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PURDUE UNIVERSITY

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# Cozero and Baire maps on products of uniform spaces

by

**Gloria J. Tashjian** (Hamilton, N. Y.)

**Abstract.** The main result is this: THEOREM. *A uniform Baire function from a product of uniform spaces into a metric space depends on countably many coordinates.* The proof uses a set-theoretic result: let $X$ be a cartesian product and let $\Sigma$ denote the collection of subsets of $X$ which depend on countably many coordinates. LEMMA. *If $\mathfrak{U}$ is a family of subsets of $X$ such that $\bigcup \mathfrak{V} \in \Sigma$ whenever $\mathfrak{V} \subset \mathfrak{U}$, then $\mathfrak{U}$ depends on countably many coordinates.* Corollaries describe the cozero and Baire-fine coreflections of a product of uniform spaces in terms of its countable subproducts, and it follows in particular that metric-fine coreflections of products of metric spaces and measurable coreflections of products of complete metric spaces are proximally fine. (The various classes of uniform spaces mentioned above are discussed in recent papers of A. W. Hager, Z. Frolík, and M. D. Rice.)

In this paper we study cozero and Baire-measurable functions defined on uniform products and the cozero-fine and Baire-fine uniformities derived from these functions. The basic result is that these mappings, with metric range, depend on countably many coordinates. We then prove that the cozero-fine, Baire-fine, and proximally-fine coreflections of a product of uniform spaces are generated by the cozero maps, Baire maps, or proximally continuous maps when the coreflections on each countable subproduct are so generated.

**1. Basic result.** We begin with some definitions. Let $uX$ be a uniform space where $\mathcal{U}$ denotes a family of covers of $X$ satisfying the usual axioms for a uniformity, as in [11]. A *cozero set* in $uX$ is a set of the form $\{x \in X: f(x) \neq 0\}$ for some uniformly continuous real-valued function $f: uX \to R$. Let $\mathrm{Coz}(uX)$ denote the family of all cozero sets in $uX$, and let Baire $(uX)$ be the $\sigma$-algebra on the set $X$ generated by $\mathrm{Coz}(uX)$. Baire $(uX)$ is classified in the same way that the Borel sets of a metric space are classified, replacing open set by cozero set, in [5], [9], or [13]. Then, a family $\mathcal{U}$ of Baire sets is said to be of bounded class if there exists an ordinal $\alpha < w_1$ such that each element of $\mathcal{U}$ belongs to additive or multiplicative Baire class $\leqslant \alpha$. Given uniform spaces $uX$ and $vY$, a function $f: uX \to vY$ is a cozero or Baire map if $f^{-1}(U)$ is a cozero or Baire set in $uX$ for every cozero or Baire set $U$ in $vY$.

Now let $X = \prod_{a \in A} X_a$ be a cartesian product of sets. A function $f: X \to Y$ to another set $Y$ depends on the index set $I \subseteq A$ if $f(x) = f(y)$ whenever $\pi_I(x) = \pi_I(y)$.

(Here, $\pi_I\colon X\to\prod_{a\in I} X_a$ denotes the projection mapping to the subproduct $\prod_{a\in I} X_a$, which we sometimes denote by $X_I$. Similarly for a point $x$ of $X$ we may write $x_I$ for $\pi_I(x)$.) When $I$ is countable, we say that the function $f$ depends on countably many coordinates.

A subset $U$ of $X$ depends on $I$ if $\pi_I^{-1}(\pi_I(U)) = U$, i.e., if $x\in U$ and $x_I = y_I$, then $y\in U$. A collection $\mathcal{U}$ of subsets of $X$ depends on $I$ if each element of $\mathcal{U}$ depends on $I$. If $\mathcal{U}$ is a family of subsets of $X$, there exists a smallest cardinal $\gamma$ such that $\mathcal{U}$ depends on a set of power $\gamma$. We denote this cardinal by $\gamma(\mathcal{U})$. Similarly, we write $\gamma(f)$ for the smallest cardinal $\gamma$ such that the function $f$ depends on a set of power $\gamma$.

With these definitions, it is easy to prove the following:

(i) *A cozero set in a uniform product depends on countably many coordinates.* (Use the theorem of Vidossich in [17] which says that a uniformly continuous map from a product into a metric space depends on countably many coordinates.) *By transfinite induction, Baire sets in a uniform product depend on countably many coordinates.*

(ii) *Suppose $X=\prod_{a\in A} X_a$ is a product of topological spaces $X_a$ and $U$ is an open subset of $X$. Let $J_a = A-\{a\}$, for each $a\in A$. Let $r(U) = \{a\in A\colon \pi_{J_a}^{-1}(\pi_{J_a}(U)) \neq U\}$. This is called the restriction set of $U$. Then $U$ depends on $r(U)$.*

Proof. Write $U$ as the union of basic open subsets of the product topology: $U=\bigcup_{s\in S} B_s$, where each set $B_s$ has a finite restriction set $r(B_s)$; let $I = r(U)$. Claim $\pi_I^{-1}(\pi_I(B_s))\subseteq U$, for every $s\in S$; to prove this, we write $r(B_s)-I = \{a_1, \ldots, a_n\}$, and use induction on this finite set. Let $R_1 = r(B_s)-\{a_1\}$. Then $\pi_{R_1}^{-1}(\pi_{R_1}(B_s))\subseteq U$: let $y\in \pi_{R_1}^{-1}(\pi_{R_1}(B_s))$. Then $\exists x\in B_s$ such that $y_a = x_a \forall a\in R_1$. Define a point $z$ in $X$ by $z_a = x_a \forall a\in r(B_s)$ and $z_a = y_a \forall a\notin r(B_s)$. Then $z\in B_s\subseteq U$; also $z_a = y_a \forall a\neq a_1$. Since $a_1\notin r(U)$, this means $y\in U$. By induction, then, we may eliminate all indices in $r(B_s)-I$, so that $\pi_I^{-1}(\pi_I(B_s))\subseteq U$. So $U$ can be expressed as the union of basic open sets, each having restriction set contained in $r(U) = I$. Hence, $\pi_I^{-1}(\pi_I(U)) = U$, i.e., $U$ depends on $I$.

Definition. If $X=\prod_{a\in A} X_a$ is a product set, let $\Sigma(X)$ denote the family of all subsets of $X$ which depend on countably many coordinates. Then it is easy to see that $\Sigma(X)$ is a $\sigma$-algebra on the set $X$. We say that a family $\mathcal{U}$ of subsets of $X$ is a *completely additive* (c.a.) *$\Sigma(X)$ family* if $\mathcal{U}\subseteq\Sigma(X)$ and the union of each subfamily of $\mathcal{U}$ belongs to $\Sigma(X)$.

We are now ready to state the basic result:

Lemma 1. *Let $X=\prod_{a\in A} X_a$ be a product of sets. A disjoint completely additive $\Sigma(X)$-family depends on countably many coordinates.*

Proof. Let $\mathcal{U}$ be a disjoint completely additive $\Sigma(X)$-family, and let $\gamma(\mathcal{U})$ denote the minimal cardinality of all index sets on which $\mathcal{U}$ depends.



Suppose that $\gamma(\mathcal{U})>\aleph_0$. Then we can define by transfinite induction two sequences of sets,

$$A_0\subseteq A_1\subseteq \ldots \subseteq A_i\subseteq \ldots \subseteq A$$

and

$$U_0, U_1, \ldots, U_i, \ldots \in\mathcal{U},$$

each of length $\omega_1$, and two sequences of points of $X$, $\{x_i\colon i<\omega_1\}$ and $\{y_i\colon i<\omega_1\}$, such that the following conditions hold for each $i<\omega_1$:

(a) $|A_i| = \aleph_0$;

(b) $U_i$ depends on $A_i$, but $U_i$ does not depend on the set $B_i = \bigcup_{j<i} A_j$;

(c) $x_i\in U_i$, $y_i\notin \bigcup_{j\leqslant i} U_j$, and $\pi_{B_i}(x_i) = \pi_{B_i}(y_i)$;

(d) $U_i\cap\{y_j\colon j<i\} = \emptyset$.

To do this, let $i<\omega_1$ and suppose that the sets $A_j$, $U_j$ and the points $x_j$, $y_j$ are defined for all $j<i$ so that conditions (a)-(d) hold for each $j$. Since $\mathcal{U}$ is a disjoint family and $\{y_j\colon j<i\}$ is countable, there are at most countably many elements of $\mathcal{U}$ which contain some $y_j$ for $j<i$. However, there are uncountably many elements of $\mathcal{U}$ which do not depend on $B_i$, since $\gamma(\mathcal{U})>\aleph_0$. Hence, there exists $U_i\in\mathcal{U}$ such that $U_i\cap\{y_j\colon j<i\} = \emptyset$ and $U_i$ does not depend on $B_i$. Then we may choose $x_i\in U_i$ and $y_i\notin U_i$ such that $\pi_{B_i}(x_i) = \pi_{B_i}(y_i)$. Note that $y_i\notin U_j$ for any $j<i$, since $x_i\notin U_j$ and $U_j$ depends on $B_i$. Finally, let $I$ be a countable set on which $U_i$ depends, and let $A_i = I\cup B_i$. This completes the induction, so that conditions (a)-(d) hold.

Now let $U=\bigcup_{i<\omega_1} U_i$. Then $U\in\Sigma(X)$ by complete additivity of $\mathcal{U}$, so it depends on some countable set $I\subseteq A$. But $U$ also depends on the set $\bigcup_{i<\omega_1} A_i$, so $U$ depends on $I\cap\bigcup_{i<\omega_1} A_i$. Hence there exists $i<\omega_1$ such that $U$ depends on $A_i$. Note that conditions (c) and (d) for every $i<\omega_1$ imply that $U\cap\{y_i\colon i<\omega_1\} = \emptyset$, so in particular we have points $x_{i+1}\in U$ and $y_{i+1}\notin U$ such that $\pi_{B_{i+1}}(x_{i+1}) = \pi_{B_{i+1}}(y_{i+1})$. But $B_{i+1} = A_i$, so $U$ does not depend on $A_i$, yielding a contradiction.

Remark. The above lemma, whose proof was suggested to me by the referee's comments, generalizes and simplifies an earlier version. The proof actually intended by the referee uses a recent set-theoretic result of D. Preiss [14, Lemma 1 p. 342] which we present below:

Theorem (Preiss). *Let $a$ be a regular ordinal, let $A$ be a set and let $\Psi\colon \exp(A)\to\alpha$ be a function. Suppose that there is a function $f\colon \alpha\times\alpha\to\alpha$ such that $\Psi(A_1\cap A_2)\leqslant f(\Psi(A_1), \Psi(A_2))$ for all $A_1, A_2$ in $\exp(A)$. Then there exists $i<\alpha$ such that $\Psi(\{a\})\leqslant i$ for each $a\in A$.*

To use this result we suppose $\gamma(\mathcal{U})>\aleph_0$, and we define two sequences of sets, $A_0\subseteq A_1\subseteq \ldots A_i\subseteq \ldots A$ and $U_0, U_1, \ldots, U_i, \ldots\in\mathcal{U}$, of length $\omega_1$, such that $|A_i|\leqslant\aleph_0$ and $U_i$ depends on $A_i$ but not on $\bigcup_{j<i} A_j$. For any $E\subseteq\omega_1$ the set $U_E = \bigcup\{U_i\colon i\in E\}$

depends on some countable set $I$, and it also depends on $\bigcup\{A_i: i<\omega_1\}$. Hence, there exists $i<\omega_1$ such that $U_E$ depends on $A_i$. Thus we may define a function $\Psi$: $\exp(\omega_1)\to\omega_1$ by $\Psi(E) = \min\{i: U_E \text{ depends on } A_i\}$. Now if $E_1, E_2\subseteq\omega_1$, then $\Psi(E_1\cap E_2)\leqslant\max\{\Psi(E_1), \Psi(E_2)\}$. Hence, by the theorem of Preiss there exists $i<\omega_1$ such that $\Psi(\{j\})\leqslant i$ for all $j<\omega_1$. This implies, for $j = i+1$, that $U_{i+1}$ depends on $A_i$, a contradiction.

As we noted earlier, the family $\Sigma(X)$ is a $\sigma$-algebra on the set $X$; hence we may define $\Sigma(X)$-measurable functions from $X$ into a metric space $M$ to be those $f$ such that $f^{-1}(U)\in\Sigma(X)$ for every open set $U\subseteq M$. Then we have the following result:

PROPOSITION 1. *Let* $X=\prod_{a\in A}X_a$ *be a product of sets. A map* $f$: $X\to M$, *where* $M$ *is metric, is* $\Sigma(X)$*-measurable iff it depends on countably many coordinates.*

Proof. Suppose $f$ is $\Sigma(X)$-measurable. The metric space $M$ has a countable base of uniform covers $\{\mathscr{S}_n\}$, where $\mathscr{S}_n$ is the family of spheres of radius $1/n$. Each cover $\mathscr{S}_n$ has a $\sigma$-uniformly discrete open refinement $\mathscr{V}_n$, by Stone's theorem. Now write $\mathscr{V}_n=\bigcup_{m\in N}\mathscr{V}_{nm}$, where $\mathscr{V}_{nm}$ is a discrete family of sets in $M$. Then $f^{-1}(\mathscr{V}_{nm})$ is a disjoint c.a. $\Sigma(X)$ family in $X$, so it depends on some countable set $I_{nm}$, by Lemma 1. Let $I=\bigcup_{n,m}I_{nm}$. Then $f$ depends on $I$.

Conversely, if $f$ depends on the countable set $I$, then $f^{-1}(U)$ depends on $I$ for every subset $U$ of $M$, so $f$ is $\Sigma(X)$-measurable.

**2. Application to Baire-fine and cozero-fine spaces.** We consider Baire and cozero sets in products of uniform spaces. A completely additive Baire (cozero) family is a family of Baire (cozero) sets such that the union of every subfamily is also a Baire (cozero) set. Such families are all completely additive $\Sigma(X)$ families, so that we have immediately the following result:

PROPOSITION 2. *Let* $uX$ *be a product of uniform spaces, where* $u$ *denotes the product uniformity. If* $\mathscr{U}$ *is a completely additive disjoint Baire family in* $uX$, *then* $\mathscr{U}$ *depends on countably many coordinates.*

We may rephrase Proposition 2 in terms of Baire and cozero mappings on a product. If $X=\prod_{a\in A}X_a$ is a product of topological spaces, define a uniformity $\varrho$ on $X$ to have base $\mathscr{B}=\{\mathscr{U}: \mathscr{U}$ is an open normal cover of $X$ which depends on $\aleph_0$ coordinates$\}$. Note that $\varrho$ does not change the topology on $X$.

PROPOSITION 3. *Let* $uX$ *be a product of uniform spaces, and let* $\delta$ *be any uniformity on* $X$ *such that* $u\subseteq\delta\subseteq\varrho$.

(a) *If* $f$: $\delta X\to vY$ *is a Baire map to a uniform space* $vY$ *and if* $\mathscr{V}\in v$, *then* $f^{-1}(\mathscr{V})$ *is refined by a completely additive Baire partition of* $\delta X$ *which depends on countably many coordinates.*



(b) *If* $f$: $\delta X\to vY$ *is a cozero map and* $\mathscr{V}\in v$, *then* $f^{-1}(\mathscr{V})$ *is refined by a completely additive cozero cover of* $\delta X$ *which depends on* $\aleph_0$ *coordinates and which starts a normal sequence of such covers.*

Proof. Since $\delta\subseteq\varrho$, $\delta X$ has a basis consisting of covers which depend on $\aleph_0$ coordinates, so any cozero or Baire set in $\delta X$ must depend on $\aleph_0$ coordinates.

(a) Given $\mathscr{V}\in v$, there exists a c.a. Baire partition $\mathscr{W}$ of $vY$ which refines $\mathscr{V}$ (as in [7] Lemma 3.3). Then $f^{-1}(\mathscr{W})$ is a c.a. Baire partition of $\delta X$ which refines $f^{-1}(\mathscr{V})$, and by Proposition 2, $f^{-1}(\mathscr{W})$ depends on $\aleph_0$ coordinates.

(b) Given $\mathscr{V}\in v$, there exists a $\sigma$-uniformly discrete c.a.-cozero cover $\mathscr{W}$ of $vY$ which star-refines $\mathscr{V}$, by Stone's theorem. Write $\mathscr{W}=\bigcup_{n\in N}\mathscr{W}_n$, where each $\mathscr{W}_n$ is uniformly discrete. Then $f^{-1}(\mathscr{W})=\bigcup_{n\in N}f^{-1}(\mathscr{W}_n)$, and each family $f^{-1}(\mathscr{W}_n)$ is a disjoint c.a. cozero family in $\delta X$. By Proposition 2, each $f^{-1}(\mathscr{W}_n)$ depends on $\aleph_0$ coordinates, so $f^{-1}(\mathscr{W})$ depends on $\aleph_0$ coordinates. Clearly, $f^{-1}(\mathscr{W})$ starts a normal sequence of such covers.

We now consider the cozero-fine and Baire-fine coreflections for products of uniform spaces. A uniform space $uX$ is cozero-fine, Baire-fine, or proximally-fine ($p$-fine) if every cozero-map, Baire map, or proximally continuous map ($p$-map) $f$ on $uX$ is uniformly continuous. The collection of all cozero-fine, Baire-fine, or $p$-fine spaces forms a coreflective subcategory of uniform spaces. Let $\Phi_{\mathrm{coz}}$, $\Phi_{\mathrm{Ba}}$, and $\Phi_p$ denote these coreflective functors; we write $\Phi(uX)$ for the coreflection of a uniform space $uX$.

These coreflections may be described as follows; given a uniform space $uX$, let $\bar{u}^{\mathrm{coz}}X$ denote the uniformity on $X$ projectively generated by the family of all cozero maps on $uX$. Repeat for $\bar{u}X$: let $\bar{\bar{u}}^{\mathrm{coz}}X$ be the set $X$ with the uniformity generated by all cozero maps on $\bar{u}^{\mathrm{coz}}X$. Continue this operation inductively, taking suprema at limit ordinals. A limit uniformity is obtained, and this is clearly the cozero-fine coreflection of $uX$. The Baire-fine and $p$-fine coreflections are obtained in a similar way, replacing cozero maps by Baire or $p$-maps. (We write $\bar{u}^{\mathrm{Ba}}X$, $\bar{u}^{\mathrm{P}}X$ for the intermediate uniformities, and when there is no confusion about the coreflection being discussed, we write $\bar{u}X$ or even $\bar{X}$.) (These facts are all proved by Hager and Frolík in [2] and [7].)

It is not known whether $\Phi(uX)=\bar{u}X$ always holds, for any of the three coreflections. In the case of cozero-fine spaces, Z. Frolík has given a description of the coreflection in [2]: it is asserted that $\Phi_{\mathrm{coz}}(uX)$, for any space $uX$, is projectively generated by all mappings $f$: $X\to M$ to metric spaces $M$ which satisfy the following:

1. $f=h\circ g$, where $g: X\to\prod_{a\in A}S_a$, $S_a$ is metric $\forall a$, and $\pi_a\circ g$ is a cozero map $\forall a$.

2. $h$: $g(X)\to M$ is a cozero map. (Note that any cozero map on $\bar{X}$ is such a generating map.) In showing that the space so generated is cozero-fine, it is claimed that the reduced product of two generating maps is a generating map. This statement implies that the reduced product of two cozero maps on $\bar{X}$ is a cozero map on $\bar{X}$: let $f_1, f_2$: $\bar{X}\to M_1, M_2$ be cozero maps to metric spaces $M_1, M_2$.

Now $\bar{X}$ is generated by all cozero maps on $X$, so the evaluation map $e\colon \bar{X}\to\prod_{a\in A} S_a$ of all cozero maps on $X$ is an embedding of $\bar{X}$ into $\prod S_a$, (where $\{S_a\colon a\in A\}$ is the collection of all cozero images of $X$). Then the mappings $f_i\circ e\colon X\to M_i$ are generating maps, for $i=1,2$. (We identify $\bar{X}$ with the image of $X$ under $e$, so $\bar{X}\subseteq\prod_{a\in A} S_a$). Hence the reduced product $(f_1\circ e)\times(f_2\circ e)\colon X\to M_1\times M_2$ is a generating map. Note that $(f_1\circ e)\times(f_2\circ e)=(f_1\times f_2)\circ e$.

$$\begin{tikzcd}
 & \prod S_a & \\
 X \arrow[r, "e"] \arrow[dr, "g"'] & \bar{X} \arrow[u, phantom, "\cup"] \arrow[r, "f_1\times f_2"] & M_1\times M_2 \\
 & g(X) \arrow[ur, "h"'] & \\
 & \prod R_i \arrow[u, phantom, "\cap"] &
\end{tikzcd}$$

By definition of generating map, there exists a product of metric spaces $\prod_{i\in I} R_i$ and maps $g\colon X\to\prod R_i$, $h\colon g(X)\to M_1\times M_2$ satisfying:

(1) $\pi_i\circ g$ is a cozero map on $X$ $\forall i\in I$;

(2) $h$ is a cozero map on $g(X)$,

(3) $h\circ g=(f_1\times f_2)\circ e$.

For each $i\in I$, there exists $a_i\in A$ such that $\pi_{a_i}\circ e=\pi_i\circ g$, since $e$ is the evaluation of all cozero maps on $X$. Therefore, if we let $B=\{a_i\colon i\in I\}$ and $C=A-B$, we have $e=g\times j$ for some function $j\colon X\to\prod_{a\in C} S_a$. Hence, the map $f_1\times f_2$ depends on the index set $B$. Let $U\in\operatorname{coz}(M_1\times M_2)$, and let $V=(f_1\times f_2)^{-1}(U)$. Then $\pi_B(V)=h^{-1}(U)$, where $\pi_B(V)\subseteq\prod_{a\in B} S_a$, which equals $\prod_{i\in I} R_i$; $\pi_B$ is defined on $\bar{X}$. Hence $\pi_B(V)$ is a cozero set since $h$ is a cozero map, so $V=\pi_B^{-1}(\pi_B(V))$ is a cozero set in $\bar{X}$. Hence, the reduced product $f_1\times f_2$ of cozero maps $f_1$ and $f_2$ is a cozero map on $\bar{X}$.

Now, using Proposition 1.5 of [7], this implies that $\bar{X}$ is cozero-fine. This shows that for any space $X$, if the reduced product of two generating maps as defined in [2] is a generating map, then $\bar{X}$ is cozero-fine.

We will show that if $uX$ is a product of metric spaces, then $\bar{u}X$ is cozero-fine.

DEFINITION. Let $uX$ be a product of uniform spaces. Define two uniformities $v$, $\sigma$ on $X$ as follows:

(i) Let $v$ have subbase $\{\mathcal{U}\colon \mathcal{U}$ is a c.a. cozero cover of $uX$, $\gamma(\mathcal{U})=\aleph_0$, and $\mathcal{U}$ starts a normal sequence of such covers$\}$. (Note that $u\subseteq v\subseteq\varrho$.)

(ii) Let $\sigma$ have subbase $\{\mathcal{U}\colon \mathcal{U}$ is a c.a. Baire partition of $uX$ and $\gamma(\mathcal{U})=\aleph_0\}$.

COROLLARY 1. *Let $uX$ be a product of uniform spaces. Let $\bar{u}$ be the uniformity on $X$ generated by all cozero maps on $uX$. Then:*

(a) $\bar{u}=v$.



(b) *If $\operatorname{coz}(\bar{X}_I)=\operatorname{coz}(X_I)$ for every countable subproduct $X_I$ of $X$, then* $\operatorname{coz}(uX)=\operatorname{coz}(\bar{u}X)$.

(c) *If $\operatorname{coz}(uX)=\operatorname{coz}(\bar{u}X)$, then $\bar{u}X$ is cozero-fine.*

Proof. (a) By Proposition 3.2 of [7], $\bar{u}X$ is generated by all c.a. cozero covers of $uX$ which start a normal sequence of such covers. Hence $v\subseteq\bar{u}$. Also, by Proposition 3(b) any cozero map on $uX$ is uniformly continuous on $vX$. Hence $\bar{u}\subseteq v$.

(b) Suppose $\operatorname{coz}(\bar{X}_I)=\operatorname{coz}(X_I)$ $\forall$ countable $I\subset A$. Let $f\colon \bar{u}X\to R$ be uniformly continuous, and let $U=\operatorname{coz}(f)$. To show that $U$ is a cozero set in $uX$, we need the following fact:

LEMMA 2. *Let $uX=\prod_{a\in A} X_a$ be a product of uniform spaces. If $U$ is a cozero set in $uX$ which depends on the countable set $I$, then $\pi_I(U)$ is a cozero set in $X_I$.*

Proof. Suppose $f\colon uX\to R$ is u.c. and $U=\operatorname{coz}(f)$. There exists a countable set $J\subseteq A$ such that $f$ depends on $J$, and $I\subseteq J$. Let $\mathcal{S}_n$ be a basic cover of $R$. Choose finitely many indices $\alpha_1,\dots,\alpha_k$ from $J$ and covers $\mathcal{V}_i^n$ in $X_{a_i}$, $i\leqslant k$, such that $\bigwedge_{i=1}^{k}\pi_{a_i}^{-1}(\mathcal{V}_i)<f^{-1}(\mathcal{S}_n)$. If we do this for each $n\in N$, we obtain, for each $a\in J$, at most countably many covers $\mathcal{V}_i^n$ in $X_a$ which are required to make $f$ uniformly continuous. For each $a\in J$, let $Y_a$ be the set $X_a$ with the uniformity generated by these countably many covers. Then $Y_a$ is a pseudometric space for each $a\in J$. Let $Y=\prod_{a\in A-J} X_a\times\prod_{a\in J} Y_a$. Then $f\colon Y\to R$ is uniformly continuous. Also, $\pi_I(U)$ is an open set in $\prod_{a\in I} Y_a$, which is a pseudometric space; hence $\pi_I(U)$ is a cozero set in that space. Since $\prod_{a\in I} Y_a$ is coarser than $\prod_{a\in I} X_a$, then $\pi_I(U)$ is a cozero set in $X_I$. This proves the lemma.

Returning to the proof of (b) let $\{\mathcal{S}_n\}$ be a countable base for $R$. For each $n\in N$ we may choose finitely many subbasic covers $\mathcal{U}_1^n,\dots,\mathcal{U}_k^n$ of $\bar{u}X$ such that $\bigwedge_{i=1}^{k}\mathcal{U}_i^n<f^{-1}(\mathcal{S}_n)$. Each cover $\mathcal{U}_1^n$ depends on a countable index set, so there exists a countable set $I$ such that all covers $\mathcal{U}_i^n$ depend on $I$. Then we have

$$\bigwedge_{i=1}^{k}\pi_I(\mathcal{U}_i^n)=\pi_I\Big(\bigwedge_{i=1}^{k}\mathcal{U}_i^n\Big)<\pi_I\big(f^{-1}(\mathcal{S}_n)\big).$$

By Lemma 2, $\pi_I(\mathcal{U}_i^n)$ is a c.a. cozero cover of $X_I$, so it belongs to $\bar{X}_I$. Hence $\pi_I f^{-1}(\mathcal{S}_n)$ is a uniform cover of $\bar{X}_I$ also, $\forall n\in N$. Now $U$ is a union of elements chosen from $\bigcup_{n\in N} f^{-1}(\mathcal{S}_n)$, so $\pi_I(U)$ is a union of elements from $\bigcup_{n\in N}\pi_I f^{-1}(\mathcal{S}_n)$. Therefore, $\pi_I(U)$ is a cozero set in $\bar{X}_I$. Now $\operatorname{coz}\bar{X}_I=\operatorname{coz}X_I$ by assumption, so $\pi_I(U)$ is a cozero set in $X_I$. It follows that $U$ is a cozero set in $uX$, since $U=\pi_I^{-1}(\pi_I(U))$. Hence, $\operatorname{coz}(\bar{u}X)\subseteq\operatorname{coz}(uX)$.

(c) If $\operatorname{coz}(uX)=\operatorname{coz}(\bar{u}X)$, then any cozero map on $\bar{u}X$ is a cozero map on $uX$, and therefore it is uniformly continuous on $\bar{u}X$. Hence $\bar{u}X$ is cozero-fine.

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Corollary 2. (a) *A product $uX$ of uniform spaces is cozero-fine if and only if each countable subproduct is cozero-fine.*

(b) *If $uX$ is a product of metric spaces, then $\Phi_{\mathrm{coz}}(uX) = \bar{u}X$.*

Proof. (a) If the product is cozero-fine, then the countable subproducts are cozero-fine since they are quotients of the product. Conversely, any metric-valued cozero map on $uX$ factors as a cozero map through a countable subproduct, using Proposition 3(b) and Lemma 2. Hence if each countable subproduct is cozero-fine, a cozero map on the whole product must be uniformly continuous.

(b) Any countable subproduct $X_I$ of $X$ is metric, so $\mathrm{coz}(X_I)$ is the family of all open subsets of $X_I$. Hence $\mathrm{coz}(\overline{X}_I) = \mathrm{coz}(X_I)$ since $\overline{X}_I$ has the same topology as $X_I$. By Corollary 2, $\mathrm{coz}(uX) = \mathrm{coz}(\bar{u}X)$, so $\bar{u}X$ is cozero-fine.

Remark. $\mathrm{Coz}(\overline{X}_I) = \mathrm{coz}(X_I)$ whenever $X_I$ has a Lindelöf topology. Hence by Corollary 1, $\Phi_{\mathrm{coz}}(uX) = \bar{u}X$ whenever $uX$ is a product of uniform spaces such that each countable subproduct is Lindelöf.

We now prove analogous statements about the $p$-fine and Baire-fine coreflections. If $uX$ is a uniform space, let $puX$ denote the set $X$ with the uniformity consisting of all finite uniform covers of $uX$.

Corollary 3. *Let $uX$ be a product of uniform spaces.*

(a) *If $p\overline{X}_I^p = pX_I$ for every countable $I$, then $p\bar{u}X = puX$.*

(b) *If $p\bar{u}X = puX$, then $\bar{u}X$ is $p$-fine.*

(c) *The product $uX$ is $p$-fine if and only if each countable subproduct is $p$-fine.*

Proof. (a) Let $\mathscr{U} \in p\bar{u}X$. We may assume that $\mathscr{U}$ is a finite cozero cover of $\bar{u}X$. By definition of $\bar{u}X$, there exist $p$-maps $f_1, \ldots, f_n$: $uX \to Y_i$, for $i \leqslant n$, and uniform covers $\mathscr{V}_i$ of $Y_i$ such that $\bigwedge_{i=1}^{n} f_i^{-1}(\mathscr{V}_i) < \mathscr{U}$. The maps $f_i$ are all cozero maps on $uX$, so by Proposition 3 each one depends on a countable set; choose a countable set $I$ such that all the $f_i$ depend on $I$ and also $\mathscr{U}$ depends on $I$. (We may do this since cozero sets in $\bar{u}X$ depend on $\aleph_0$ coordinates). Then

$$\pi_I\Big(\bigwedge_{i=1}^{n} f_i^{-1}(\mathscr{V}_i)\Big) = \bigwedge_{i=1}^{n} \pi_I\big(f_i^{-1}(\mathscr{V}_i)\big) < \pi_I(\mathscr{U}).$$

The maps $f_i$ depend on $I$, so they factor as $p$-maps through $X_I$; i.e., there exist $p$-maps $g_i$: $X_I \to Y_i$ such that $g_i \circ \pi_I = f_i \ \forall i \leqslant n$. Then $\pi_I(f_i^{-1}(\mathscr{V}_i)) = g_i^{-1}(\mathscr{V}_i)$, a uniform cover of $\overline{X}_I$. Hence $\pi_I(\mathscr{U}) \in p\overline{X}_I$ since $\pi_I(\mathscr{U})$ is finite. By the assumption $p\overline{X}_I = pX_I$, we have $\pi_I(\mathscr{U}) \in pX_I$. Then $\mathscr{U} \in puX$, since $\mathscr{U} = \pi_I^{-1}(\pi_I(\mathscr{U}))$. So, $p\bar{u}X = puX$.

(b) Trivial.

(c) Suppose each countable subproduct $X_I$ is $p$-fine. Let $f$: $uX \to Y$ be a $p$-map to a metric space $Y$. Then $f = g \circ \pi_I$ for some countable $I$ and $p$-map $g$: $X_I \to Y$. Since $g$ is uniformly continuous if $X_I$ is $p$-fine, $f$ is also uniformly continuous. Hence, $uX$ is $p$-fine.



Remark. M. Hušek has proved independently in [10]: a product of uniform spaces is $p$-fine if and only if each finite subproduct is $p$-fine; a proximally continuous map from a uniform product into a metric space depends on countably many coordinates; the statement of Corollary 4(a) below.

Definition. A uniform space $uX$ is metric-fine if each uniformly continuous map $f$: $uX \to M$ to a metric space $M$ is uniformly continuous with respect to the fine uniformity $\alpha$ on $M$. (Here $\alpha$ has base consisting of all open covers of $M$.) The metric-fine spaces form a coreflective subcategory of uniform spaces. (Metric-fine spaces were originally defined by Hager in [6], where separable metric-fine spaces are discussed. The non-separable theory is developed by Frolík in [4] and by Rice in [15].)

Corollary 4. (a) *A product of metric spaces is $p$-fine.*

(b) *The metric-fine coreflection of a product of metric spaces is $p$-fine.*

Proof. (a) All countable subproducts of $uX$ are metric. By a theorem of Švarc (in [11] p. 38) metric spaces are $p$-fine. Since all $p$-maps on $uX$ factor through a countable subproduct, they are then uniformly continuous on the whole product. Hence $uX$ is $p$-fine.

(b) It is easy to see that if $uX$ is a product of metric spaces, the metric-fine coreflection of $uX$ is $\varrho X$. We have shown that $\varrho X$ is cozero-fine (here $\varrho X = vX = \bar{u}X$) in Corollary 1. Hence, $\varrho X$ is $p$-fine, since all $p$-maps are cozero maps.

Corollary 5. *Let $uX$ be a product of uniform spaces.*

(a) *$\bar{u}^{\mathrm{Ba}}X = \sigma X$, and if $\mathrm{Ba}(X_I) = \mathrm{Ba}(\overline{X}_I^{\mathrm{Ba}})$, for every countable $I$, then $\mathrm{Ba}(uX) = \mathrm{Ba}(\bar{u}X)$.*

(b) *If $\mathrm{Ba}(uX) = \mathrm{Ba}(\bar{u}X)$, then $\bar{u}X$ is Baire-fine.*

Proof. (a) $\bar{u}$ has sub-base consisting of all completely additive Baire partitions of $uX$ which, by Proposition 2, all depend on $\aleph_0$ coordinates. Hence $\bar{u} = \sigma$.

Assume now that $\mathrm{Ba}(X_I) = \mathrm{Ba}(\overline{X}_I)$ for every countable subproduct $X_I$. To prove that $\mathrm{Ba}(uX) = \mathrm{Ba}(\bar{u}X)$, we need the following fact:

Lemma 3. *Let $uX$ be a product of uniform spaces. If $B$ is a Baire set in $X$ of class $\alpha$ which depends on the countable set $I$, the $\pi_I(B)$ is a Baire set in $X_I$ of class $\leqslant \alpha+1$.*

Proof. First, suppose $uX$ is a product of metric spaces. The Baire set $B$ originates from a countable number of cozero sets in $uX$, so we may choose a countable set $J \subseteq A$ such that all of these cozero sets depend on $J$, and $I \subseteq J$. Since $\pi_J$: $X \to X_J$ preserves unions and intersections of sets in $X$ which depend on $J$, $\pi_J(B)$ is a Baire set in $X_J$. Now let $K = J - I$. Fix a point $p$ in $X_k$ and let $C = \{p\} \times X_I$.

Since $X_K$ is metric, $\{p\}$ is a Baire set in $X_K$, so $C$ is a Baire set in $X_J$. Then $C \cap \pi_J(B)$ is a Baire set in $X_J$, where $C \cap \pi_J(B) = \{p\} \times \pi_I(B)$. Finally, we note that $\pi_1(B)$ is a Baire set in $X_I$, since it is the inverse image of the Baire set $\{p\} \times \pi_I(B)$ under the uniformly continuous map $f$: $X_I \to X_J$ defined by $f(x) = (p, x)$, (where $X_J = X_K \times X_I$).

Now if $uX$ is an arbitrary product of uniform spaces and if $B \in \mathrm{Ba}(uX)$ and $J$ is a countable index set as above, we may define pseudometric spaces $Y_a$ weaker than $X_a$, for each $a \in J$, such that $B$ is a Baire set in the product $\prod_{a \notin J} X_a \times \prod_{a \in J} Y_a$ (as in Lemma 2). Then $\pi_J(B)$ is a Baire set in the pseudometric space $Y = \prod_{a \in J} Y_a$. If we let $M, M_a$ be the metric identifications of $Y, Y_a$ respectively, $\forall a \in J$, then $M = \prod_{a \in J} M_a$. Let $f\colon Y \to M$ be the canonical map. Then $f(\pi_J(B))$ is a Baire set in the metric space $M$, and it depends on the index set $I$, so that by the first part it projects onto a Baire set in $M_I$. This implies that $\pi_I(B)$ is a Baire set in $Y_I$, hence it is a Baire set in the finer space $X_I$. This completes the lemma.

Now, to show $\mathrm{Ba}(uX) = \mathrm{Ba}(\bar{u}X)$, it suffices to show that $\mathrm{coz}(\bar{u}X) \subseteq \mathrm{Ba}(uX)$. Let $\mathrm{coz}(f) \in \mathrm{coz}(\bar{u}X)$. For each basic cover $\mathscr{I}_n$ of $R$ choose finitely many subbasic covers $\mathscr{U}_1^n, \dots, \mathscr{U}_K^n$ of $\bar{u}X$ such that $\bigwedge_{i=1}^{K} \mathscr{U}_i^n < f^{-1}(\mathscr{I}_n)$. Each cover $\mathscr{U}_i^n$ is a completely additive Baire partition of $uX$, so by Proposition 2 it depends on a countable set $I_{in}$. Let $I = \bigcup \{I_{in}\colon n \in N, i \leqslant K\}$. Then $I$ is countable and each cover $\mathscr{U}_{in}$ depends on $I$, so by Lemma 3 $\pi_I(\mathscr{U}_{in})$ is a c.a. Baire cover of $X_I$, i.e., a uniform cover of $X_I$. Now

$$\bigwedge_{i=1}^{K} \pi_I(\mathscr{U}_i^n) = \pi_I\Big(\bigwedge_{i=1}^{K} \mathscr{U}_i^n\Big) < \pi_I(f^{-1}(\mathscr{I}_n)) .$$

So, $\pi_I(f^{-1}(\mathscr{I}_n))$ is a uniform cover of $\bar{X}_I$. Then $\pi_I(\mathrm{coz}(f))$ is a cozero set in $X_I$, so by our assumption it is a Baire set in $X_I$. Therefore, $\mathrm{coz}(f) = \pi_I^{-1}\big(\pi_I(\mathrm{coz}(f))\big)$ is a Baire set in $uX$.

Definition. A uniform space $uX$ is *measurable* if every pointwise limit of uniformly continuous metric-valued functions is uniformly continuous. The measurable spaces form a coreflective subcategory of uniform spaces. If $uX$ is a uniform space, its measurable coreflection $m_* uX$ is generated by all $\sigma$-uniformly discrete c.a. Baire covers on $uX$. (Measurable spaces are discussed in [3], [5], and [15].)

Corollary 6. *Assume* GCH. *Let $uX$ be a product of uniform spaces such that $|Baire(X_I)| \leqslant c$ for every countable $I$. Then the measurable coreflection $m_* uX$ is Baire-fine.*

Proof. It is known that if $|\mathrm{Ba}(uX)| \leqslant c$, then each c.a. Baire partition of $uX$ is countable (Corollary 7.9 in [7]). Hence for each countable $I$, all c.a. Baire partitions of $X_I$ are countable. Then $m_* X_I = \bar{X}_I$. Now, $\mathrm{Ba}(uX) = \mathrm{Ba}(m_* uX)$ for any space $uX$, as in [3] or [16], so $\mathrm{Ba}(uX_I) = \mathrm{Ba}(\bar{X}_I)$ $\forall$ countable $I$. Hence $\bar{X}_I$ is Baire-fine. Now if $f\colon m_* uX \to Y$ is a Baire map, it is a Baire map on $uX$; using Proposition 3 and Lemma 3, there exists a countable set $I$ and a Baire map $g\colon X_I \to Y$ such that $f = g \circ \pi_I$. Now $g\colon m_* X_I \to Y$ is u.c. since $m_* X_I$ is Baire-fine, so $f\colon m_* uX \to Y$ is u.c. Hence $m_* uX$ is Baire-fine.

Remark. Corollary 6 applies to products of separable metric spaces since $|\mathrm{Ba}(X_I)| \leqslant c$ $\forall$ countable subproduct $I$.



**3. Further applications.** We now consider the recent result of D. Preiss in [14]:

(a) *Any completely additive disjoint system of Baire sets in a topological space is of bounded class.*

It is known that this is equivalent to the following three conditions, for complete metric spaces $\varrho M$:

(b) *Any Baire map $f\colon \varrho M \to Y$ to a metric space $Y$ is of bounded class* (*i.e., there exists $\alpha < w_1$ such that for any open subset $U$ of $Y$, $f^{-1}(U)$ is a Baire set of class $\alpha$*).

(c) *The measurable coreflection $m_* \varrho M$ is $p$-fine.*

(d) *The meet of two* c.a. *Baire covers is a* c.a. *Baire cover on $\varrho M$.*

Proof. The equivalence of (b) and (c) was proved by M. Rice in [14, II, Theorem 2.2, p. 6]. The equivalence of (c) and (d) was shown by M. Rice in [14, I, Theorem 2.2, p. 8].

(b)→(a) Let $\mathscr{U}$ be a c.a. disjoint Baire family in $\varrho M$, where $\mathscr{U} = \{U_a\colon a \in A\}$. Let $D$ be the discrete metric space of power $|A|$. Define $f\colon \varrho M \to D$ by $f(x) = 0$ if $x \notin \bigcup \mathscr{U}$ and $f(x) = a$ if $x \in U_a$. ($D = \{0\} \cup A$.) Then $f$ is a Baire map, so by (b) it is of bounded class. Hence $\exists \alpha < w_1$ such that $U_a = f^{-1}(\{a\})$ is a Baire set of class $\alpha$ $\forall a \in A$.

(a)→(d) Let $\mathscr{U}, \mathscr{V}$ be c.a. Baire covers in $\varrho M$, let $D, E$ be the discrete spaces of power $|\mathscr{U}|, |\mathscr{V}|$ respectively, and let $f, g\colon \varrho M \to D, E$ be maps which are constant on each element of $\mathscr{U}$ or $\mathscr{V}$. Now the meet $\mathscr{U} \wedge \mathscr{V}$ is a c.a. Baire cover if and only if the reduced product $f \times g\colon M \to D \times E$ is a Baire map. By a theorem of Hansell [8, Theorem 5, p. 163] it suffices to show that $f$ and $g$ are $\sigma$-discrete and of class $\alpha$, some $\alpha < w_1$. Again, we apply theorems of Hansell: $f$ and $g$ are of class $\alpha$ by Corollary 5, p. 159 in [8], and $\mathscr{U}, \mathscr{V}$ are $\sigma$-discretely decomposable by Theorem 2, p. 156 in [8], so it follows that $f$ and $g$ are $\sigma$-discrete.

This shows that conditions (a)-(d) are true for complete metric spaces $\varrho M$, and it is not hard to see that they hold for products of complete metric spaces also.

Proposition 4. *Let $uX = \prod_{a \in A} X_a$ be a product of complete metric spaces.*

(i) *Any Baire map $f\colon uX \to Y$ to a metric space $Y$ is of bounded class.*

(ii) *The measurable coreflection $m_* uX$ is $p$-fine.*

(iii) *The meet of two completely additive Baire covers on $uX$ is a* c.a. *Baire cover.*

Proof. (i) By Proposition 3(a), $f$ depends on a countable index set $I \subseteq A$. Let $g\colon X_I \to Y$ be the map such that $f = g \circ \pi_I$. Then $g$ is a Baire map on $X_I$, where $X_I$ is a complete metric space, so by condition (b) above $g$ is of bounded class. Hence, $f$ is of bounded class.

(ii) Let $f\colon m_* uX \to Y$ be a $p$-map, where $Y$ is metric. Then $f$ is a Baire map on $m_* uX$; by a theorem of M. Rice in [15], $\mathrm{Ba}(m_* uX) = \mathrm{Ba}(uX)$, so $f\colon uX \to Y$ is a Baire map. By Proposition 3, $f$ depends on a countable index set $I$, so there exists a Baire map $g\colon X_I \to Y$ such that $f = g \circ \pi_I$. We know that $m_* X_I$ is $p$-fine; hence, it is Baire-fine, by a theorem of A. W. Hager in [7]. Thus $g\colon m_* X_I \to Y$ is uniformly continuous, so $f\colon m_* X \to Y$ is uniformly continuous.

(iii) If $\mathscr{U}, \mathscr{V}$ are completely additive Baire covers on $uX$, there exists a countable index set $I$ on which $\mathscr{U}$ and $\mathscr{V}$ depend, so $\mathscr{U} = \pi_I^{-1}(\pi_I(\mathscr{U}))$ and $\mathscr{V} = (\pi_I^{-1}(\pi_I(\mathscr{V}))$. By Lemma 3, $\pi_I(\mathscr{U})$ and $\pi_I(\mathscr{V})$ are c.a. Baire covers on $X_I$, so by condition (d), $\pi_I(\mathscr{U}) \wedge \pi_I(\mathscr{V})$ is a c.a. Baire cover on $X_I$. Then

$$\mathscr{U} \wedge \mathscr{V} = \pi_I^{-1}(\pi_I(\mathscr{U}) \wedge \pi_I(\mathscr{V}))$$

is a c.a. Baire cover on $uX$.

Finally, we make some observations about other subclasses of uniform spaces. Let $\mathscr{F}$ be a class of functions each having domain and range in uniform spaces. Let $\mathscr{R}$ be a reflective subcategory of uniform spaces (i.e., $\mathscr{R}$ is closed under product and closed subspace formation). Define the class of uniform spaces $\mathscr{F}$-$\mathscr{R}$ as follows: $uX \in \mathscr{F}$-$\mathscr{R}$ iff each image of $uX$ under a function in $\mathscr{F}$ belongs to $\mathscr{R}$. For example, we may let $\mathscr{F}$ = cozero, Baire, or $p$-maps, and $\mathscr{R} = \mathscr{P}$ or $\mathscr{E}$. (Here $\mathscr{P}$ denotes the precompact uniform spaces, and $\mathscr{E}$ denotes the separable spaces, i.e. those uniform spaces having a basis of countable covers). Then, a space is in Coz-$\mathscr{E}$ or Ba-$\mathscr{E}$ iff each c.a. cozero cover or c.a. Baire partition has a countable subcover. These notions are discussed by Hager in [7].

It is known that Coz-$\mathscr{P}$ and $\mathscr{P}$-$\mathscr{P}$ are closed under arbitrary product formation (see [12] for $\mathscr{P}$-$\mathscr{P}$ and [7] for Coz-$\mathscr{P}$.) In addition, we have the following:

(i) *A product of uniform spaces is in* Coz-$\mathscr{E}$ *iff each countable subproduct is in* Coz-$\mathscr{E}$. *A product of separable metric spaces is in* Coz-$\mathscr{E}$.

(ii) *Assume* GCH. *A product of uniform spaces is in* Ba-$\mathscr{E}$ *iff each countable subproduct is. A product of separable metric spaces is in* Ba-$\mathscr{E}$.

Proof. (i) Let $uX$ be a product of uniform spaces. Assume that $uX \in$ Coz-$\mathscr{E}$. If $X_I$ is a countable subproduct and $f\colon X_I \to Y$ a cozero map with image $Y$, then $f \circ \pi_I\colon X \to Y$ is a cozero map, so $Y \in \mathscr{E}$. Hence $X_I \in$ Coz-$\mathscr{E}$. Conversely, suppose that $X_I \in$ Coz-$\mathscr{E}$ $\forall$ countable $I$. Let $f\colon uX \to Y$ be a cozero map. Then $\mathscr{F}$ factors through a countable subproduct as a cozero map, so $Y \in \mathscr{E}$. Hence $uX \in$ Coz-$\mathscr{E}$.

Suppose $uX$ is a product of separable metric spaces. Then each countable subproduct $X_I$ is a separable metric space, so it is topologically separable. Then any cozero (i.e. continuous) image $Y$ of $X_I$ is topologically separable, hence any compatible uniformity on $Y$ is in $\mathscr{E}$. So $X_I \in$ Coz-$\mathscr{E}$. By the first part, $uX \in$ Coz-$\mathscr{E}$.

(ii) The proof is similar to the proof for (i). Note that we must factor Baire maps through countable subproducts. For the second statement, we use the fact that, assuming the continuum hypothesis, any c.a. Baire family in a separable metric space is countable. (This fact also holds for any uniform product $uX$ whose countable subproducts satisfy $|\mathrm{Baire}(uX)| \leqslant c$.)

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COLGATE UNIVERSITY
Hamilton, New York

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