinary action $\xi \mapsto \xi^u$ $(u \in \mathsf{U})$. For rational integers a and $m \in \boldsymbol{N}$ with $(a, m) = 1$ we denote by

$$D_m(a) = \{p \in P \mid p \equiv a \bmod m\}$$

the arithmetic prime progression of $a \bmod m$.

Congruence filters are all systems

$$\{D_m(a_m) \mid m \in \boldsymbol{N}\}$$

which form a filter basis on the set of primes P. They determine Cauchy filters $\{a_m + m\hat{\boldsymbol{Z}} \mid m \in \boldsymbol{N}\}$ on $\hat{\boldsymbol{Z}}$ which converge to an idele unit $u \in \mathsf{U}$.

We can now formulate

(5.4) Corollary. *Each superprime U on the set P of primes contains exactly one congruence filter*

(i) $$U \supset \{D_m(a_m) \mid m \in \boldsymbol{N}\}.$$

If u is the uniquely determined $u \in \mathsf{U}$:

(ii) $$u \equiv a_m \bmod m \quad \textit{for all } m \in \boldsymbol{N},$$

then for the maximal abelian subfield holds

(iii) $$\mathrm{Ab}\Big(\prod_p \boldsymbol{F}_p/U\Big) = \mathrm{Fix}_{\mathsf{P}}(u).$$

Hence the ultraproduct $\prod_p \boldsymbol{F}_p/U$ contains the Kronecker field P or the field $\mathsf{P}_{q\infty}$ of all q^ν-th roots of unity if and only if U contains all prime sets $\{p \equiv 1 \bmod m\}$ for all m or all sets $\{p \equiv 1 (q^\nu)\}$ for all ν respectively.

Proof. By definition of the invariant one has $u = \mathrm{inv}_{\boldsymbol{Q}}(U)$ if (i) and (ii) holds. Here, in the rational case, we can identify $\bar{C}_{\boldsymbol{Q}} = \mathsf{U}$. But the invariant-mapping is surjective and the invariant $\mathrm{inv}_{\boldsymbol{Q}}(U)$ determines the congruence filter in (i) uniquely according to (ii). This proves the first assertion. The second part follows using $u = \mathrm{inv}_{\boldsymbol{Q}}(U)$ and theorem (5.2), (b).

Finally we mention the following theorem which partially is a consequence of theorem (5.2). The proof uses the continuum hypothesis $2^{\aleph_0} = \aleph_1$ (CH).

(5.5) Theorem (CH). *Let k be an algebraic number field and $U_1, U_2 \in \Omega_k$ superprimes of k. Then the following statements are equivalent:*

(a) *The ultraproducts k_{U_1} and k_{U_2} of the local fields $k_\mathfrak{p}$ are k-isomorphic.*

(b) *The ultraproducts $\overline{k_{U_1}}$ and $\overline{k_{U_2}}$ of the finite fields $\bar{k}_\mathfrak{p}$ are k-isomorphic.*

(c) *U_1 and U_2 belong to the same division of $G_k = G(\tilde{\boldsymbol{Q}}|k)$.*

By the *division* D of $\sigma \in G(\tilde{\boldsymbol{Q}}|k)$ we mean the set of all $\tau \in G(\tilde{\boldsymbol{Q}}|k)$ which generate the same closed subgroups as some conjugate of σ, and we say that *U belongs to the division D* of G_k if $\left(\frac{\tilde{\boldsymbol{Q}}|k}{U}\right) \subseteq D$.

Proof of (5.5). Clearly, (a) implies (b) and (b) implies $\mathrm{Abs}(\overline{k_{U_1}}) \underset{k}{\simeq} \mathrm{Abs}(\overline{k_{U_2}})$. From (5.2) we conclude that $\left(\frac{\tilde{\boldsymbol{Q}}|k}{U_1}\right)$ and $\left(\frac{\tilde{\boldsymbol{Q}}|k}{U_2}\right)$ determine the same set of conjugate closed subgroups of $G(\tilde{\boldsymbol{Q}}|k)$; hence (b) implies (c). In the same way, (c) implies $\mathrm{Abs}(\overline{k_{U_1}}) \underset{k}{\simeq} \mathrm{Abs}(\overline{k_{U_2}})$. Now, one main result of J. Ax [2] says that $\overline{k_{U_1}}$ and $\overline{k_{U_2}}$ are k-isomorphic if and only if $\mathrm{Abs}(\overline{k_{U_1}})$ and $\mathrm{Abs}(\overline{k_{U_2}})$ are. Hence (c) implies (b). Let us now assume (b). Then the ultraproducts $\prod_{\mathfrak{p} \in P_k} k_\mathfrak{p}/U_i = k_{U_i}$ have the following properties:

(a) k_{U_1} and k_{U_2} are valued fields with cross sections and the same value group ${}^*\boldsymbol{Z} = \boldsymbol{Z}^{P_k}/U_i$ (see the same argument in Section 1).

(b) k_{U_i} is ω-pseudo-complete. In fact, every non-principal ultraproduct of a countable family of fields has this property (Ax–Kochen [3], L. 9).

(c) As ultraproducts of henselian valued fields the k_{U_i} are also henselian.

(d) $\overline{k_{U_1}}$ and $\overline{k_{U_2}}$ are isomorphic fields of characteristic 0. Moreover $\# k_{U_i} = 2^{\aleph_0} = \aleph_1$ (CH).

Hence the assumptions of M. Armbrust's version [1] of the central theorem of Ax–Kochen are fulfilled and it follows

$$k_{U_1} \text{ and } k_{U_2} \text{ are } k\text{-isomorphic}.$$

This proves the theorem.

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Some remarks on a number theoretic problem of Graham

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In considering generalizations of van der Waerden's theorem, R. L. Graham [1] was led to consider finite sequences of positive integers $a_1 < a_2 < \ldots < a_n$ and certain ratios, namely, $a_i/(a_i, a_j)$ where (x, y) denotes the g.c.d. of x and y. He proposed the following conjecture.

CONJECTURE I. *If* $0 < a_1 < a_2 < \ldots < a_n$, *then*

$$\max_{i,j}\{a_i/(a_i, a_j)\} \geqslant n.$$

The conjecture has been verified in some special cases:

(a) a_i is square-free for all i (Marica and Schönheim [2]),

(b) a_1 is prime (Winterle [4]),

(c) n is prime (Szemerédi [3]).

One of the results of this note is to prove Conjecture I when $n-1$ is prime.

A natural question to ask is: For what sequences is equality achieved? Before going into this question we make some remarks.

1. If we multiply a sequence by a constant we obtain the same set of ratios, so we may assume g.c.d. $(a_1, a_2, \ldots, a_n) = 1$.

2. Given a sequence $Q = \{a_1 < a_2 < \ldots < a_n\}$, let $A =$ l.c.m. $\{a_1, a_2, \ldots, a_n\}$ and form

$$Q^{-1} = \{A/a_n < A/a_{n-1} < \ldots < A/a_1\}.$$

It is easy to show that Q and Q^{-1} have the same set of ratios.

Notation. Let $M_n =$ l.c.m. $\{1, 2, \ldots, n\}$ and $b_i^{(n)} = M_n/(n-i+1)$, so $M_n/n < M_n/(n-1) < \ldots < M_n/2 < M_n/1$ is the "inverse" of $\{1 < 2 < \ldots < n\}$.

DEFINITION. Given a sequence $a_1 < a_2 < \ldots < a_n$, we say it is a *standard sequence* if it is a multiple of $\{1 < 2 < \ldots < n\}$ or of $\{b_1^{(n)} < b_2^{(n)} < \ldots \ldots < b_n^{(n)}\}$. That is, either

$$a_i = ki \quad \text{for all } i,$$