Weak contractibility and hyperspaces

by

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Abstract. Weak contractibility is studied for compact connected Hausdorff spaces and their hyperspaces. The two hyperspaces dealt with in this paper are the space of all nonempty closed subsets of a given compact connected Hausdorff space and the space of all nonempty closed connected subsets. These hyperspaces are known to be acyclic. M. Wojdyslawski has shown that the hyperspaces of any locally connected compact connected metric space are absolute retracts. An example is presented of a locally connected compact connected Hausdorff space that does not have weakly contractible hyperspaces. A weakly contractible compact connected Hausdorff space has weakly contractible hyperspaces.

1. Introduction. In this paper we examine some theorems of M. Wojdyslawski [24] and J. L. Kelley [9] and establish whether or not they generalize to nonmetric spaces. Some results in [23, Chapter XI] are extended to give an affirmative answer to a question of A. D. Wallace [20]. Part of the material presented here is summarized in [15] and is contained in the author's master's thesis at Dalhousie University under S. B. Nadler, Jr. The author is pleased to acknowledge the considerable assistance given by Professor S. B. Nadler, Jr., and Professor L. E. Ward, Jr., in the preparation of this paper.

2. Definitions and preliminary results. All spaces in this paper are nonempty Hausdorff spaces. A *continuum* is a compact connected space. A map is a continuous function.

An *arc* A is a nondegenerate continuum with a simple order "$\leqslant$" such that A has a first point c and a last point d and the topology on A is the order topology. Let $x, y \in A$, $x<y$. We use the notation:

$$[x, y] = \{a \in A\colon\ x\leqslant a\leqslant y\}\,, \quad]x, y[= \{a \in A\colon\ x<a<y\}\,.$$

Thus $[c, d] = A$. The notation $[x, x]$ stands for $\{x\}$. The notation (x, y) does not stand for an open interval, but for a point in $A\times A$.

A *simple closed curve* is a space obtained by identifying the endpoints of some arc. We note that an arc may also be defined as a nondegenerate continuum with exactly two non-cutpoints and a simple closed curve as a nondegenerate continuum Z such that the removal of any of distinct points disconnects Z. A separable arc is homeomorphic to $[0, 1]$, the unit interval in the real line R^1, and a separable simple closed curve is homeomorphic to S^1, the unit circle in the complex plane (see [6, pp. 54–55]).

Two maps $f, g\colon X \to Y$ are said to be *homotopic* (*weakly homotopic*) if there exist an arc (continuum) W, distinct points $c, d \in W$, and a map $h\colon X \times W \to Y$ such that, for each $x \in X$, $h(x, c) = f(x)$ and $h(x, d) = g(x)$. If W is arcwise connected (in particular, if W is an arc) we may restrict the map h to the product of X and an arc $A \subset W$ with endpoints c and d. Let $X \subset Y$, X nonempty. The subspace X is said to be (*weakly*) *contractible* in Y if there is a constant map $f\colon X \to Y$ which is (weakly) homotopic to the inclusion map $g\colon X \to Y$. The subspace Z is said to be a (*weak*) *deformation retract* of Y if there is a retraction $r\colon Y \to Z$ such that $f\colon Y \to Y$ (defined by $f(y) = r(y)$, for each $y \in Y$) is (weakly) homotopic to the identity map $g\colon Y \to Y$. Any space Y is a deformation retract of itself. A space Y is (weakly) contractible in itself if and only if there is a point $x \in Y$ such that $\{x\}$ is a (weak) deformation retract of Y.

Note that the definitions of "arc", "simple closed curve", "homotopic", and "contractible" given in this paper are nonstandard. Two maps $f, g\colon X \to Y$ are said to be $[0, 1]$-*homotopic* if there exists a map $h\colon X \times [0, 1] \to Y$ such that, for each $x \in X$, $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. We define "$[0, 1]$-contractible" in the natural way. Our "$[0, 1]$-homotopic" is the same as Whyburn's "homotopic" (see [23, p. 225] also [4, p. 315] and [6, p. 150]). Any nonmetrizable arc serves as an example of a contractible continuum which is not $[0, 1]$-contractible.

For any two spaces X and Y, Y^X denotes the space of all maps from X to Y, with the compact-open topology [4, p. 257]. If X is compact and Y is metrizable, then Y^X is metrizable (using the supremum metric, see [4, 8.2(3), p. 270]). Let $h\colon X \times W \to Y$ be an arbitrary function, where X is compact. Define $h'\colon W \to Y^X$ by $(h'(w))(x) = h(x, w)$, for all $w \in W$ and $x \in X$. Then h is continuous if and only if h' is continuous [4, Theorem 3.1, p. 261]. Thus a continuum X is weakly contractible if and only if there exists a continuum in X^X containing both the identity map and some constant map. Since the continuous image of an arc is arcwise connected (for a proof by J. K. Harris see [17]) we have the result that a nondegenerate continuum X is contractible if and only if there exists an arc in X^X with the identity map and some constant map as endpoints. If X is a contractible metric continuum, then X is $[0, 1]$-contractible (recall that X^X is metrizable).

G. T. Whyburn [23, Chapter XI] proves several results on unicoherence and property (b) for metric continua which easily generalize to arbitrary continua. Among these are the following. If a continuum X has property (b), then so does any retract of X. If a continuum X has property (b), then X is unicoherent. A continuum X has property (b) if and only if $(S^1)^X$ is arcwise connected. A continuum X has property (b) if and only if $(S^1)^X$ is connected.

THEOREM 1. *A weakly contractible continuum has property* (b).

Proof. Let X be a continuum which is weakly contractible to the point $p \in X$. Let $f \in (S^1)^X$. The map f is weakly homotopic to the constant map sending all of X to $f(p)$. Since any two constant maps in $(S^1)^X$ are weakly homotopic, we conclude that any two maps in $(S^1)^X$ are weakly homotopic. Thus $(S^1)^X$ is connected, and so X has property (b).

3. Semigroups and hyperspaces. A continuum is said to be *acyclic* if it has the Alexander cohomology groups of a point-space. Any weakly contractible continuum is acyclic [7, 1.3, p. 336]. If X is an acyclic continuum, then X also has the Čech cohomology groups of a point-space [18] and so $(S^1)^X$ is arcwise connected [1, Theorem 9.5, p. 230]. By results stated in Section 2, X has property (b) and is therefore unicoherent.

Let X be a space and $m\colon X \times X \to X$ an associative map. Then (X, m) is called a *topological semigroup*. It follows directly from the definitions that a compact connected topological semigroup with a right identity d and a right zero c is weakly contractible. A. D. Wallace obtained the result that a compact connected topological semigroup with a right identity and a zero is acyclic (and so is unicoherent) as a consequence [19, Corollary 1, p. 48] of a theorem on topological semigroups. In [20, P. 333] he asks if there is a proof of the fact that any compact connected topological semigroup with identity and zero is unicoherent using only set theoretic topology. Since only set theoretic topology is used in Section 2 of this paper, Theorem 1 provides an affirmative answer to Wallace's question.

Let X be an arcwise connected continuum such that for any map $f\colon X \to X$ which is homotopic to the identity map, $f(X) = X$. If (X, m) is a topological semigroup with identity it follows that $m(X \times \{c\}) = X$, for each $c \in X$, and so (X, m) is a group. Then (X, m) is a topological group [7, 2.3, p. 17].

Let Z be a simple closed curve which admits the structure of a topological semigroup with identity. We shall prove that Z is homeomorphic to S^1. By the remarks in the preceding paragraph, Z is a topological group. This fact also follows from [10, Theorem 3, p. 279]. Then Z is homogeneous and so is first countable (a proof of this statement can easily be adapted from the proof of [8, 4.3, p. 326]). But a first countable (Hausdorff) topological group is metrizable [5, Theorem 8.3, p. 70]. Thus Z is homeomorphic to S^1.

For any compact space X, $S(X)$ denotes the space of all nonempty closed subsets of X (with the finite topology [12]) and $C(X)$ denotes the subspace of $S(X)$ consisting of all nonempty closed connected subsets of X. The spaces $S(X)$ and $C(X)$ are compact, and are connected (locally connected) if X is connected (locally connected). The subspace $J_1(X) = \{\{x\}\colon x \in X\}$ is homeomorphic to X. It is known [12] that $J_1(S(X))$ is always a retract of $S(S(X))$.

A *semilattice* is a commutative idempotent semigroup. M. M. McWaters [11] proves that $(S(X), \cup)$ is a topological semilattice for any continuum X and shows that $S(X)$ and $C(X)$ are arcwise connected.

THEOREM 2. *Let X be a continuum. Then $S(X)$ and $C(X)$ are acyclic.*

Proof. By the remark preceding this theorem, $(S(X), \cup)$ is a compact connected topological semilattice and so is acyclic [21, Corollary 1, p. 103]. Since the union of two subcontinua of X may fail to be connected, $C(X)$ is not, in general, a subsemilattice of $(S(X), \cup)$. However, for any $B \in C(X)$, $\boldsymbol{B} = \{D \in C(X): D \supset B\}$ is a compact connected subsemilattice of $(S(X), \cup)$ and so is acyclic (indeed, $\boldsymbol{B}$ has an identity B and a zero X and so $\boldsymbol{B}$ is weakly contractible). We can now apply [22, Theorem 3.1, p. 149] to the partially ordered space $(C(X), \supset)$ to conclude that $C(X)$ is acyclic.

There is an alternate proof of Theorem 2. In [16] J. Segal proves that $C(X)$ acyclic, for any metric continuum X. Using results from [3], together with [14, Theorem VII. 3, p. 303], one can similarly establish that $S(X)$ and $C(X)$ are acyclic, for any continuum X. This fact was pointed out to the author in 1969 by S. B. Nadler, Jr., who independently proved some of the results in [3].

THEOREM 3. *Let Y be a compact space such that $J_1(Y)$ is a retract of $C(Y)$. Let X be a compact subspace of Y. If X is weakly contractible in Y, then X is contractible in Y.*

Proof. By hypothesis, there exist a continuum W, distinct points $c, d \in W$, a point $p \in Y$, and a map $f: X\times W \to Y$ such that, for each $x \in X$, $f(x, c) = p$ and $f(x, d) = x$. Define g: $C(X\times W) \to C(Y)$ by $g(B) = \{f(z): z \in B\}$, for each $B \in C(X\times W)$. The function g is continuous [12, Theorem 5. 10.1, p. 170]. Clearly $C(X)\times C(W)$ is homeomorphic to a subspace of $C(X\times W)$. Since $C(W)$ is arcwise connected [11], there exists an arc $\boldsymbol{A}$ contained in $C(W)$ with endpoints $\{c\}$ and $\{d\}$. Define h: $J_1(X)\times \boldsymbol{A} \to C(Y)$ by $h(\{x\}, B) = g(\{x\}\times B)$, for each $x \in X$ and each $B \in \boldsymbol{A}$. Thus $J_1(X)$ is contractible in $C(Y)$. But $J_1(X)$ is contained in $J_1(Y)$, which is a retract of $C(Y)$. It follows that $J_1(X)$ is contractible in $J_1(Y)$.

Let X be a continuum. Then $S(X)$ is said to be *contractible using an order preserving homotopy* if there exist an arc $[c, d]$ and a map h: $S(X)\times[c, d] \to S(X)$ such that, for each $B \in S(X)$, $h(B, c) = X$ and $h(B, d) = B$, and given points $B \supset D$ in $S(X)$ and $s \leqslant t$ in $[c, d]$, then $h(B, s) \supset h(D, t)$. The proof of the following theorem is omitted, as it is a straightforward generalization of the proof of 3.1 Lemma of [9].

THEOREM 4. *Let X be a continuum. The following three statements are equivalent*:

(a) *$J_1(X)$ is contractible in $S(X)$.*

(b) *$S(X)$ is contractible using an order preserving homotopy.*

(c) *$C(X)$ is contractible (in itself) using an order preserving homotopy.*

COROLLARY. *If X is a weakly contractible continuum, then $S(X)$ and $C(X)$ are contractible (using order preserving homotopies).*

Proof. If X is weakly contractible, then we certainly have $J_1(X)$ weakly contractible in $S(X)$. But $J_1(S(X))$ is a retract of $C(S(X))$. Now we can apply Theorem 3 (with $Y = S(X)$) to conclude that $J_1(X)$ is contractible in $S(X)$. Now Theorem 4 applies.

Remark. It is known (see [2], [13]) that $C(S^1)$ is a two cell with $J_1(S^1)$ as its boundary. This result can be generalized to arbitrary simple closed curves.

Let $A = [a, b]$ be an arbitrary arc and let Z be the simple closed curve obtained by identifying the endpoints of A, with p: $A \to Z$ the identification map. We note that $C(A)$ is clearly homeomorphic to $T = \{(x, y) \in A\times A: x \leqslant y\}$. Let $\{0, 1\}$ have the discrete topology and let $T\times\{0, 1\}$ have the product topology. Let $M = T\times \times\{0, 1\}$ and define f: $M \to C(Z)$ by, for all $(x, y) \in T$:

$$f(x, y, 0) = p([x, y]), \quad f(x, y, 1) = p([a, x] \cup [y, b]).$$

Define the equivalence relation $R \subset M\times M$ by $(m, n) \in R$ if and only if $f(m) = f(n)$. Let $D = M/R$ (with the quotient topology), with q: $M \to D$ the projection map. Then h: $D \to C(Z)$ is a homeomorphism, where h is defined by $h(q(m)) = f(m)$, for each $m \in M$. While all of the results of this paper can be proved without referring to the space D, it is convenient to have this representation of $C(Z)$ at hand.

Let β be a cardinal number (= an ordinal number such that if $\alpha < \beta$, then α is not equipollent to β). Let A have the dictionary order and the order topology, where:

$$A = (\{\alpha: 0 \leqslant \alpha < \beta\}\times[0, 1[) \cup \{(\beta, 0)\}.$$

Then A is an arc $[0, \beta]$, where 0 and β denote the points $(0, 0)$ and $(\beta, 0)$, respectively.

Consider the case where $\beta = \Omega$, the first uncountable ordinal. Let Z be the simple closed curve obtained by identifying the endpoints of $A = [0, \Omega]$ and let p: $A \to Z$ be the identification map. The space Z is connected by separable arcs but is not itself separable.

Let

$$\boldsymbol{C} = \{p([x, y]): 0 \leqslant x \leqslant y < \Omega\},$$
$$\boldsymbol{D} = \{p([0, x] \cup [y, \Omega]): 0 \leqslant x < y < \Omega\} \cup \{Z\}.$$

Let $E = \{p(0)\}$. Then $E \in \boldsymbol{C}$, $Z \in \boldsymbol{D}$, $C(Z) = \boldsymbol{C} \cup \boldsymbol{D}$, and $\boldsymbol{C} \cap \boldsymbol{D}$ is empty. It is easily verified that $\boldsymbol{C}$ and $\boldsymbol{D}$ are each connected by separable arcs, but $C(Z)$ is not connected by separable arcs.

Consider the "picture" of $C(Z)$ given in the preceding Remark. If $C(Z)$ was contractible to the point Z, then the "bounding" simple closed curve $J_1(Z)$ could be continuously deformed to the point Z. But this means that at some time the image of $J_1(Z)$ (which is connected by separable arcs) would have nonempty intersection with both $\boldsymbol{C}$ and $\boldsymbol{D}$, thus contradicting the fact that $C(Z)$ is not arcwise connected by separable arcs. This is the basic idea underlying the proof of the following theorem.

THEOREM 5. *Let $A = [0, \Omega]$ and let Z be the simple closed curve obtained by identifying the endpoints of A, with p: $A \to Z$ the identification map. Then Z is a locally connected continuum and neither $S(Z)$ nor $C(Z)$ is weakly contractible.*

Proof. Suppose that either $S(Z)$ or $C(Z)$ is weakly contractible. Then $J_1(Z)$ is weakly contractible in $S(Z)$. By the proof of the Corollary to Theorem 4, $C(Z)$ is contractible (in itself) using an order preserving homotopy. Thus there exist an arc $[c, d]$ and a map $f: C(Z)\times[c, d]\to C(Z)$ such that, for each $B\in C(Z)$, $f(B, c) = Z$, $f(B, d) = B$, and given points $s\leqslant t$ in $[c, d]$, $f(B, s)\supset f(B, t)$. For each t in $[c, d]$ let $\boldsymbol{Z}_t$ denote $\{f(\{x\}, t)\colon x\in Z\}$. Then $\boldsymbol{Z}_t$ is a subcontinuum of $C(Z)$ which is connected by separable arcs, and so $\boldsymbol{Z}_t$ is contained in either $\boldsymbol{C}$ or $\boldsymbol{D}$. Let t denote the greatest lower bound of $\{w\in[c, d]\colon f(E, w) = E\}$. Then $f(E, t) = E$ and $c<t\leqslant d$. Thus $E\in\boldsymbol{Z}_t$ and so $\boldsymbol{Z}_t\subset\boldsymbol{C}$.

Let

$$\boldsymbol{C}' = \{p([x, y])\colon 0<x\leqslant y<\Omega\},$$

$$\boldsymbol{C}'' = \{p([0, y])\colon 0\leqslant y<\Omega\}.$$

Then $\boldsymbol{C}'$ is an open subcollection of $C(Z)$, as $\boldsymbol{C}'$ is the collection of all $K\in C(Z)$ such that K is contained in $(Z-\{p(0)\})$, which is an open subset of Z. Also $E\in\boldsymbol{C}''$, $\boldsymbol{C} = \boldsymbol{C}'\cup\boldsymbol{C}''$, and $\boldsymbol{C}'\cap\boldsymbol{C}''$ is empty.

Suppose $\boldsymbol{Z}_t\subset\boldsymbol{C}''$. If $0<x<\Omega$ then $f(\{p(x)\}, t) = p([0, y])$, for some y such that $x\leqslant y<\Omega$. Since $E\in\boldsymbol{Z}_t$ and $\boldsymbol{Z}_t$ is connected, it follows that $\boldsymbol{Z}_t = \boldsymbol{C}''$. But $\boldsymbol{Z}_t$ is compact and $\boldsymbol{C}''$ is not.

Thus there exists a point $x\in Z$ $(x\neq p(0))$ such that $f(\{x\}, t)\in\boldsymbol{C}'$. By the continuity of f, there exists a point $s\in[c, d]$ such that $c<s<t$ and $f(\{x\}, s)\in\boldsymbol{C}'$. Then $\boldsymbol{Z}_s\subset\boldsymbol{C}$. By the definition of t, $f(E, s)\neq E$. Let $B = f(E, s)$. Now $p(0)\in B\in\boldsymbol{C}$, and so $B = p([0, x])$, for some x such that $0<x<\Omega$. Consider the net $\{\{p(y)\}\colon y\in[0, \Omega[\}$ converging to E in $C(Z)$. Let $B(y) = f(\{p(y)\}, s)$. Then the net $\{B(y)\colon y\in[0, \Omega[\}$ converges to $B = p([0, x])$. Pick a point w such that $x<w<\Omega$. Let $\boldsymbol{U}$ denote the subcollection of $C(Z)$ consisting of all $K\in C(Z)$ such that $K\subset p([0, w[)$. Then $\boldsymbol{U}$ is open in the space $\boldsymbol{C}$ and $B\in\boldsymbol{U}$. Thus the net $\{B(y)\colon y\in[0, \Omega[\}$ is eventually in $\boldsymbol{U}$. But for all y such that $w<y<\Omega$, $p(y)\in B(y)$ and $p(y)$ is not in $p([0, w[)$. Thus for all y such that $w<y<\Omega$, $B(y)$ is not in $\boldsymbol{U}$, and so a contradiction has been reached.

Let β be a cardinal number greater than Ω. Let Z be the simple closed curve obtained by identifying the endpoints of $[0, \beta]$. The proof of Theorem 5 generalizes to give the result that neither $S(Z)$ nor $C(Z)$ is weakly contractible. We note that nonmetrizable simple closed curves exist which have weakly contractible hyperspaces.

THEOREM 6. *Let $P = \{0\}\cup\{1/n\colon n = 1, 2, \ldots\}$ with the topology inherited from the real line. Let X be obtained from $[0, \Omega]\times P$ by collapsing the set $([0, \Omega]\times\{0\})\cup\cup(\{0, \Omega\}\times P)$ to a single point, denoted by $(0, 0)\in X$. Then X is a locally connected continuum such that neither $S(X)$ nor $C(X)$ is weakly contractible or locally weakly contractible.*

Proof. Any neighborhood of $(0, 0)\in X$ contains (as a retract) a copy of the simple closed curve Z referred to in Theorem 5. Thus any neighborhood of $\{(0, 0)\}$ in $S(X)$ contains a retract which is homeomorphic to $S(Z)$, and so the neighborhood is not weakly contractible. Thus $S(X)$ is not weakly contractible and is not locally weakly contractible. The space $C(X)$ can be handled by a similar proof. We note that the procedure outlined in the Remark following Theorem 4 can be extended to give a "model" of $C(X)$.

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