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General multiplicative functions

by

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0. Concepts and notations. Here I list (a) some definitions and notations, which are used in the paper without any more mentioning, (b) the number of definitions, which are used in more sections.

(a)

p, q, p_n	generally mean primes, p_n the nth prime, beginning with $p_1 = 2$.
$\boldsymbol{N}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$	the set of natural, rational, real and complex numbers.
AG$\boldsymbol{Q}$, MS$\boldsymbol{R}$ etc.:	structures on these sets. The first letter shows that the operation is addition or multiplication, the third the fundamental set, the second stands for group or semigroup. MG$\boldsymbol{Q}$ etc. means the group, got by omitting the number 0.
$\boldsymbol{G} = (G, \circ)$	a structure on the set G with the operation $\circ$.
e	generally the unity of a structure.
$\operatorname{gen} K$	the subgroup, generated by the subset K.
$o(g)$	the order of the element g (in a group).
$C(n)$, $C(\infty)$	cyclic groups.
$C(p^\infty)$	quasicyclic group.
$C(\infty^\infty)$	a group, isomorphic to AG$\boldsymbol{Q}$.
$\operatorname{deninf} S$, $\operatorname{densup} S$,	$\operatorname{den} S$: the lower density, the upper density, resp. the asymptotic density (if exists) of the sequence $S \subset \boldsymbol{N}$.

(b)

The number in the left shows the place of the definition.

(1.1)	arithmetical function,
(1.2)	G-multiplicative and strongly G-multiplicative function,
(1.5)	$M(f)$, mean value of the function f,
(3.1)	$N(f, K, x)$,
(3.2)	$ds(f, K)$, $di(f, K)$,
(3.3)	$d(f, K)$,
(3.4)	density set,
(3.5)	density class,
(3.6)	uniform density class,
(3.8)	concentration complex and concentration group,
(3.9)	concentrated and deconcentrated function,
(7.1)	$\varrho(f_1, f_2)$ and $f_n \to f$,
(9.9)	superconcentrated function,
(9.10)	f/G_1 factor-function.

1. Introduction. Multiplicative and additive functions were investigated in hundreds of papers. As far as I know, these terms always denoted a complex-valued arithmetical function f satisfying

$$f(nm) = f(n)f(m)$$

resp.

$$f(nm) = f(n)+f(m)$$

for all $(n, m) = 1$. In this paper we shall show that sometimes it is more natural to regard them as particular cases of a more general concept.

(1.1) DEFINITION. An *arithmetical function* f is an arbitrary function with

$$\operatorname{dom} f = \boldsymbol{N}.$$

The most general possible concept of multiplicativity is the following.

(1.2) DEFINITION. Let $\boldsymbol{G} = (G, \circ)$ be a groupoid and $f\colon \boldsymbol{N} \to G$ an arithmetical function. (We must distinguish the structure $\boldsymbol{G}$ from the fundamental set G.) We call f *$\boldsymbol{G}$-multiplicative* if

$$f(nm) = f(n)\circ f(m) \tag{1.3}$$

for all $(n, m) = 1$. We call f *strongly $\boldsymbol{G}$-multiplicative* if (1.3) holds without any restriction on n and m. (Generally we shall write simply ab instead of $a\circ b$.)

Our main interest will be the *local limit theorems*, that is statements concerning the number of natural numbers $n \leqslant x$ for which $f(n) = g$. For example,

(1.4) THEOREM. *If $\boldsymbol{G}$ is an Abelian group and f is $\boldsymbol{G}$-multiplicative, then the sequence $f^{-1}(g)$ has an asymptotic density for all $g \in G$.*

Observe that for functions, assuming only the values ± 1, it is equivalent to the existence of the mean value

$$M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n). \tag{1.5}$$

This was conjectured long ago by P. Erdös and proved by E. Wirsing [11] under much more general conditions. Our theorem generalizes in another direction.

The following more complicated example gives a better illustration.

(1.6) COROLLARY. *Let f_k, $k = 1, 2, \ldots$, be complex-valued multiplicative functions, $f_k(n) \neq 0$, and $g_k \in \boldsymbol{C}$ arbitrary. The sequence of the solutions of the infinite system of equations*

$$f_k(n) = g_k$$

has an asymptotic density.

The nonzero complex numbers form a group and we can apply Theorem (1.4) to the (complete) direct product of such groups. I note that it does not remain true if we allow $f_k(n) = 0$ (albeit this does not matter if we regard only finitely many f_k). This shows the importance that $\boldsymbol{G}$ be a group. (See also Section 2.)

At this point it is natural to ask what is the inner meaning of this theorem, that is, which sequences are proved to have a density.

(1.7) PROBLEM. Given a sequence $S \subset \boldsymbol{N}$, under what conditions can one find an Abelian group $\boldsymbol{G}$, a $\boldsymbol{G}$-multiplicative f and a $g \in G$ such that

$$S = f^{-1}(g)?$$

This way we get the purely number-theoretic content of our local limit theorem, with no "alien" concept as a group. One might hope that this leads closer to the proof, but this is not the case. Indeed, (1.7) can easily be solved.

(1.8) STATEMENT. *Let $S \subset \boldsymbol{N}$. The following three conditions are equivalent to each other.*

(a) *There exists an Abelian group $\boldsymbol{G}$, a $g \in G$ and a stronly $\boldsymbol{G}$-multiplicative f such that*

$$S = f^{-1}(g).$$

(b) *There exists a $G_0 < \mathrm{MG}\boldsymbol{Q}$ and a $0 \neq q \in \boldsymbol{Q}$ such that*

$$S = qG_0 \cap \boldsymbol{N} \tag{1.9}$$

(that is, S consists of the natural numbers contained in a coset of a subgroup of the multiplicative group of the rational numbers).

(c) *For arbitrary $s_1, \ldots, s_{n+1}, t_1, \ldots, t_n \in S$*

$$\prod t_i \Big| \prod s_j \Rightarrow s = \prod s_j \Big/ \prod t_i \in S. \tag{1.10}$$

Proof in Section 6.

It does not matter that we spoke of strongly multiplicative functions. In Sections 7–8 we shall formulate and prove a theorem which states that a set of prime-powers T with

$$\sum_{t\in T} \frac{1}{t} < \infty$$

such as the set of all p^i, $i \geqslant 2$, is always negligible.

Condition (1.10) is no good. I wonder if there will be anybody to prove directly that every sequence satisfying (1.10) has a density. The strongly multiplicative function f plays the same role here as the characters in the proof of Dirichlet's theorem on the primes in arithmetical progressions.

There is one more fact emphasizing our conclusion. Erdös [3] asked the following

(1.11) PROBLEM. Let $S \subset N$ be a sequence satisfying

(1.12) $$\left(s_1, \ldots, s_m, t_1, \ldots, t_n \in S, \prod s_i = \prod t_j\right) \Rightarrow m = n.$$

What is

$$\beta = \sup_S (\operatorname{densup} S)?$$

He proved $\beta \geqslant 1/e$.

We can reformulate this question for functions.

(1.13) STATEMENT. *Condition* (1.12) *is equivalent to the existence of a real-valued strongly additive* (*that is, strongly* A$G\boldsymbol{R}$*-multiplicative*) f *such that*

$$S \subset f^{-1}(1).$$

Proof in Section 6.

Using (1.13) I could prove $\beta = 1/e$ (see Section 4). Moreover, I proved this several years ago (stated without proof in Erdös–Ruzsa–Sárközi [4]), but nobody observed that the statement concerning strongly additive functions solved (1.11). So the real worth of conditions like (1.10) and (1.13) is that they make possible to deal with functions instead of sequences.

In Sections 2–5 we state other problems and results, and Sections 6–14 contain the proofs.

2. Algebraic and analytic questions. The existence of the density of a sequence, defined by a multiplicative function, is a typical analytic question. On the other hand, we made an algebraic generalization, regarding functions, mapping into an abstract structure. Present paper emphasizes the analytic aspects (though it does not contain a single sign of integration), but we must face some algebraic problems.

First, if G is an arbitrary groupoid, very strange things may happen: the product of two multiplicative functions may not be multiplicative, or to a multiplicative f we cannot define its "strongly multiplicative brother" $\tilde{f}_1$ by $f_1(p) = f(p)$, or maybe that there are no G-multiplicative functions at all. To avoid these pathologic cases, except of this section we suppose the following restriction.

(A) G is a commutative semigroup with unity e.

(2.1) STATEMENT. *Suppose we are given the values* $F(p^i)$ *for every prime-power* ($i \geqslant 1$). *If* (A) *holds, then there exists exactly one G-multiplicative f satisfying*

$$f(p^i) = F(p^i), \quad f(1) = e. \quad \blacksquare$$

Of course, this is not true without $f(1) = e$; for example, if $G = \text{MS}\boldsymbol{R}$ and $F(p^i) = 0$, then $f(n) \equiv 0$ is also good. For sake of unicity we always suppose

(B) $f(1) = e$.

Under these natural assumptions we may — and shall — define a multiplicative f by giving $f(p^i)$ and a strongly multiplicative one by $f(p)$.

The distance between (A) and the Abelian groups is still enormous. I cannot decide in which structures do all the G-multiplicative functions obey a local limit law (this simply means the existence of $\operatorname{den} f^{-1}(g)$ for all $g \in G$), but I have some results in both directions.

(2.2) THEOREM. *Let* $\boldsymbol{D}_2$ *be the multiplicative semigroup of the numbers* 0 *and* 1, $\boldsymbol{D} = \boldsymbol{D}_2^{\infty}$ *and* $\mathbf{0} = (0, 0, \ldots) \in D$. *There exists a* $\boldsymbol{D}$*-multiplicative* f *for which the sequence* $f^{-1}(\mathbf{0})$ *does not have a density.*

Let G satisfy (A), $a, b \in G$. We write aHb if both the equations $ax = b$ and $by = a$ are solvable in G. H is a congruence-relation, so we can form the factor-semigroup $G_1 = G/H$.

(2.3) THEOREM. *Suppose* (A) *and let* G_1 *be as above. If in* G_1 *every element has only a finite number of divisors, then every G-multiplicative function obeys a local limit law.*

For example, if $f_1, \ldots, f_m$ are complex-valued multiplicative functions (allowed to assume 0) and $g_i \in \boldsymbol{C}$, then the sequence of the solutions of the system of equations

$$f_i(n) = g_i$$

has a density.

There are also some results if we do not suppose (A). I can prove, that all the results, which were and will be stated for Abelian groups, are valid also in non-commutative groups. Or if G is an arbitrary *finite* groupoid, then all the G-multiplicative functions obey a local limit law.

Theorem (2.2) will be proved in Section 6. The other results are put off to another paper, dealing with the algebraic problems of multiplicativity.

3. The local limit laws for multiplicative functions. Let $f\colon N \to G$ be an arithmetical function and $K \subset G$. We introduce some notations.

(3.1) $$N(f, K, x) = |\{n\colon n \leqslant x, f(n) \in K\}|.$$

(3.2) $$ds(f, K) = \operatorname{densup} f^{-1}(K),$$
$$di(f, K) = \operatorname{deninf} f^{-1}(K).$$

(3.3) $$d(f, K) = \operatorname{den} f^{-1}(K) \text{ if it exists.}$$

Most frequently $K = \{g\}$; in this case we omit the braces and write for example $d(f, g)$ instead of $d(f, \{g\})$.

(3.4) DEFINITION. $K \subset G$ is a *density set* for f if $d(f, K)$ exists.

(3.5) DEFINITION. Let $\mathfrak{K}$ be a class of subsets of G. We call $\mathfrak{K}$ a *density class* for f if every $K \in \mathfrak{K}$ is a density set for f.

In this terms Theorem (1.4) can be formulated as "$\mathfrak{G} = \{\{g\}: g \in G\}$ is a density class". We prove a little more.

(3.6) DEFINITION. $\mathfrak{K}$ is a *uniform density class* for f if the convergence

$$\frac{N(f, K, x)}{x} \to d(f, K) \quad (K \in \mathfrak{K})$$

is uniform in K.

(3.7) THEOREM. *Let G be an Abelian group. $\mathfrak{G} = \{\{g\}: g \in G\}$ is a uniform density class for every G-multiplicative function.*

We can give some information about the behaviour of the values $(d(f, g))$.

(3.8) DEFINITION. Let G be a group and f a G-multiplicative function. We call

$$K = \Big\{g: \sum_{f(p)=g} \frac{1}{p} = \infty\Big\}$$

the *concentration complex* and

$$G_1 = \operatorname{gen} K$$

the *concentration group* of f.

(3.9) DEFINITION. Let G be a group and f G-multiplicative with the concentration group G_1. We call f *concentrated* if

$$|G_1| < \infty \quad \text{and} \quad \sum_{f(p) \notin G_1} \frac{1}{p} < \infty,$$

and *deconcentrated* otherwise.

(3.10) THEOREM. *Let G be an Abelian group and f G-multiplicative. One of the following two possibilities holds.*

(a) *f is deconcentrated, $d(f, g) \equiv 0$.*

(b) *f is concentrated, $d(f, g) > 0$ for every $g \in \operatorname{im} f$ and*

$$\sum_{g \in G} d(f, g) = 1.$$

The next result is a generalization of many uniformity theorems.

(3.11) UNIFORM DISTRIBUTION THEOREM. *Let G be an Abelian group and f G-multiplicative with the concentration group G_1. $d(f, g)$ depends only on the coset of G_1 in which g lies.*

At last we state our only non-local result.

(3.12) THEOREM. *Let G be an Abelian group and let $\mathfrak{P}$ denote the set algebra generated by all the cosets of all the subgroups of G. $\mathfrak{P}$ is a density class for every G-multiplicative function.*

4. Erdös's problem and the maximum-theorems

(4.1) THEOREM. *There exists a constant $c > 0$ such that for every Abelian group G, $e \neq g \in G$ and G-multiplicative f we have*

$$N(f, g, x) \leqslant x(1-c).$$

(4.2) MAXIMUM-THEOREM FOR MULTIPLICATIVE FUNCTIONS. *For every Abelian group G, $e \neq g \in G$ and G-multiplicative f*

$$d(f, g) \leqslant \tfrac{1}{2}.$$

Equality holds only in the following cases.

(a) *$f(2^i) = g$ for all $1 \leqslant i < \infty$; there is a prime q for which $f(q^j) = g$ or e (depending on j) and $f(p^k) = e$ for every other prime p.*

(b) *$o(g) = 2$, $f(p^i) = g$ or e for every prime-power and either $f(2^i) = g$ for all $i \geqslant 1$ or*

$$\sum_{f(p)=g} \frac{1}{p} = \infty.$$

(4.3) SUPREMUM-THEOREM FOR STRONGLY MULTIPLICATIVE FUNCTIONS *Let G be an Abelian group, f a strongly G-multiplicative function and*

$$p_0 = \min\{o(g): g \in G, g \neq e\}.$$

(p_0 is a prime or ∞.)

(a) *In case $p_0 = 2$ we have*

$$d(f, g) \leqslant \tfrac{1}{2}$$

for all $g \neq e$. Equality holds only if $f(p^i) = g$ or e,

$$\sum_{f(p)=g} \frac{1}{p} = \infty.$$

(b) *In case $p_0 > 2$ we have for $g \neq e$*

$$d(f, g) < W_{p_0},$$

where W_n is the maximum of the function

$$w_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{kn+1}}{(kn+1)!} \quad (x \in \boldsymbol{R}).$$

W_{p_0} can be arbitrarily approached. $(w_\infty(x) = xe^{-x},\ W_\infty = 1/e.)$

Let S be a sequence satisfying Erdös's condition (1.12). From Statement (1.13) and Theorems (4.1) and (4.3) we have

(4.4) COROLLARY. (a) *There exists an absolute constant $c > 0$ such that for every x*

$$S(x) \leqslant x(1-c),$$

where

$$S(x) = |\{n:\ n \leqslant x, n \in S\}|.$$

(b) $\operatorname{densup} S < 1/e$; *moreover, S is contained in a sequence which has a density $< 1/e$.* ■

For real-valued additive functions (that is, for the case $G = \mathrm{AG}\boldsymbol{R}$) Theorems (4.1) and (4.2) were proved and (4.3) was stated without proof in a joint paper with Erdös and Sárközi [4]. For Theorem (4.1) the same proof applies, so we shall not prove it here. Theorems (4.2) and (4.3) will be proved in Sections 13 and 14.

5. Unsolved problems

(5.1) For what structures G do all the G-multiplicative functions obey a local limit law?

In this generality probably there is no simple condition. Perhaps the (most interesting) case when G is a commutative semigroup can be solved. The result, mentioned in Section 2, is far from being best possible; it does not contain even the case when G can be imbedded into a group. I have no plausible conjecture.

(5.2) Let G be a group. Which subsets of G are density sets for all the G-multiplicative functions?

I cannot solve this even in the simplest non-trivial case $G = C(\infty)$. I can show that the sets, given in Theorem (3.12), do not exhaust this class.

(5.3) Can one formulate and prove the analogues of the well-known global limit theorems?

I think nothing can be stated for arbitrary topologic groups. I failed even to find a reasonable definition for a global limit law. However this can be done without difficulty for metric groups. For example, probably an analogue of the "three-series-theorem" of Erdös and Wintner [5] is true.

6. Inverse image sequences. The aim of this section is to prove Statements (1.8), (1.13) and Theorem (2.2). (1.8) and (1.13) are connected with the inner characterization of sequences of the form $f^{-1}(g)$; (2.2) is different, but we prove it through a similar lemma.

Proof of (1.8).
(a) ⇒ (c) is trivial.
(c) ⇒ (b). Let

$$G_0 = \left\{\frac{a_1 \dots a_n}{b_1 \dots b_n}:\ a_i,\ b_j \in S\right\}, \qquad \boldsymbol{G}_0 = (G_0, \cdot) < \mathrm{MG}\boldsymbol{Q}.$$

Let $r \in S$ be arbitrary. Since with an $s \in S$

$$s = r\frac{s}{r}, \qquad \frac{s}{r} \in G_0,$$

hence

$$S \subset rG_0 \cap \boldsymbol{N}.$$

On the other hand, let $s \in rG_0 \cap \boldsymbol{N}$. $s \in rG_0 \Rightarrow s/r \in G_0$, so

$$\frac{s}{r} = \frac{a_1 \dots a_n}{b_1 \dots b_n}, \qquad a_i,\ b_j \in S.$$

Now

$$s = \frac{ra_1 \dots a_n}{b_1 \dots b_n} \in S$$

by condition (c).
(b) ⇒ (a). Let

$$\varphi:\ \mathrm{MG}\boldsymbol{Q} \to \mathrm{MG}\boldsymbol{Q}/\boldsymbol{G}_0$$

be the natural homomorphism; $f = \varphi|_{\boldsymbol{N}}$ will do. ■

Now we prove (1.13) in a more general form.

(6.1) STATEMENT. (a) *Let $\boldsymbol{G}$ be a group, $g \in G$ and f a strongly $\boldsymbol{G}$-multiplicative function. If $S \subset f^{-1}(g)$, then*

$$\prod_{i=1}^{m} s_i = \prod_{j=1}^{n} t_j, \qquad s_i, t_j \in S \Rightarrow n \equiv m \pmod{o(g)}. \tag{6.2}$$

(In case $o(g) = \infty$ we mean equality.)

(b) *If $\boldsymbol{G}$ is a divisible Abelian group, $g \in G$ and $S \subset \boldsymbol{N}$ satisfies* (6.2), *then there exists a $\boldsymbol{G}$-multiplicative f with $S \subset f^{-1}(g)$.*

Proof. (a) is trivial.

To prove (b) let $\boldsymbol{G}_0 = \operatorname{gen} S$, where this is meant in $\mathrm{MG}\boldsymbol{Q}$. G_0 consists of the elements of the form

$$x = \frac{s_1 \dots s_m}{t_1 \dots t_n}, \qquad s_i, t_j \in S.$$

Let for this x

$$\varphi_0(x) = g^{m-n}.$$

The condition (6.2) just means that this value is unique. $\varphi_0\colon \boldsymbol{G}_0 \to \boldsymbol{G}$ is a homomorphism. By a well-known theorem of Baer [1] (see also Fuchs [6]) a divisible group is injective, that is, this homomorphism can be extended to a homomorphism $\varphi\colon \mathrm{M}G\boldsymbol{Q} \to \boldsymbol{G}$. Since for $s \in S$, we have

$$\varphi(s) = \varphi_0(s) = g,$$

$f = \varphi|_{\boldsymbol{N}}$ will do. ■

To prove Theorem (2.2) we need

(6.3) LEMMA. *Let* $D_2 = \{0, 1\}$, $\boldsymbol{D}_2 = (D_2, \cdot)$, $\boldsymbol{D} = \boldsymbol{D}_2^\infty$ *(complete direc product) and* $\boldsymbol{0} = (0, 0, \ldots) \in D$. *If* $S \subset \boldsymbol{N}$ *satisfies*

$$n|m, \quad n \in S \Rightarrow m \in S, \tag{6.4}$$

then there exists a $\boldsymbol{D}$*-multiplicative* f *with* $S = f^{-1}(\boldsymbol{0})$.

The lemma implies Theorem (2.2) immediately, since Besicovitch [2] constructed a sequence satisfying (6.4) and having no density.

Proof of Lemma (6.3). A $\boldsymbol{D}$-multiplicative f can be given via a sequence $f_1, f_2, \ldots$ of $\boldsymbol{D}_2$-multiplicative functions. Let

$$f_k(n) = \begin{cases} 1 & \text{if} \quad k \notin S \text{ and } n|k, \\ 0 & \text{otherwise}. \end{cases}$$

This f_k is obviously $\boldsymbol{D}_2$-multiplicative. Evidently for $s \in S$ $f_k(s) = 0$, but if $z \notin S$, then $f_z(z) = 1$. ■

7. The metric space of multiplicative functions. Throughout this and the next section we shall always speak of multiplicative functions, mapping into a fixed structure $\boldsymbol{G}$, and we shall assume conditions (A) and (B) of Section 2, without any more mentioning.

(7.1) DEFINITION. Let for the multiplicative functions f_1, f_2

$$\varrho(f_1, f_2) = \sum_{f_1(p^k) \neq f_2(p^k)} p^{-k}. \tag{7.2}$$

This is obviously a metrics (with the exception that it may be infinite). *Convergence will always be meant in this metrics.*

Our aim is to prove

(7.3) NEGLECTION THEOREM. *Suppose* $\varrho(f, f_0) < \infty$ *and that one of the following conditions holds:*

(a) $\quad f(2^k) = f_0(2^k)$ *for all* k,

(b) $\quad f_0(2^{k+1}) = f_0(2) f_0(2^k)$ *for all* k.

Let $\mathfrak{K}$ *be a class of subsets of* G, *satisfying the following condition:*

$$K \in \mathfrak{K}, \quad g \in G \Rightarrow g^{-1}K \in \mathfrak{K}, \tag{7.4}$$

where

$$g^{-1}K = \{h\colon h \in G,\ gh \in K\}.$$

(i) *If* $\mathfrak{K}$ *is a density class for* f_0, *then so is for* f.

(ii) *If* $\mathfrak{K}$ *is a uniform density class for* f_0, *then so is for* f.

(iii) *If* $d(f_0, K) = 0$ *for all* $K \in \mathfrak{K}$, *then* $d(f, K) = 0$ *as well.*

The theorem is not true without any condition on $f_0(2^k)$. (The counterexample is complicated.) But it remains true for arbitrary groupoids; the generalization of the proof presents only some algebraic difficulties, due to the fact that some auxiliary functions may not exist.

The theorem will be proved in the next section; here we make some preliminary investigations.

(7.5) STATEMENT. *For arbitrary* $f_1, f_2, K \subset G$ *and* $x \geq 0$ *we have*

$$|N(f_1, K, x) - N(f_2, K, x)| \leq x\varrho(f_1, f_2).$$

Proof. If $f_1(n) \neq f_2(n)$, then n must be divisible by at least one p^k for which $f_1(p^k) \neq f_2(p^k)$. For a fixed p^k the number of these n is $[xp^{-k}]$. ■

(7.6) STATEMENT. *For every* f_1, f_2 *and* $K \subset G$

$$|ds(f_1, K) - ds(f_2, K)| \leq \varrho(f_1, f_2),$$

$$|di(f_1, K) - di(f_2, K)| \leq \varrho(f_1, f_2),$$

and if $d(f_1, K)$ *and* $d(f_2, K)$ *exist, then*

$$|d(f_1, K) - d(f_2, K)| \leq \varrho(f_1, f_2). \ ■$$

(7.7) STATEMENT. *If* $f_n \to f$ *and* $K \subset G$, *then*

$$ds(f_n, K) \to ds(f, K),$$

$$di(f_n, K) \to di(f, K). \ ■$$

(7.8) STATEMENT. *If* $f_n \to f$, $K \subset G$ *and* $d(f_n, K)$ *exists for all* n, *then* $d(f, K)$ *exists and*

$$d(f_n, K) \to d(f, K). \ ■$$

(7.9) STATEMENT. *If* $f_n \to f$ *and* $\mathfrak{K}$ *is a uniform density class for all* f_n, *then* $\mathfrak{K}$ *is a uniform density class for* f *as well.*

Proof. We must prove

$$a = \limsup_{x \to \infty} \sup_{K \in \mathfrak{K}} \left| \frac{N(f, K, x)}{x} - d(f, K) \right| = 0.$$

But from (7.5) and (7.6)

$$\left|\frac{N(f,K,x)}{x}-d(f,K)\right|$$
$$\leqslant\left|\frac{N(f_n,K,x)}{x}-d(f_n,K)\right|+\left|\frac{N(f_n,K,x)-N(f,K,x)}{x}\right|+$$
$$+|d(f_n,K)-d(f,K)|$$
$$\leqslant\left|\frac{N(f_n,K,x)}{x}-d(f_n,K)\right|+2\varrho(f_n,f).$$

By the assumption

$$a\leqslant 2\varrho(f_n,f);$$

making $n\to\infty$ we get the desired result. ■

The metrics ϱ is the most comfortable, but not the best one. For example, for $\varrho(f_1,f_2)>1$ the statements (7.5) and (7.6) are vacuous. For sake of completeness I describe the best possible metrics $\tilde\varrho$ for which (7.6) is true.

(7.10) Definition. Let $\tilde\varrho(f_1,f_2)$ be the density of natural numbers that are of the form p^km, $p\nmid m$, $f_1(p^k)\neq f_2(p^k)$.

Well-known elementary methods yield

(7.11) Statement. *$\tilde\varrho(f_1,f_2)$ exists for every f_1 and f_2, and*

$$\tilde\varrho(f_1,f_2)=1-\prod_p\Big(1-\sum_{\substack{k\\ f_1(p^k)\neq f_2(p^k)}}(p-1)p^{-k-1}\Big).\ ■$$

From the definition easily follows that $\tilde\varrho$ is really a metrics. (7.5) does not remain valid, but

(7.12) Statement. *For every f_1,f_2, $K\subset G$ and $x\geqslant 0$*

$$|N(f_1,K,x)-N(f_2,K,x)|\leqslant x\cdot\tilde\varrho(f_1,f_2)+o(x),$$

where the o depends only on x, f_1 and f_2 and not on K. ■

In Statements (7.6)–(7.9) ϱ can be replaced by $\tilde\varrho$.

ϱ and $\tilde\varrho$ are connected by the following relations, that can be got from (7.2), (7.10) and (7.11) by an easy computation.

(7.13) Statement. *For every f_1 and f_2*

$$\tilde\varrho(f_1,f_2)\leqslant\varrho(f_1,f_2),$$
$$1-e^{-\frac12\varrho(f_1,f_2)}\leqslant\tilde\varrho(f_1,f_2)\leqslant 1-e^{-\varrho(f_1,f_2)},$$
$$\tilde\varrho(f_1,f_2)=1 \text{ if and only if } \varrho(f_1,f_2)=\infty.\ ■$$

8. The proof of the neglection theorem

(8.1) Lemma. *Let f be a multiplicative function, p a prime and $\mathfrak{K}$ a density class for f, satisfying (7.4). If $p=2$, we suppose also condition (b) of (7.3), with f_0 replaced by f. Let*

$$M(K,x)=|\{n:\ n\leqslant x,\ f(n)\in K,\ p\nmid n\}|.$$

(a) *For $K\in\mathfrak{K}$ there exists*

(8.2) $$d'(K)=\lim_{x\to\infty}\frac{M(K,x)}{x},$$

and we have

(8.3) $$d(f,K)=\sum_{i=0}^{\infty}p^{-i}d'\big(f(p^i)^{-1}K\big).$$

(b) *If $\mathfrak{K}$ was a uniform density class for f, then (8.2) is uniform in $K\in\mathfrak{K}$.*

Proof. Let $n=p^im$, $p\nmid m$. $f(n)\in K$ is equivalent to $f(m)\in f(p^i)^{-1}K$, so we have

(8.4) $$N(f,K,x)=\sum_{i=0}^{\infty}M\big(f(p^i)^{-1}K,\ xp^{-i}\big)$$
$$=M(K,x)+\sum_{i=1}^{\infty}M\big(f(p^i)^{-1}K,\ xp^{-i}\big).$$

From (8.4)

(8.5) $$M(K,x)=N(f,K,x)-\sum_{i=1}^{\infty}M\big(f(p^i)^{-1}K,\ xp^{-1}\big).$$

Let first $p\geqslant 3$ and let

$$a(K)=\limsup_{x\to\infty}\frac{M(K,x)}{x}-\liminf_{x\to\infty}\frac{M(K,x)}{x},\qquad a=\sup_{K\in\mathfrak{K}}a(K).$$

Evidently $0\leqslant a(K)\leqslant 1$, $0\leqslant a\leqslant 1$. From (8.5), using

$$N(f,K,x)=xd(f,K)+o(x)\quad\text{and}\quad 0\leqslant M(K,x)\leqslant x,$$

we get with an arbitrary n

$$a(K)\leqslant\sum_{i=1}^{n}a\big(f(p^i)^{-1}K\big)p^{-i}+\sum_{i=n+1}^{\infty}p^{-i}.$$

Making $n\to\infty$

$$a(K)\leqslant\sum_{i=1}^{\infty}a\big(f(p^i)^{-1}K\big)p^{-i}\leqslant\frac{a}{p-1},$$

so $a \leqslant a/(p-1)$, which implies $a = 0$, which is (a). (8.3) is clear from (8.4). (Observe that a sum like

$$\sum_{i=0}^{\infty} \frac{1}{x} M(K_i, xp^{-i})$$

must be uniformly convergent, since it has the numerical majorant $\sum p^{-i}$).

To prove (b) let

$$b(x) = \sup_{K \in \mathfrak{K}} |M(K, x) - xd'(K)| \quad \text{and} \quad b = \limsup_{x \to \infty} \frac{b(x)}{x}.$$

From (8.5) and (8.3) we have

$$|M(K, x) - xd'(K)|$$
$$\leqslant |N(f, K, x) - xd(f, K)| + \sum_{i=1}^{\infty} \left| M\big(f(p^i)^{-1}K, xp^{-i}\big) - xp^{-i}d'(K) \right|.$$

Taking $\sup_{K \in \mathfrak{K}}$ and applying the fact that $\mathfrak{K}$ was a uniform density class, we get

$$b(x) \leqslant \sum_{i=1}^{\infty} b(xp^{-i}) + o(x).$$

Since evidently $|b(x)| \leqslant x$, we have with an arbitrary n

$$b(x) \leqslant \sum_{i=1}^{n} b(xp^{-i}) + x\Big(\sum_{i=n+1}^{\infty} p^{-i} + o(1)\Big).$$

From the definition of b

$$b \leqslant \sum_{i=1}^{n} bp^{-i} + \sum_{i=n+1}^{\infty} p^{-i};$$

making $n \to \infty$ we get $b \leqslant b/(p-1)$, that is $b = 0$, which is (b).

If $p = 2$, then by condition (b) of (7.3) we have simply

$$M(K, x) = N(f, K, x) - N\left(f, f(2)^{-1}K, \frac{x}{2}\right),$$

which implies (a) and (b) immediately. ■

(8.6) Lemma. *Let all the letters mean the same as in* (8.1). *If we suppose* (8.1(a)), *resp.* (8.1(b)), *then $\mathfrak{K}$ is a density class, resp. a uniform density class for f.* (Here we do not suppose (7.3(b)).) *The numbers $d(f, K)$ satisfy* (8.3).

This follows directly from (8.4). ■

Proof of the theorem. Let

$$f_n(p^i) = \begin{cases} f(p^i), & p \leqslant p_n, \\ f_0(p^i), & p > p_n. \end{cases}$$

First we show our statements for f_n by induction. For f_0 this is the assumption. Going from f_n to f_{n+1} we first apply Lemma (8.1) for $f = f_n$, $p = p_{n+1}$, then Lemma (8.6) for $f = f_{n+1}$, $p = p_{n+1}$. (Since f_n and f_{n+1} are equal at all the prime-powers, except of the powers of p_{n+1}, their M-functions are the same.)

Here the only critical point is $n = 0$, $p_1 = 2$. But in this case either $f_1 = f_0$ (case (7.2(a))) or (7.2(b)) holds, which justifies the application of Lemma (8.1).

Evidently $f_n \to f$, so we get (i) from (7.8) and (ii) from (7.9). In case $d(f_0, K) = 0$ for all $K \in \mathfrak{K}$ we first get $d(f_n, K) = 0$ by (8.3) and (iii) by (7.8). ■

9. Concentrated functions. In this section we prove the local limit theorems for concentrated functions. We begin with the case $|G| < \infty$. We have

(9.1) $$N(f, g, x) = \frac{1}{|G|} \sum_{\chi} \bar{\chi}(g) \sum_{n \leqslant x} \chi\big(f(n)\big),$$

where χ runs over all the characters of G.

$\psi(n) = \chi\big(f(n)\big)$ is a complex-valued multiplicative function, assuming only the values $e^{2k\pi i/T}$, $0 \leqslant k \leqslant T-1$, $T = |G|$. Therefore we can apply the following result of Wirsing [11] (Satz 1.2.1 and 1.2.2).

9.2) Lemma. *Let ψ be a complex-valued multiplicative function, $|\psi(n)| = 1$. tuppose that there is an arc of the unit circle which contains no $\psi(p)$. Then here exists*

(9.3) $$M(\psi) = \lim \frac{1}{x} \sum_{n \leqslant x} \psi(n) = \prod_p \left(\frac{p-1}{p} \sum_{k=1}^{\infty} \frac{\psi(p^k)}{p^k}\right).$$

$M(\psi) = 0$ if and only if either $\psi(2^k) = -1$ for all k or

(9.4) $$\sum_p \frac{1}{p}\big(1 - \operatorname{Re}\psi(p)\big) = \infty.$$

(9.5) Lemma. *Let G be a finite Abelian group, $|G| = T$, and let f be G-multiplicative. $d(f, g)$ exists for all $g \in G$. If the concentration group of f is G, then $d(f, g) = \frac{1}{T}$ for all $g \in G$.*

Proof. (9.1) and (9.2) imply that $d(f, g)$ exists and

$$d(f, g) = \frac{1}{T} \sum_{\chi} \bar{\chi}(g) M\big(\chi(f)\big).$$

To prove the second statement let us observe that the principal character $\chi_0 \equiv 1$ gives $1/T$. We prove that for $\chi \neq \chi_0$ we have $M(\psi) = 0$, where $\psi = \chi(f)$. If no, then, since $\psi(p) \neq 1$ implies

$$1 - \operatorname{Re}\psi(p) \geqslant 1 - \cos\frac{2\pi}{T} > 0,$$

using (9.4) we get

$$\sum_{\psi(p)\neq 1} \frac{1}{p} < \infty.$$

This means

$$\sum_{f(p)\notin\ker\chi} \frac{1}{p} < \infty,$$

that is, if K denotes the concentration complex of f, we have $K \subset \ker\chi$. Since $\ker\chi$ is a subgroup,

$$\ker\chi \supset \operatorname{gen} K = \boldsymbol{G}, \qquad \chi = \chi_0. \ \blacksquare$$

(9.6) Lemma. *Let $\boldsymbol{G}$ be an Abelian group and f a concentrated $\boldsymbol{G}$-multiplicative function.*

$$\mathfrak{G} = \{\{g\}:\ g \in G\}$$

is a uniform density class for f.

Proof. Let G_1 be the concentration group of f and

$$f_1(p^k) = \begin{cases} f(p^k) & \text{if} \quad f(p^k) \in G_1, \\ e & \text{otherwise}. \end{cases}$$

We have $\varrho(f, f_1) < \infty$ so we can apply Lemma (9.5) and the neglection theorem. (In a finite group uniformity is fulfilled automatically.)

(9.7) Lemma. *Let f be a concentrated function with the concentration group $\boldsymbol{G}_1$. If $f(p^k) \in G_1$ for $p \geqslant p_n$, then $d(f, g)$ depends only on the coset of $\boldsymbol{G}_1$ in which g lies.*

Proof. Induction on n. For $n = 1$ this is contained in (9.5). Suppose we know it for $n-1$. Let

$$f_1(p^k) = \begin{cases} e & \text{if} \quad p = p_n, \\ f(p^k) & \text{otherwise}. \end{cases}$$

We know the statement for f_1. Setting

$$M(g, x) = |\{m:\ m \leqslant x, f(m) = g, p_n \nmid m\}|$$

we have

$$M(g, x) = N(f_1, g, x) - N(f_1, g, xp_n^{-1})$$

and

$$N(f, g, x) = \sum_{k=0}^{\infty} M\big(gf(p_n^k)^{-1}, xp_n^{-k}\big).$$

From these formulas

$$d(f, g) = \frac{p-1}{p} \sum_{k=0}^{\infty} d\big(f_1, gf(p_n^k)^{-1}\big) p_n^{-k},$$

and the right-hand side depends only on the coset of $\boldsymbol{G}_1$, containing g. $\blacksquare$

(9.8) Lemma. *If f is concentrated with the concentration group $\boldsymbol{G}_1$, then $d(f, g)$ depends only on the coset of $\boldsymbol{G}_1$, containing g.*

Proof. Let

$$f_n(p^k) = \begin{cases} e & \text{if} \quad p \geqslant p_n \text{ and } f(p^k) \notin G_1, \\ f(p^k) & \text{otherwise}. \end{cases}$$

Obviously $f_n \to f$. If g_1 and g_2 are in the same coset of $\boldsymbol{G}_1$, then by (9.7) and (7.8)

$$d(f, g_1) - d(f, g_2) = \lim\big(d(f_n, g_1) - d(f_n, g_2)\big) = 0. \ \blacksquare$$

(9.9) Definition. Let $\boldsymbol{G}$ be a group and f $\boldsymbol{G}$-multiplicative. We call f *superconcentrated* if

$$\sum_{f(p^k)\neq e} p^{-k} < \infty.$$

We can reduce the investigation of concentrated functions to superconcentrated ones.

(9.10) Definition. Let $\boldsymbol{G}$ be an Abelian group, $\boldsymbol{G}_1 < \boldsymbol{G}$ and

$$\varphi:\ \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{G}_1$$

the natural homomorphism. We define the *factor-function* of a $\boldsymbol{G}$-multiplicative f with respect to $\boldsymbol{G}_1$ as

$$f_1 = f/\boldsymbol{G}_1 = \varphi(f).$$

(That is, $f_1(n)$ is the coset of $\boldsymbol{G}_1$ containing $f(n)$.)

(9.11) Statement. *If f is concentrated with the concentration group $\boldsymbol{G}_1$, then $f/\boldsymbol{G}_1$ is superconcentrated.* $\blacksquare$

By (9.8) we have

(9.12) Statement. *If f is concentrated with the concentration group $\boldsymbol{G}_1$ and $f_1 = f/\boldsymbol{G}_1$, then for all $g \in G$*

$$d(f, g) = \frac{1}{|G_1|} d(f_1, g\boldsymbol{G}_1). \ \blacksquare$$

(9.13) LEMMA. *If f is superconcentrated, then*

(a) $$d(f,g)>0 \quad \text{for all } g\in \operatorname{im} f,$$

(b) $$\sum_{g\in G} d(f,g)=1.$$

Proof. Let $g\in\operatorname{im} f$, that is, $g=f(n_0)$ with some n_0. Let Q be the set of prime-powers p^k for which either $f(p^k)\neq e$ or $p|n_0$. Obviously

$$\sum_{q\in Q}\frac{1}{q}<\infty.$$

The well-known inequality of Heilbronn [9] and Rohrbach [10] easily implies that the sequence S of numbers which are divisible by no element of Q has a density

$$\operatorname{den} S>\prod_{q\in Q}\left(1-\frac{1}{q}\right)>0.$$

Since $n_0S\subset f^{-1}(g)$, we have

$$d(f,g)\geqslant\frac{\operatorname{den}S}{n_0}>0,$$

which is (a).

To prove (b) let Q be the set of prime-powers p^k for which $f(p^k)\neq e$ and let Q_1 be a *finite* subset of Q satisfying

(9.14) $$\sum_{q\in Q\setminus Q_1}\frac{1}{q}<\varepsilon.$$

Let K denote the set of elements of G which can be written as

$$f(q_1\ldots q_n),\quad q_i\in Q_1,\ q_i\neq q_j.$$

K is finite, so there exists $d(f,K)$ and $d(f,G\setminus K)$. If

$$n=\prod p_i^{k_i},\quad f(n)\notin K,$$

then we must have

$$p_i^{k_i}\in Q\setminus Q_1$$

for at least one i.

Now because of (9.14) we have

$$d(f,G\setminus K)<\varepsilon,\quad d(f,K)>1-\varepsilon;$$

ε was arbitrary, which implies (b). ■

Combining (9.12) and (9.13) we get

(9.15) STATEMENT. *Lemma* (9.13) *is true for every concentrated function.*

10. Deconcentrated functions. Let G be an infinite Abelian group. By a *character* of G we mean a homomorphism

$$\chi\colon G\to \mathrm{M}G\,C;\quad |\chi(g)|=1.$$

The group of characters will be denoted by $\operatorname{char} G$.

If G is infinite, we cannot express $N(f,g,x)$ by a finite combination of characters, but we can do the following.

Let $\chi_1,\ldots,\chi_m$ be arbitrary characters and

(10.1) $$\vartheta(n)=\Big|\sum_{j=1}^{m}\bar\chi_j(g)\chi_j\big(f(n)\big)\Big|^2.$$

Obviously

(10.2) $$\vartheta(n)\begin{cases}=m^2 & \text{if } f(n)=g,\\ \geqslant 0 & \text{otherwise,}\end{cases}$$

hence we have

(10.3) $$N(f,g,x)\leqslant m^{-2}\sum_{n\leqslant x}\vartheta(n).$$

We have to estimate

(10.4) $$\theta(x)=\sum_{n\leqslant x}\vartheta(n)=\sum_{j,k=1}^{m}c_{jk}\sum_{n\leqslant x}\psi_{jk}(n),$$

where

$$c_{jk}=\bar\chi_j(g)\chi_k(g)$$

and

(10.5) $$\psi_{jk}(n)=\chi_jf\big((n)\big)\bar\chi_k\big(f(n)\big)$$

is a multiplicative function satisfying

$$|\psi_{jk}(n)|=1.$$

Since $|c_{jk}|=1$, for every g

(10.6) $$N(f,g,x)\leqslant m^{-2}\sum_{j,k=1}^{m}\Big|\sum_{n\leqslant x}\psi_{jk}(n)\Big|.$$

We shall estimate $\sum\psi_{jk}(n)$ using the theorems of Halász. The main difficulty will be to find suitable characters, which yield a good estimate.

Halász [7] proved

(10.7) LEMMA. *Let φ be a complex-valued multiplicative function, $|\varphi(n)| = 1$,*

$$\Phi(x) = \sum_{n\leqslant x} \varphi(n).$$

One of the following two possibilities must hold.

(a) *For arbitrary real h*

$$\sum_p \frac{1}{p}\left(1-\mathrm{Re}\left(p^{-ih}\varphi(p)\right)\right) = \infty,$$

$$\Phi(x) = o(x).$$

(b) *There exists exactly one $h = h(\varphi)$, for which*

$$\sum_p \frac{1}{p}\left(1-\mathrm{Re}\left(p^{-ih}\varphi(p)\right)\right) < \infty.$$

In this case

$$\Phi(x) = cx^{1+ih}L(\log x)+o(x),$$

where L is a complex-valued function satisfying

$$|L(x)| = 1, \qquad \frac{L(u_1)}{L(u)} \to 1$$

uniformly if $u\to\infty$ and $u \leqslant u_1 \leqslant 2u$.

(10.8) LEMMA. *If $h(\varphi)$ and $h(\psi)$ exist, then so do $h(\bar\varphi)$ and $h(\varphi\psi)$ as well, and*

$$h(\bar\varphi) = -h(\varphi),$$

$$h(\varphi\psi) = h(\varphi)+h(\psi).$$

Proof. The first is trivial. The second is due to the inequality

$$|a| = |b| = 1 \Rightarrow 1-\mathrm{Re}\,ab \leqslant 2\left((1-\mathrm{Re}\,a)+(1-\mathrm{Re}\,b)\right). \ \blacksquare$$

(10.9) LEMMA. *In* (10.7(b))

$$|c| \leqslant \frac{1}{\sqrt{1+h^2}}.$$

Proof. Summation by parts yields (in virtue of $L(u_1)/L(u)\to 1$)

$$\sum_{n\leqslant x} \varphi(n)n^{-ih} = cL(\log x)(1+ih)x+o(x),$$

and the lemma follows from

$$\left|\sum_{n\leqslant x}\varphi(n)n^{-ih}\right| \leqslant x. \ \blacksquare$$

We need a lemma, which will be proved in the next section.

(10.10) LEMMA. *Let G be an Abelian group and f a deconcentrated G-multiplicative function. Either there is a character χ for which $h(\chi(f))$ exists and differs from 0, or there is an infinite sequence $\chi_1, \dots, \chi_n, \dots$ of characters such that*

$$h\left(\chi_j(f)\bar\chi_k(f)\right)$$

never exists if $j \neq k$.

(10.11) LEMMA. *If f is deconcentrated, then*

$$\frac{N(f, g, x)}{x} \to 0$$

uniformly in g when $x\to\infty$.

Proof. We distinguish two cases according the previous lemma. Suppose first that χ is a character for which there exists

$$h = h\left(\chi(f)\right) \neq 0.$$

Apply (10.6) with

$$\chi_j = \chi^{jm}.$$

If $j = k$, we have the trivial estimate

$$\left|\sum_{n\leqslant x}\psi_{jk}(n)\right| \leqslant x.$$

If $j \neq k$, then by (10.7(b)), (10.8) and (10.9)

$$\left|\sum_{n\leqslant x}\psi_{jk}(n)\right| \leqslant \frac{x}{|(j-k)mh|}+o(x) \leqslant \frac{x}{m|h|}+o(x).$$

Summing up we get

$$N(f, g, x) \leqslant \frac{1}{m}\left(1+\frac{1}{|h|}\right)x+o(x);$$

since m was arbitrary, we are ready.

Now suppose the second possibility and apply (10.6) with the χ_j's of Lemma (10.11). In case $j = k$ we have the trivial estimate. In case $j \neq k$ by (10.7(a))

$$\sum_{n\leqslant x}\psi_{jk}(n) = o(x),$$

so we get again

$$N(f, g, x) \leqslant \frac{x}{m}+o(x). \ \blacksquare$$

11. Random characters. In this section we prove Lemma (10.10). We shall define a probability measure on the set of characters and then prove that if we choose χ_j at random, then with probability 1 either one of them satisfies the first possibility, or the whole sequence satisfies the second.

First we define a topology on $\operatorname{char} G$. Let the subbase of χ consist of the sets

$$K_\chi(g, \varepsilon) = \{\psi\colon |\psi(g)-\chi(g)| < \varepsilon\}, \quad g \in G,\ \varepsilon > 0.$$

This makes $\operatorname{char} G$ a compact Hausdorff topological group. (The space of all the functions, bounded by 1, is compact, and the characters form a closed subset.)

Now we can regard the (normed) Haar-measure. (See, for example, Halmos [8].) This measure — which we shall denote by P — is known to be invariant under multiplication.

(11.1) STATEMENT. *Let $g \in G$. If $o(g) < \infty$, then*

$$P\left(\chi(g) = \exp\frac{2\pi ik}{o(g)}\right) = \frac{1}{o(g)}$$

for all k. If $o(g) = \infty$, then $\chi(g)$ is uniformly distributed in the unit circle. ∎

(11.2) STATEMENT. *For $g \in G$*

$$\boldsymbol{E}(\chi(g)) = \begin{cases} 0 & \text{if } g \neq e, \\ 1 & \text{if } g = e. \end{cases}$$ ∎

(11.3) LEMMA. *Let f be a deconcentrated G-multiplicative function. We have with probability 1*

$$\sum_p \frac{1}{p}\big(1-\operatorname{Re}\chi(f(p))\big) = \infty.$$

Proof. Introduce the notation

$$\alpha(g) = \sum_{f(p)=g} \frac{1}{p}.$$

We must prove

$$\sum_{g \in G} \alpha(g)\big(1-\operatorname{Re}\chi(g)\big) = \infty$$

with probability 1.

It is easily seen that the condition that f be deconcentrated is equivalent to that

(a) either there exists a g_0 with $o(g_0) = \infty$ and $\alpha(g_0) = \infty$,

(b) or

$$\sum_{g \in G} \beta(g) = \infty, \quad \beta(g) = \min\big(1, \alpha(g)\big).$$

In case (a)

$$\sum \alpha(g)\big(1-\operatorname{Re}\chi(g)\big) \geqslant \alpha(g_0)\big(1-\operatorname{Re}\chi(g_0)\big) = \infty$$

unless $\chi(g_0) = 1$, which has probability 0 by (11.1).

In case (b) we use a second-moment method. It is sufficient to prove that

$$P\Big(\sum \beta(g)\big(1-\operatorname{Re}\chi(g)\big) < c\Big) < \varepsilon$$

for every c and $\varepsilon > 0$. Choose a $H \subset G\setminus\{e\}$ finite subset such that

$$B = \sum_{g \in H} \beta(g)$$

be large and let

$$B(\chi) = \sum_{g \in H} \beta(g)\big(1-\operatorname{Re}\chi(g)\big) = B - L(\chi).$$

We shall compute $\boldsymbol{E}\big(L^2(\chi)\big)$.

Since

$$\operatorname{Re} z = \tfrac{1}{2}(z+\bar z), \quad \operatorname{Re} z_1 \operatorname{Re} z_2 = \tfrac{1}{4}(z_1 z_2 + z_1 \bar z_2 + \bar z_1 z_2 + \bar z_1 \bar z_2),$$

and

$$\overline{\chi(g)} = \chi(g^{-1}),$$

we have

$$\boldsymbol{E}\big(\operatorname{Re}\chi(g_1)\operatorname{Re}\chi(g_2)\big) = \tfrac{1}{4}\boldsymbol{E}\big(\chi(g_1g_2)+\chi(g_1g_2^{-1})+\chi(g_1^{-1}g_2)+\chi(g_1^{-1}g_2^{-1})\big)$$
$$= \begin{cases} 1 & \text{if } g_1 = g_2 = e, \\ \tfrac{1}{2} & \text{if } g_1g_2 = e \text{ or } g_1 = g_2, \text{ but } g_1, g_2 \neq e, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore

$$\boldsymbol{E}\big(L^2(\chi)\big) = \tfrac{1}{2}\Big(\sum_{g \in H} \beta^2(g) + \sum_{g, g^{-1} \in H} \beta(g)\beta(g^{-1})\Big) \leqslant \sum_{g \in H} \beta^2(g) \leqslant B.$$

(We used the fact $\beta(g) \leqslant 1$.)

From Tshebyshev's inequality

$$P\left(B(\chi) \leqslant \frac{B}{2}\right) \leqslant P\left(|L(\chi)| \geqslant \frac{B}{2}\right) \leqslant \frac{4}{B}.$$

Since B can be arbitrary large, we are ready. ∎

Proof of Lemma (10.10). Observe that Lemma (11.3) just means that the event

$$\text{“}h(\chi(f)) = 0\text{”}$$

has probability 0. If there is a χ with

$$h(\chi(f)) \neq 0,$$

we are ready; if not, we know that the event

$$\text{“}h(\chi(f)) \text{ does not exist”}$$

happens with probability 1.

Now we choose χ_1 at will. If we have chosen $\chi_1, \ldots, \chi_{n-1}$, we have to find a χ_n such that

$$h(\chi_j(f)\chi_n(f)), \quad 1 \leqslant j \leqslant n-1,$$

should not exist. But — since the measure P was invariant — this requirement is fulfilled by almost all χ. ∎

12. Completion of proofs of Theorems (1.4), (3.7), (3.10), (3.11) and (3.12). The first four theorems are ready, we only have to gather the details. For deconcentrated functions they all follow from Lemma (10.11). For concentrated functions (3.7) and its weaker form (1.4) follow from (9.6), (3.10) from (9.15) and (3.11) from (9.8).

To prove (3.12) we require a lemma. (We leave the easy proof to the reader.)

(12.1) LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two classes of subsets of a set X. Suppose $\mathfrak{A}$ is closed under intersection and $\mathfrak{B}$ is closed under proper difference and disjoint union, that is*

$$A, B \in \mathfrak{A} \Rightarrow A \cap B \in \mathfrak{A},$$
$$(A, B \in \mathfrak{B}, A \supset B) \Rightarrow A \setminus B \in \mathfrak{B},$$
$$(A, B \in \mathfrak{B}, A \cap B = \emptyset) \Rightarrow A \cup B \in \mathfrak{B}.$$

If $\mathfrak{B} \supset \mathfrak{A}$ then $\mathfrak{B}$ contains the set algebra generated by $\mathfrak{A}$ (if $X \in \mathfrak{A}$). ∎

Proof of (3.13). Let G be an Abelian group, f a G-multiplicative function, $\mathfrak{A}$ the class of all cosets of all subgroups of G and $\mathfrak{B}$ the class of all density sets for f. $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the conditions of Lemma (12.1), so we are ready if we can prove $\mathfrak{B} \supset \mathfrak{A}$. That is, we must prove that a coset is a density set for f. But let $G_1 < G$; applying Theorem (1.4) for the factor-function

$$f_1 = f/G_1$$

(see the definition in (9.10)) we get the desired result. ∎

13. Proof of Theorem (4.2). Because of Theorem (3.10) we may suppose f concentrated. Let its concentration group be G_0. By the uniform distribution theorem (3.11) we have for all $g \in G$

$$d(f, g) \leqslant \frac{1}{|G_0|}, \tag{13.1}$$

equality only in case $\operatorname{im} f = G_0$.

If $|G_0| \geqslant 3$, we are ready. If $|G_0| = 2$, then in (13.1) equality must hold, that is, $\operatorname{im} f = G_0$, which is contained in possibility (b).

Remains the case of superconcentrated f. In this case $f_n \to f$, where

$$f_n(p^k) = \begin{cases} f(p^k) & \text{if} \quad p \leqslant p_n, \\ e & \text{if} \quad p > p_n. \end{cases}$$

Furthermore

$$d(f_n, g) = \left(1 - \frac{1}{p_n}\right) \sum_{k=0}^{\infty} d\left(f_{n-1}, f(p_n^k)^{-1} g\right) p_n^{-k}. \tag{13.2}$$

Introduce the notations

$$S_n = \max_g d(f_n, g), \qquad s_n = \max_{g \neq e} d(f_n, g).$$

(13.2) implies

$$S_n \leqslant S_{n-1}, \tag{13.3}$$

$$s_n \leqslant \left(1 - \frac{1}{p_n}\right)\left(s_{n-1} + \sum_{k=1}^{\infty} S_{n-1} p_n^{-k}\right) = \left(1 - \frac{1}{p_n}\right) s_{n-1} + \frac{1}{p_n} S_{n-1}. \tag{13.4}$$

We prove $s_n \leqslant 1/2$ by induction. It is true for $n = 0$. If $s_{n-1} \leqslant 1/2$, then

$$S_{n-1} = \max\left(s_{n-1}, d(f_{n-1}, e)\right) \leqslant \max(s_{n-1}, 1 - s_{n-1}) = 1 - s_{n-1}.$$

From (13.4)

$$s_n \leqslant \left(1 - \frac{1}{p_n}\right) s_{n-1} + \frac{1}{p_n}(1 - s_{n-1}) \tag{13.5}$$
$$= \frac{1}{p_n} + \left(1 - \frac{2}{p_n}\right) s_{n-1} \leqslant \frac{1}{p_n} + \frac{1}{2}\left(1 - \frac{2}{p_n}\right) = \frac{1}{2}.$$

So in case $g \neq e$

$$d(f, g) = \lim d(f_n, g) \leqslant \tfrac{1}{2}.$$

Now we have to determine the extremal functions. Suppose $d(f, g) = 1/2$. We have

$$S_n \geqslant d(f_n, g) \to \tfrac{1}{2},$$

so by (13.3) $S_n \geqslant 1/2$ for all n.

Regard first the case that f assumes not only the values e and g. In this case

$$d(f, e) < \tfrac{1}{2},$$

so for $n > n_0$

$$d(f_n, e) < \tfrac{1}{2}.$$

This implies $S_n = s_n$; since $S_n \geqslant 1/2$, $s_n \leqslant 1/2$, we have

$$S_n = s_n = \tfrac{1}{2} \quad (n > n_0).$$

Since $d(f_n, g) \to 1/2$ and $d(f_n, e) \to d(f, e) > 0$, hence for $g' \neq g$, e and $n > n_0$

$$d(f_n, g') \leqslant 1 - d(f_n, g) - d(f_n, e) < \tfrac{1}{2},$$

so $s_n = 1/2$ implies

$$d(f_n, g) = \tfrac{1}{2}.$$

Summing up

$$n > n_0 \Rightarrow d(f_n, g) = \tfrac{1}{2}, \quad d(f_n, g') < \tfrac{1}{2} \quad (g' \neq g).$$

(This n_0 is the greater of the previous two — different — n_0's.)

For $n > n_0$ in (13.4) equality must hold, and this is possible only if $f(p_n^k) = e$, $k = 1, 2, \ldots$ This means that the sequence stabilizes. Let n be the greatest suffix for which $f_n \neq f_{n-1}$.

If $n = 1$, we are ready. If $n \geqslant 2$, $p_n > 2$, hence equality in (13.5) holds only under the condition

$$s_{n-1} = \tfrac{1}{2},$$

and in (13.4) only if

$$d\big(f_{n-1}, f(p_n^k)^{-1} g\big) = \tfrac{1}{2}, \quad k = 0, 1, \ldots \tag{13.6}$$

Setting $k = 0$ we get

$$d(f_{n-1}, g) = \tfrac{1}{2}.$$

Let k be such that $f(p_n^k) \neq e$ (such a k must exist, otherwise $f_n = f_{n-1}$). We have

$$d(f_{n-1}, g') = \tfrac{1}{2}, \quad g' = f(p_n^k)^{-1} g \neq g.$$

$g' \neq e$ would imply (since $d(f_{n-1}, e) > 0$)

$$\sum_{h \in G} d(f_{n-1}, h) > 1;$$

a contradiction, so $g' = e$. This means that

$$f(p_n^k) = e \text{ or } g, \quad k = 1, 2, \ldots$$

We got

$$d(f_{n-1}, e) = d(f_{n-1}, g) = \tfrac{1}{2}.$$

Therefore $\operatorname{im} f_{n-1} = \{e, g\}$. There can be only one prime p for which $f_{n-1}(p^k) \not\equiv e$, since

$$f_{n-1}(p^k) = f_{n-1}(q^m) = g, \quad p \neq q$$

would imply

$$f_{n-1}(p^k q^m) = g^2 \neq e, g.$$

(If $g^2 = e$, then f_n would not assume values different from e and g.) Let this prime be q.

$$d(f_{n-1}, g) = \left(1 - \frac{1}{q}\right) \sum_{f_{n-1}(q^k) = g} q^{-k} = \frac{1}{2}$$

is possible only if $q = 2$ and $f_{n-1}(2^k) = g$, $k = 1, 2, \ldots$; this is case (a).

There remained the case $\operatorname{im} f = \{e, g\}$.

If there is only one prime for which $f(p^k) \not\equiv e$, we get no new solution. If there are more, we have again $g^2 = e$. Without loss of generality we can restrict ourselves to the case $g = -1$, $e = +1$, f multiplicative in the usual sense. Now

$$d(f, g) = \tfrac{1}{2}$$

is equivalent to

$$\sum_{n \leqslant x} f(n) = o(x).$$

By Wirsing's theorem (see our Lemma (9.2)) we have for some p

$$1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^k} = 0.$$

This is impossible unless $p = 2$, $f(2^k) = -1$, $k = 1, 2, \ldots$, which is contained in case (b).

14. Proof of Theorem (4.3). This proof will not be a beautiful one. It is full of routine calculations, which I shall never detail. A sign (RC) is used to denote where such details are omitted. The proof of the most complicated lemmas is left to the next section.

First we settle the trivial cases. The case $p_0 = 2$ follows from Theorem (4.2). Suppose $p_0 \geqslant 3$.

If f is deconcentrated, we have nothing to prove. If f is concentrated with the concentration group $G_0 \neq \{e\}$, then by $p_0 \geqslant 3$ we have $|G_0| \geqslant 3$, so (13.1) gives $d(f, g) \leqslant 1/3$. Since

$$W_{p_0} \geqslant w_{p_0}(1) \geqslant 1/e > 1/3,$$

we are ready. Remains the case f is superconcentrated.

(14.1) LEMMA. *Let $o(g) = n$, n a natural number or ∞, P a set of primes and*

$$\min_{p\in P} p = M, \qquad \sum_{p\in P} \frac{1}{p} = x.$$

Let f denote the stronly multiplicative function defined by

$$f(p) = \begin{cases} g & \text{if} \quad p\in P, \\ e & \text{if} \quad p\notin P. \end{cases}$$

We have

$$|d(f, g) - w_n(x)| < \frac{c}{M}, \tag{14.2}$$

with an absolute constant c.

Proof in the next section. This lemma shows that W_{p_0} can be arbitrarily approached.

(14.3) LEMMA. *Suppose we know*

$$d(f, g) < W_p$$

for strongly $C(p^\infty)$-multiplicative functions f and elements g of order p, p prime or ∞. This implies Theorem (4.3).

Proof. Let G be arbitrary, $e \neq g\in G$ and f strongly G-multiplicative. Let $n = o(g)$ and p_1 a prime divisor of n if $n < \infty$ and ∞ if $n = \infty$. Let $g_1 \in C(p_1^\infty)$, $o(g_1) = p_1$. By (6.1) there exists a strongly $C(p_1^\infty)$-multiplicative f_1 such that

$$f_1^{-1}(g_1) \supset f^{-1}(g).$$

Now we have

$$d(f, g) \leqslant d(f_1, g_1) < W_{p_1} \leqslant W_{p_0},$$

since $p_0 \leqslant p_1$ and W_n is decreasing (RC). ∎

By now on we restrict ourselves to groups $C(p_0^\infty)$ and elements $o(g) = p_0$.

First we regard functions with $f(p) = e$ with the exception of a finite set of primes, P. Let

$$F = F(P) = \{f\colon f(p) \neq e \Rightarrow p\in P\}, \tag{14.4}$$

where it is to be involved that f is strongly G-multiplicative, G is a group of type $C(p_0^\infty)$,

$$\begin{aligned} r(f) &= |\{p\colon p\in P, f(p) = e\}|, \\ s(f) &= |\{p\colon p\in P, f(p) = g\}|, \end{aligned} \tag{14.5}$$

where g is the investigated element of order p_0.

(14.6) LEMMA. *Let $f\in F$. At least one of the following three possibilities must hold.*

(a) $d(f, g) \leqslant 1/3$,

(b) $r(f) + s(f) = |P|$,

(c) *there exists an $f^*\in F$, for which*

$$d(f^*, g) > d(f, g), \qquad r(f^*) > r(f).$$

Proof in the next section.

(14.7) LEMMA. *If*

$$\sup_{f\in F} d(f, g) > 1/3,$$

then there exists

$$\max_{f\in F} d(f, g),$$

and it is reached only at functions satisfying $r(f) + s(f) = |P|$.

Proof. Start with a function f_1, for which $d(f_1, g) > 1/3$. Iterating Lemma (14.6) we get a sequence of functions $f_1, f_2, \ldots$ such that both $d(f_n, g)$ and $r(f_n)$ are strictly increasing. Since $r(f_n) \leqslant |P|$, this sequence must terminate. The last function f^* satisfies

$$r(f^*) + s(f^*) = |P|;$$

since there are only finitely many such functions, we can select the ones with maximal $d(f, g)$ from them. ∎

(14.8) LEMMA. *Let P be a finite set of primes and $f\in F$ the function for which*

$$f(p) = \begin{cases} g & \text{if} \quad p\in P, \\ e & \text{if} \quad p\notin P. \end{cases}$$

Suppose

$$d(f', h) < d(f, g)$$

for every $e \neq h\in G$ and $f \neq f'\in F$. Then there exists another finite set P_1 of primes for which

$$q_1 = \min_{p\in P_1} p > q_0 = \min_{p\in P} p$$

and

$$d(f_1, g) > d(f, g) + 10^{-3} q_0^{-2},$$

where

$$f_1(p) = \begin{cases} g & \text{if} \quad p\in P_1, \\ e & \text{if} \quad p\notin P_1. \end{cases}$$

Proof in the next section.

(14.9) LEMMA. *For the function f of Lemma* (14.8)

$$d(f,g) < W_{p_0} - 10^{-3} q_0^{-2}.$$

Proof. If $d(f,g) \leqslant 1/3$, we are ready ($W_{p_0} \geqslant 1/e$). If not, we define a sequence f_n. Let f_1 be the function, given by Lemma (14.8). Given f_n, let

(14.10) $$P_n = \{p\colon f_n(p) \neq e\}$$

and apply Lemma (14.7) for $F = F(P_n)$. Let f'_n be one of the extremal functions with maximal $r(f'_n)$. Now apply Lemma (14.8) for $f = f'_n$; the produced function will be f_{n+1}.

These functions satisfy

$$d(f_n, g) \leqslant d(f'_n, g) < d(f_{n+1}, g),$$
$$d(f_1, g) > d(f,g) + 10^{-3} q_0^{-2},$$

therefore

$$d(f,g) < d(f_n, g) - 10^{-3} q_0^{-2}.$$

On the other hand,

$$\lim_n \big(\min_{f_n(q)\neq e} q\big) = \infty,$$

so by Lemma (14.1)

$$\lim d(f_n, g) = \lim w_{p_0}\Big(\sum_{f_n(q)\neq e} \frac{1}{q}\Big) \leqslant W_{p_0}. \ \blacksquare$$

Completion of the proof of the theorem. Let f be strongly multiplicative. If $d(f,g) \leqslant 1/3$, we are ready. If not, let

$$f_n(p) = \begin{cases} f(p) & \text{if} \quad p \leqslant p_n, \\ e & \text{if} \quad p > p_n, \end{cases}$$

P_n given by (14.10). Applying Lemma (14.7) for $F = F(P_n)$, we get a function $f'_n \in F(P_n)$ with maximal $d(f'_n, g)$ and the greatest possible $r(f'_n)$. Lemma (14.9) yields

(14.11) $$d(f_n, g) \leqslant d(f'_n, g) < W_{p_0} - 10^{-3} q_n^{-2}$$

with

$$q_n = \min_{f'_n(q)\neq e} q.$$

Obviously

$$f_n \to f, \quad d(f_n, g) \to d(f,g);$$

if $q_n \not\to \infty$, this and (14.11) immediately imply the theorem.

If $q_n \to \infty$, then $f'_n \to E$, where $E(n) \equiv e$, so

$$d(f,g) = \lim d(f_n, g) = \lim d(f'_n, g) = d(E, g) = 0,$$

a contradiction. $\blacksquare$

15. Proof of Lemmas (14.1), (14.6) and (14.8)

Proof of Lemma (14.1). First let $n < \infty$. We can assume $g = \exp\frac{2\pi i}{n}$. Then we have

$$N(f,g,x) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j\leqslant x}\big(f(j)\bar{g}\big)^k = \frac{1}{n}\sum_{k=0}^{n-1}\bar{g}^k\sum_{j\leqslant x} f^k(j),$$

Fo according to Lemma (9.2)

(15.1) $$d(f,g) = \frac{1}{n}\sum_{k=0}^{n-1}\bar{g}^k\prod_{p\in P}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{g^k}{p}\Big)^{-1}.$$

srom (15.1), using

$$\log\Big(1-\frac{1}{p}\Big)\Big(1-\frac{g^k}{p}\Big)^{-1} = \frac{g^k-1}{p} + O(p^{-2})$$

we get (RC)

$$d(f,g) = \frac{1}{n}\sum_{k=0}^{n-1}\bar{g}^k\exp(g^k-1)x + O\Big(\sum_{p\in P} p^{-2}\Big).$$

Now we get (14.2) using the formulas (RC)

$$\sum_{p\in P} p^{-2} = O(M^{-1}), \qquad \sum_{k=0}^{n-1}\bar{g}^k\exp(g^k-1)x = nw_n(x).$$

In case $n = \infty$ we have to start with the formulas

$$d(f,g) = \prod_{p\in P}\Big(1-\frac{1}{p}\Big)\sum_{p\in P}\frac{1}{p}, \qquad \log\Big(1-\frac{1}{p}\Big) = -\frac{1}{p} + O(p^{-2}). \ \blacksquare$$

Proof of Lemma (14.6). If (b) does not hold, then there is a $q_0 \in P$ such that

$$h = f(q_0) \neq e, g.$$

Now let

$$f_1(q) = \begin{cases} f(q) & \text{if} \quad q \neq q_0, \\ e & \text{if} \quad q = q_0. \end{cases}$$

If there exists a $g' \in G$, $g' \neq e$ such that

$$d(f_1, g') > d(f,g),$$

then let η be an endomorphism of G such that $\eta g' = g$. (Such an endomorphism must exist, since G was a group of type $C(p_0^\infty)$ and $o(g) = p_0$.) Now $f^* = \eta f_1$ satisfies (c).

If there is no such g', regard the following equality:

$$d(f, g) = (1-q_0^{-1}) \sum_{i=0}^{\infty} q_0^{-i} d(f_1, gh^{-i}). \tag{15.2}$$

If $d(f_1, gh^{-i}) \leqslant d(f, g)$ for all i, then equality in (15.2) cannot hold unless $d(f_1, gh^{-i}) = d(f, g)$ for all i. In this case

$$d(f, g) = \tfrac{1}{3}\big(d(f_1, g) + d(f_1, gh^{-1}) + d(f_1, gh^{-2})\big) \leqslant \tfrac{1}{3},$$

since the elements g, gh^{-1}, gh^{-2} must be different because $o(h) \geqslant p_0 \geqslant 3$. This is case (a).

Remains the case $d(f_1, gh^{-i}) > d(f, g)$ for some i. Above we asserted that this implies $gh^{-i} = e$, that is,

$$d(f_1, e) > d(f, g). \tag{15.3}$$

Let

$$a = d(f_1, e), \quad b = d(f_1, g), \quad c = 1-a-b.$$

Let $t = o(g)$ and let j be the least natural number with $h^j = g$. Now we have

$$d(f_1, gh^{-i}) \begin{cases} = a & \text{if} \quad i = j+kt, \\ = b & \text{if} \quad i = kt, \\ \leqslant c & \text{otherwise}. \end{cases}$$

Substituting into (15.2) we get

$$d(f, g) \leqslant (1-q_0^{-1}) \left[\frac{b}{1-q_0^{-t}} + \frac{aq_0^{-j}}{1-q_0^{-t}} + c\left(\frac{1}{1-q_0^{-1}} - \frac{1+q_0^{-j}}{1-q_0^{-t}}\right)\right]. \tag{15.4}$$

Using the conditions

$$a+b+c = 1, \quad a > d(f, g) \geqslant b$$

from (15.4) we get (RC) $d(f, g) \leqslant 1/3$, that is again we got the possibility (a). ∎

Proof of Lemma (14.8). Let

$$f_0(p) = \begin{cases} f(p) & \text{if} \quad p \neq q_0, \\ e & \text{if} \quad p = q_0. \end{cases}$$

We have, with the notation $q_0^{-1} = x$,

$$d(f, g) = (1-x) \sum_{i=0}^{\infty} x^i d(f_0, g^{1-i}). \tag{15.5}$$

Now, if

$$d(f_0, e) + 2d(f_0, g) \leqslant 1.01,$$

then (RC) $d(f, g) \leqslant 0.34$ and we are ready. Suppose

$$d(f_0, e) + 2d(f_0, g) > 1.01.$$

If $h \neq e, g$, then

$$d(f_0, e) + d(f_0, g) + d(f_0, h) \leqslant 1,$$

so

$$d(f_0, h) < d(f_0, g) - 0.01.$$

If $d(f_0, e) \leqslant d(f, g)$ were, then, since $d(f_0, h) < d(f, g)$ for $h \neq e$, (15.5) would imply $d(f, g) < d(f, g)$. So we must have

$$d(f_0, e) > d(f, g) > d(f_0, g).$$

Summing up, for $h \neq e, g$ we got

$$d(f_0, e) > d(f_0, g) > d(f_0, h) + 0.01. \tag{15.6}$$

Now choose a finite set of primes Q such that

$$q_2 = \min_{q \in Q} q > \max_{p \in P} p, \qquad \sum_{q \in Q} \frac{1}{q} = y, \qquad |y-x| < \delta.$$

Let

$$f_1(p) = \begin{cases} g & \text{if} \quad p \in P \cup Q \setminus \{q_0\}, \\ e & \text{otherwise}. \end{cases}$$

We have, by a (RC), similar to Lemma (14.1),

$$\begin{aligned} d(f_1, g) &= e^{-y} \sum_{i=0}^{\infty} d(f_0, g^{1-i}) \frac{y^i}{i!} + O(q_2^{-1}) \\ &= e^{-x} \sum_{i=0}^{\infty} d(f_0, g^{1-i}) \frac{x^i}{i!} + O(q_2^{-1} + \delta). \end{aligned}$$

Since $q_2^{-1} + \delta$ can be made arbitrarily small with a suitable Q, it is sufficient to prove

$$T = \sum_{i=0}^{\infty} d(f_0, g^{1-i}) \underbrace{\left(\frac{x^i e^{-x}}{i!} - (1-x)x^i\right)}_{t_i} > 10^{-3}x^2. \tag{15.7}$$

We have (RC)

$$(15.8)\quad t_0, t_1 > 0, \quad t_n < 0, n = 2, 3, \ldots, \quad \sum_{i=0}^{\infty} t_i = 0,$$
$$t_0 + t_2 + t_3 + t_5 \leqslant 0.$$

We shall estimate using (15.6). In case $i = 4$ and $i \geqslant 6$ we use

$$d(f_0, g^{1-i}) \leqslant d(f_0, e).$$

In case $i = 2, 3, 5$ we have $g^{1-i} \neq e$, since

$$o(g) = p_0 \nmid 1, 2, 4.$$

Furthermore in case $i = 2$ we have $g^{1-i} = g^{-1} \neq g$. So from (15.5), (15.6) and (15.7)

$$T \geqslant t_0 d(f_0, g) + t_1 d(f_0, e) + t_2\big(d(f_0, g) - 0.01\big) + \\ + (t_3 + t_5) d(f_0, g) + \sum_{\substack{i=4 \\ i \neq 5}}^{\infty} t_i d(f_0, e)$$
$$= (t_0 + t_2 + t_3 + t_5)\big(d(f_0, g) - d(f_0, e)\big) - 0.01 t_2$$
$$\geqslant -0.01\, t_2 = 0.01\, x^2 \left(1 - x - \frac{e^{-x}}{2}\right) > 10^{-3} x^2,$$

i nce $x \leqslant 1/2$. ■

At last I propose one more problem: to find a proof of the half lenght.

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