ANR's and NES's in the category of mappings on metric spaces

by

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Abstract. ANR's and NES's are defined in the category of mappings on metric spaces. It is shown that many of the theorems which are true for the usual ANR's and NES's are true in this category. It is also shown that these maps are like fiber maps and several theorems are proved which give sufficient conditions for a map to be a fiber map.

1. Introduction. In [3] Massey asked the question: What is the relation between the various types of fiber maps? A particular case of this is when does a Serre fibration have the slicing structure property (SSP) (Maps with the slicing structure property over paracompact spaces are Hurewicz fibrations). In [7] Michael showed that a closed map which is n-regular for all n is a Serre fibration. Hence the question that this author has considered for many years is when does an n-regular map have the SSP. There are many examples of n-regular maps which do not have the SSP, however there are no examples with "nice" base spaces.

While working on this problem I noticed that many of the things I wanted to prove and many of the proofs had analogues in the theory of absolute neighborhood retracts. In [13] an attempt was made to put these similarities into categorical language. Using some of the ideas in [13] we were able to show that in the category of mappings on metric spaces n-regular maps are very much like LC^n spaces and maps with the SSP are very much like ANR's. From this we were able to extend the results of [11].

Section 2 contains the definitions of n-NES, n-ANR NES and ANR in the category of mappings on metric spaces (¹).

Sections 3, 4 and 5 contain results which show that the analogy mentioned above is essentially correct. A few of the more important results are summarized in the following:

Let p be a mapping from a complete metric space E onto a metric space B.

(¹) It has been pointed out to the author that Dold defined similar objects in the category of mappings and the reader should see his paper, *The fixed point index of fibre-preserving maps*, Inv. Math. 25 (1974), pp. 281–298.

(a) The map p is an $(n+1)$-NES iff B is LC^n and p is n-regular.

(b) The map p is an $(n+1)$-ES iff it is an $(n+1)$-NES and B and the fibers are C^n.

(c) If $\dim p \leqslant n$ then p is an NES iff it is an ANR iff it is open and locally contractible iff it is n-regular and B is LC^n (this gives a condition for an n-regular map to be $n+1$ regular).

(d) If B is the union of two closed sets B_1 and B_2, with $B_0 = B_1 \cap B_2$ and if each $p_i = p|p^{-1}(B_i)$ is an NES, then p is an NES.

The results of Section 5 of which (d) is a special case extend some of Arnold's (TAMS 164 (1972), pp. 179–188).

Section 6 contains the applications of this material to fiber maps. It is shown that a closed map p which is an NES has the SSP, and in some cases if we omit the condition that p is closed then p almost has the SSP (i.e. there exists a cover of the domain such that p restricted to each element of the cover has the SSP). When all of these results are pieced together one has several conditions which imply that a map has the SSP. In particular, (6.2) extends the main result of [11].

Metric was chosen as the basic category to work with because it is used in Theorem 1.2 [5] which is an important tool in this paper. With care one could reword all theorems in Sections 3, 4 and 5 so that they would be true in the category of mappings on separable metric or compact metric spaces. In order to make this paper self contained all definitions will be given as they apply to the category of mappings on metric spaces, rather than in the categorical language of [13].

2. Definitions and notations. Let Q be a class of topological spaces. The concepts of absolute neighborhood retract and neighborhood extension space for the class Q have been defined in [1] and [5]. In this section these definitions will be extended to classes of mappings. In order to see what is needed consider the following definitions from [5].

(2.1) Definition. A Q space X is called an n-AR(Q) (n-ANR(Q)) iff whenever X is embedded as a closed subspace of a Q space Y with $\dim Y-X \leqslant n$, then X is a retract of Y (X is a retract of a neighborhood in Y). X is called an AR(Q) (ANR(Q)) if it satisfies the above definition with no restriction on $\dim Y-X$.

(2.2) Definition. A space X is called an n-ES(Q) (n-NES(Q)) if whenever Y is in Q, C is a closed subspace of Y with $\dim Y-C \leqslant n$ and $f\colon C \to X$ is a continuous map, then f has a continuous extension to Y (to some neighborhood U of C). Again the space X is an ES(Q) (NES(Q)) if it satisfies the above with no restriction on $\dim Y-C$.

Let M be the full subcategory of TOP whose objects are metric spaces. Let $M(M)$ denote the category whose objects are mappings between metric spaces and whose morphisms are pairs of maps (f_1, f_2) such that there exists mappings p_1 and p_2 between metric spaces satisfying $p_2 f_1 = f_2 p_1$. We will denote this by $(f_1, f_2)\colon p_1 \to p_2$.

In order to define ANR's and NES's in this category we first have to define closed, open, dimension, complement, retract and extension for the category $M(M)$. Once these are all defined the definitions of ANR and NES in this category will be that of (2.1) and (2.2) with the word space replaced by map.

(2.3) Definition. Let $p\colon E \to B$. A *submap* of p is a map $p'\colon E' \to B'$ such that $E' \subset E$, $B' \subset B$ and (abusing notation) $p' = p|E'$. The submap p' will be called a *closed* (*open*) *submap* if both E' and B' are closed (open).

(2.4) Definition. If $p'\colon E' \to B'$ is a submap of $p\colon E \to B$ then the *complement of* p' *in* p denoted by $p-p'$ will be $p|E-E'$ (i.e. $p-p'\colon E-E' \to B$).

(2.5) Definition. If $p'\colon E' \to B'$ is a submap of $p\colon E \to B$ then p' is a *retract* of p if there exists retractions $r_1\colon E \to E'$, $r_2\colon B \to B'$ such that $p' r_1 = r_2 p$.

(2.6) Definition. If $p\colon E \to B$ with E and B metric then the *dimension of* p denoted by $\dim p$ will be the maximum of the covering dimension of E and the covering dimension of B.

(2.7) Definition. Let $q'\colon C \to D$ be a submap of $q\colon X \to Y$ and $(f, g)\colon q' \to p$. A pair of maps $(F, G)\colon q \to p$ is an extension of (f, g) if F is an extension of f and G is an extension of g.

We are now in a position to define ANR's and NES's for mappings on metric spaces.

(2.8) Definition. A map $p \in M(M)$ is called an

$$n\text{-AR}(M(M)) \quad (n\text{-ANR}(M(M)))$$

if whenever it is embedded as a closed submap of a map $q \in M(M)$ with $\dim q-p \leqslant n$ then p is a retract of q (p is a retract of an open submap of q). The map p is called an AR$(M(M))$ (ANR$(M(M))$) if it satisfies the above with no restriction on $\dim q-p$.

(2.9) Definition. A map p will be called an n-ES$(M(M))$ (n-NES$(M(M))$) if whenever q' is a closed submap of a map $q \in M(M)$ with $\dim q-q' \leqslant n$. Then any morphism $(f, g)\colon q' \to p$ has an extension to q (to some open submap of q). Again p is an ES$(M(M))$ (NES$(M(M))$) if it satisfies the above with no restriction on $\dim q-q'$.

Before proceeding we will also need the following definitions and notation.

(2.10) Notation. The unit sphere (ball) in Euclidean n space will be denoted by $S^{n-1}(B^n)$. If (X, d) is a metric spaces and x_0 is a point of X then $N(x_0, \varepsilon) = \{x \in X \mid d(x, x_0) < \varepsilon\}$. The notation $p\colon E \twoheadrightarrow B$ will mean that p is a mapping of E onto B.

(2.11) Definition. An open map $p\colon E \to B$ is n-*regular* if given any x in E and any neighborhood U of x there exists a neighborhood V of x such that if $f\colon S^m \to V \cap p^{-1}(y)$ for some y in B ($m \leqslant n$) then there exists $F\colon B^{m+1} \to U \cap p^{-1}(y)$ which is an extension of f.

(2.12) Definition. A map $p: E \to B$ has the *slicing structure property* (SSP) if for each point b in B there is a neighborhood U_b of b and a map $\psi_b: p^{-1}(U_b) \times \times U_b \to p^{-1}(U_b)$ such that (1) $\psi_b(e, p(e)) = e$ and (2) $p\psi_b = \pi_2$ (the projection on U_b). The map ψ_b is called a *slicing function*.

There is a close relation between maps which have the SSP and Hurewicz fibrations (see [2], [11]).

(2.13) Definition. If A is a category with open and closed subobjects and X and Y are objects in A then Y is an *extension object* for X, denoted $X\tau Y$, means given any closed subobject C of Y and any morphism $f: C \to Y$ there exists an extension $F: X \to Y$ of f. Y is a *neighborhood extension object* for X, denoted by $X\tau_v Y$, means given any closed subobject C of X and any morphism $f: C \to Y$ there exists an extension of f to an open subobject of X which contains C.

3. n-NES's and n-regularity. In this section we will give some characterizations of mappings that are n-ANR's and n-NES's. The main idea in this section is that n-regularity of a mapping corresponds to local n-connectivity of a space. In this section all spaces will be assumed metric.

(3.1) Lemma. *A 0-NES$(M(M))$, $p: E \to B$ is an open map.*

Proof. Let $p: E \to B$ be a 0-NES$(M(M))$ and assume that it is not open. Hence, there exist an open set U in E such that $p(U)$ is not open in B. And so there exist $x \in p(U)$ and $x_i \in B - p(U)$ such that $\{x_i\}$ converges to x.

Let $C = \bigcup \{1/i\} \cup 0$, let $g: 0 \to C$ be inclusion, let $h: 0 \to E$ be defined by $h(0)$ is some point in $p^{-1}(x)$ and define $k: C \to B$ by $k(0) = x$ and $k(1/i) = x_i$. Note that g is a closed submap of 1_C, $\dim 1_C = 0$ and $(h, k): g \to p$. Hence, there exists a map H from an open set V of C into E such that $0 \in V$, H extends h and $pH = k$. Finally, $H^{-1}(U)$ is open in C and $0 \in H^{-1}(U)$ so there exists an i such that $1/i \in H^{-1}(U)$. Therefore $x_i = k\{1/i\} = pH(1/i) \in p(U)$. This is a contradiction so the proof is complete.

(3.2) Corollary. *An n-NES$(M(M))$, $p: E \to B$ is an open map.*

Proof. This follows from (3.6) of [13] which implies that an n-NES$(M(M))$ is a 0-NES$(M(M))$ and hence by (3.1) is open.

(3.3) Lemma. *If $p: E \twoheadrightarrow B$ is an $(n+1)$-NES$(M(M))$ then B is LC^n.*

Proof. Assume that B is not LC^n. Then there exists a point b of B, an $\varepsilon > 0$, an integer $k \leqslant n$, and a sequence of maps $f_i: S^k \to N(b, 1/i)$ such that f_i has no extension to a map from B^{k+1} into $N(b, \varepsilon)$.

Let S_i be $\{x \in B^{k+1} |\ \|x\| = 1/i\}$ and let $D = \bigcup \{S_i\} \cup 0$. Note that S_i is a k sphere and we will assume that $f_i: S_i \to N(b, 1/i)$. Let X be a discrete space with cardinality of the continuum and let g be a one to one function from X onto B^{n+1}. Let $C = g^{-1}(D)$. Let $g': C \twoheadrightarrow D$ be defined by $g'(c) = g(c)$ and note that g' is a closed submap of g. Define $k: D \to B$ by $k|S_i = f_i$ and $k(0) = b$, and let h be any function from C to E such that $ph = kg'$. By the axiom of choice such function exist since $h(c)$ could be chosen to be any point of $p^{-1}kg'(c)$. Both h and k are continuous and $(h, k): g' \to p$, by construction. Since p is an $(n+1)$-NES$(M(M))$ and $\dim g \leqslant n+1$ there exists an open submap $q: U \to V$ of g and a morphism $(H, K): q \to p$ which extends (g, h). By the continuity of K there exists a neighborhood W of 0 such that $K(W) \subset N(b, \varepsilon)$. Therefore, there is an i such that $B_i = \{x \in B^{n+1} |\ \|x\| \leqslant 1/i\}$ is contained in W. Finally B_i is a $k+1$ ball and $K|B_i$ maps B_i into $N(b, \varepsilon)$ and it also extends f_i. This is a contradiction, hence B is LC^n.

(3.4) Theorem. *If p is a map from a complete metric space E onto a metric space B and $n \geqslant 0$ then the following are equivalent.*

(1) *p is an $(n+1)$-NES$(M(M))$.*

(2) *If p' is in $M(M)$ and $\dim p' \leqslant n+1$ then $p'\tau_v p$.*

(3) *B is LC^n, p is open, and if $p': I^{n+1} \to I$ then $p'\tau_v p$.*

(4) *B is LC^n and p is n-regular.*

Proof. That (1) implies (2) follows from the definition. To prove (2)⇒(3) it should be noted that (3.1) and (3.3) only used condition (2) in their proofs. Hence, we need only prove (3)⇒(4) and (4)⇒(1).

3⇒4. Let $\alpha: I \to I$ be an increasing (not strictly) map such that $\alpha(0) = 0$, $\alpha(1) = 1$, $\alpha^{-1}(1/i)$ is a closed interval $[a_i, c_i]$ of diameter less than $1/i$ for $i > 1$ and $\alpha|\alpha^{-1}(1/(i+1), 1/i)$ is a homeomorphism. Such maps are easily seen to exist. Let $g: I^{n+1} \to I$ be defined by $g(x_1, \ldots, x_{n+1}) = \alpha(x_{n+1})$.

Assume p is not n-regular. Then there is a point e of E, a neighborhood U of e, an integer $k \leqslant n$, a sequence of points $\{b_i\}$ in B, and a sequence of maps $f_i: S^k \to N(e, 1/i) \cap p^{-1}(b_i)$ such that f_i has no extension from B^{k+1} to $U \cap p^{-1}(b_i)$. For $i > 1$ let

$$S_i = \{\bar{x} = (x_1, \ldots, x_{k+1}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}(a_i+c_i)) \in I^{n+1} |\ \|\bar{x} - (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}(a_i+c_i))\| = \tfrac{1}{4}(c_i - a_i)\}.$$

Note S_i is a k-sphere of diameter less than $1/i$ in $g^{-1}(1/i)$. Let

$$B_i = \{\bar{x} = (x_1, \ldots, x_{k+1}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}(a_i+c_i)) \in I^{n+1} |\ \|\bar{x} - (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}(a_i+c_i))\| \leqslant \tfrac{1}{4}(c_i - a_i)\}.$$

Note B_i is a $k+1$ ball of diameter less than $1/i$ in $g^{-1}(1/i)$ and S_i is the (geometric) boundary of B_i.

Let $C = \{\bigcup_{i=2}^{\infty} S_i\} \cup \{(\tfrac{1}{2}, \ldots, \tfrac{1}{2}, 0)\}$. Then $g|C$ maps C onto $\{0\} \cup \cup \{1/i|\ i = 1, 2, \ldots\}$ and is a closed submap of g. Define $h: C \to E$ by $h|S_i = f_i$ (one may assume the domain of f_i is S_i) and $h(\tfrac{1}{2}, \ldots, \tfrac{1}{2}, 0) = e$. Define $k: g(C) \to B$ by $k(1/i) = b_i$ and $k(0) = p(e)$. By construction h and k are easily seen to be continuous and (h, k) is a morphism from $g|C$ to p. Hence by condition (3) there exists an open set W containing C and maps $H: W \to E$, $K: g(W) \to B$ such that

$pH = Kg|W$. Since $(\frac{1}{2}, \frac{1}{2}, \ldots, 0) \in C \subset W$ and since H is continuous there exists an open set V such that $(\frac{1}{2}, \ldots, \frac{1}{2}, 0) \in V \subset W$ and $H(V) \subset U$.

Finally there exists an integer i such that $B_i \subset V$ and by construction it is easily seen that $H|B_i$ is an extension of f_i into $U \cap p^{-1}(b_i)$. This is a contradiction and hence p is n-regular.

$4 \Rightarrow 1$. Let $f\colon X \to Y$, $\alpha\colon C \to D$ be a closed submap of f with $\dim f - \alpha \leqslant n+1$ and let $(g, h)\colon \alpha \to p$. Note $f-\alpha\colon X-C \to Y$. Therefore by the hypothesis $\dim Y-D \leqslant \dim Y \leqslant \dim f - \alpha \leqslant n+1$. Hence since B is LC^n there exists an open set V containing D and an extension $H\colon V \to B$ of h. Define $\varphi\colon f^{-1}(V) \to 2^E$ by $\varphi(x) = p^{-1}Hf$, and note $\dim f^{-1}(V) - C \leqslant \dim X - C \leqslant \dim f - \alpha \leqslant n$. Hence φ is a carrier satisfying the hypothesis of Theorem 1.2 [6] and g is a selection for $\varphi|C$. Therefore by Theorem 1.2 [6] there exists an open set U which contains C and a selection G for $\varphi|U$ which extends g. Then $q\colon U \to V$ defined by $q = f|U$ is an open submap of f, q contains α and (G, H) is an extension of (g, h). Therefore p is an n-NES$(M(M))$.

(3.5) Note. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ was proven without any use of the completeness of E. As noted in [6] the requirement that E be complete can be replaced by the weaker requirement that $p^{-1}(b)$ is complete in the induced metric for every b in B. This will automatically happen if $p^{-1}(b)$ is compact for every b.

(3.6) Lemma. *If $1_{B^{n+1}} \tau_v p$ then E is LC^n.*

Proof. Assume E is not LC^n. Then there exists a point e in E, an integer $k \leqslant n$, a neighborhood U of e, and a sequence of maps $f_i\colon S^k \to N(e, 1/i)$ which have no extension from B^{k+1} to U.

Let $B_i = \{(x_1, \ldots, x_{k+1}, 0, \ldots, 0) \in B^{k+1} |\ \sum x_j^2 \leqslant 1/i\}$. Then B_i is a $(k+1)$-dimensional ball. Let S_i be its geometric boundary, and note S_i is a k sphere. We can assume that the domain of f_i is S_i. Let $C = \bigcup S_i \cup \{0\}$, and define $g\colon C \to E$ by $f|S_i = f_i$ and $f(0) = e$. Define $h\colon C \to B$ by $h = pg$. It is easily seen that $(g, h)\colon 1_C \to p$ and 1_C is a closed submap of $1_{B^{n+1}}$. Hence, there exists an open submap $k\colon V \to W$ of $1_{B^{n+1}}$ and a morphism $(G, H)\colon k \to p$ which extends (g, h). Hence there exists a neighborhood S of 0 such that $G(S) \subset U$ and S must contain some B_i. Therefore $G|B_i$ is an extension of f_i into U which is a contradiction.

The following corollary extends (2.6) and (2.8) in [12].

(3.7) Corollary. *If $p\colon E \to B$ is an n-regular map from a complete metric space E onto an LC^n metric space B such that $p^{-1}(b)$ is complete in the induced metric for each $b \in B$ then E is LC^n.*

Proof. This follows from (3.4), (3,5) and (3.6).

(3.8) Corollary. *Let p be a completely regular map from a metric space E onto an LC^n metric space B such that p has complete LC^n points inverses. Then E is LC^n.*

Proof. A completely regular map with LC^n fibers is trivially n-regular, hence the result follows from (3.7).

(3.9) Theorem. *If p is a map from a complete metric space E onto a metric space B and $n \geqslant 0$ then the following are equivalent.*

(1) *p is an $(n+1)$-ES$(M(M))$.*

(2) *If p' is in $M(M)$ and $\dim p' \leqslant n+1$ then $p' \tau p$.*

(3) *B is LC^n and C^n, p is open, and if $p'\colon I^{n+1} \to I$ then $p' \tau p$.*

(4) *B is LC^n and C^n and p is n-regular and has C^n fibers.*

Proof. The proof follows the same lines as (3.4) and uses the last sentence of 1.2 [6].

(3.10) Note. In [13] it was shown that an n-NES$(M(M))$ is an n-ANR$(M(M))$, however I have not been able to prove the converse. The usual method for this type of proof is an embedding argument which I have not been able to complete. In the next section however, we will be able to show an ANR$(M(M))$ is an NES$(M(M))$.

4. NES$(M(M))$'s, ANR$(M(M))$'s and local contractibility.

(4.1) Theorem. *If $p\colon E \to B$ is an NES$(M(M))$ then both E and B are NES(M)'s.*

Proof. To show that E is an NES(M) let X be a metric space, E a closed subspace of X and f a map from C to E. Then $(f, pf)\colon 1_C \to p$ and 1_C is a closed submap of 1_X. Since p is an NES$(M(M))$ there exists U open in X and a morphism $(F, G)\colon 1_U \to p$ which extends (f, pf). It then follows that F is the desired extension of f.

To show that B is an NES(M) let X be a metric space, C a closed subspace of X and $f\colon C \to B$. Let $E' = \{(e, c) \in E \times C |\ p(e) = f(c)\}$. Let $g\colon E' \to X$ be defined by $g(e, c) = c$. Note $\pi_2\colon E' \to C$ is a closed submap of g and $(\pi_1, f)\colon \pi_2 \to p$. Since p is an NES$(M(M))$ there exists an extension (G, F) of (π_1, f) to an open subobject of g. Again from the definition it is clear that F is an extension of f to an open subset of X which contains C.

(4.2) Theorem. *If $p\colon E \to B$ is an ANR$(M(M))$ then E and B are ANR(M)'s.*

Proof. In order to prove that B is an ANR(M) let B be embedded as a closed subspace of a metric space X with embedding $j\colon B \to X$. Let $i\colon E \to E \times B$ be defined by $i(e) = (e, p(e))$ and let $p'\colon E \times B \to X$ be defined by $p'(e, b) = j(b)$. Then $(i, j)\colon p \to p'$ is an embedding of p onto a closed submap of p' and since p is an ANR$(M(M))$ there exists an open submap $q\colon U \to V$ of p' and a retraction $(F, G)\colon q \to p$. It then follows that G is a retraction of an open set of X onto B as desired.

In order to show that E is an ANR(M) let $i\colon E \to X$ be an embedding of E onto a closed subset of a metric space X. By III (8.2) of [1] there exists a homeomorphism h of B onto a closed subset of a metric space A such that the map hp

factors thru i. In other words the following diagram of maps of metric spaces exists.

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ B \arrow[r, "h"'] & A \end{tikzcd}$$

Since i and h are embeddings onto closed subsets (i, h): $p \to q$ is a kernel and hence there exists an open submap q': $U \to V$ of q and a retraction (F, G): $q' \to p$. Again it is clear that F is the desired retraction of an open subset of X onto E.

(4.3) DEFINITION. A map p: $E \to B$ is *contractible to the point* $e_0 \to p(e_0)$ if there exists maps g: $E \times I \to E$ and h: $B \times I \to B$ such that $g(e, 0) = e_0$, $g(e, 1) = e$, $h(p(e_0), t) = p(e_0)$ and $pg = h(p \times 1_I)$. It is *contractible* if it is contractible to every point of p.

(4.4) Note. This is weak I contractibility with respect to the points $1_\Omega \to 1_0 \to 1_r$ and $1_\Omega \to 1_r \to 1_I$ as defined in [13] where Ω is a terminal object.

(4.5) OBSERVATIONS. If p is contractible to $e_0 \to p(e_0)$ then

(1) $h(b, 1) = b$. Since if $p(e) = b$ then $h(b, 1) = h(p(e), 1) = pg(e, 1) = p(e) = b$.

(2) $h(b, 0) = p(e_0)$. Since if $p(e) = b$ then $h(b, 0) = h(p(e), 0) = pg(e, 0) = p(e_0)$.

(3) If $e \in p^{-1}p(e_0)$ then $pg(e, t) = p(e_0)$ for all t since $pg(e, t) = h(p(e), t) = h(p(e_0), t) = p(e_0)$.

(4) g is a contraction of E to e_0 (Note g does not necessarily hold e_0 fixed.).

(5) By (1) and (2) h is a contraction of B to $p(e_0)$ and h holds $p(e_0)$ fixed.

(4.6) DEFINITION. A map p is *locally contractible* if given any point e of E and any neighborhood U of e there exists a neighborhood V of e such that if e_0 is in V there exist maps $g(e_0)$: $V \times I \to U$, $h(e_0)$: $p(V) \times I \to p(U)$ such that $g(v, 0) = e_0$, $g(v, 1) = v$, $h(p(e_0), t) = p(e_0)$ and $pg = h(p|V, 1_I)$.

(4.7) OBSERVATIONS. If p is locally contractible using the notation of the definition we have

(1) $h(b, 1) = b$ (same as (4.5), (1)).

(2) $h(b, 0) = p(e_0)$ (same as (4.5), (2)).

(3) If $e' \in p^{-1}p(e_0)$ then $pg(e', t) = p(e_0)$ (same as (4.5), (3)).

(4) E is locally contractible (not necessarily holding any point fixed).

(5) If p is an open map then B is locally contractible (holding any point fixed).

(4.8) THEOREM. *If p is open and locally contractible then p is n-regular for all n.*

Proof. Let $e \in E$ and let U be a neighborhood of e. There exists a neighborhood V of e satisfying the conditions in the definition of local contractibility of p.

Let $e_0 \in V$ and let f: $S^n \to p^{-1}p(e_0) \cap V$. By the local contractibility of p and the choice of V there exist maps g: $V \times I \to U$, h: $p(V) \times I \to p(U)$ such that $g(v, 0) = e_0$, $g(v, 1) = v$, $h(p(e_0), t) = p(e_0)$ and $ph = g(p|V, 1_I)$. Define G: $S^n \times I \to U$ by $G(x, t) = g(f(x), t)$. Note since $f(x) \in p^{-1}p(e_0)$ $G(x, t) = g(f(x), t)$ also is in $p^{-1}p(e_0)$ by Observation (3). Therefore G: $S^n \times I \to U \cap p^{-1}(b_0)$. Also $G(x, 0) = g(f(x), 0) = e_0$ and $G(x, 1) = g(f(x), 1) = f(x)$. Therefore f could be extended to a map of B^{n+1} into $U \cap p^{-1}(b_0)$ as desired.

(4.9) Note. From the above proof it is easy to see that $V \cap p^{-1}(b_0)$ is contractible in $U \cap p^{-1}(b_0)$ in other words we essentially have the proof of

(4.10) THEOREM. *If p is (locally) contractible then the fibers of p are (locally) contractible.*

(4.11) LEMMA. *Let p: $E \to B$ be an open map from a T_3 space E onto a T_3 space B. If e is in E and U is a neighborhood of E then there exists a neighborhood V of e such that $\overline{V} \subset U$ and $p(\overline{V})$ is closed.*

Proof. Let W be a neighborhood of e such that $e \in \overline{W} \subset U$, and let Z be a neighborhood of $p(e)$ such that $p(e) \in \overline{Z} \subset p(W)$. Since p is open Z exists. Then $V = \overline{W} \cap p^{-1}(\overline{Z})$ is the desired neighborhood of e.

(4.12) Note. We also have the conclusion of the lemma if either of the following conditions are satisfied

(1) E is locally compact T_2 and B is T_2 (no condition on p).

(2) p is a closed map, and E is T_3.

(4.13) DEFINITION. A map p: $E \to\!\!\to B$ is *weakly locally contractible* if given any e in E and any neighborhood U of e there exists a neighborhood V of e and maps g: $V \times I \to U$, h: $p(V) \times I \to p(U)$ such that $g(v, 0) = e$, $g(v, 1) = v$, $h(p(e), t) = p(e)$ and $pg = h(p|V, 1_I)$.

(4.14) LEMMA. *Let p: $E \to B$ and assume E is first countable. If $p \times 1_I \tau_v p$ and p, E and B satisfy any of the following sets of conditions then p is weakly locally contractible.*

(1) *p is an open map, E and B are T_3.*

(2) *p is a closed map and E is T_3.*

(3) *E is locally compact T_2 and B is T_2.*

Proof. Assume that p is not weakly locally contractible. Then there exists a point e in E and neighborhood U of e such that if V is a neighborhood of e and $V \subset U$ there do not exist maps as described in the definition of weakly locally contractible. Under all three sets of conditions there exists a countable base $\{V_i\}_{i=1}^{\infty}$ for e such that $\overline{V}_{i+1} \subset V_i \subset \overline{V}_i \subset U$ and $p(\overline{V}_i)$ is closed. Let

$$C = \bigcup \{\overline{V}_i \times 1/i\} \cup (E \times 0) \cup e \times I \subset E \times I$$

and

$$D = \bigcup \{p(\overline{V}_i) \times 1/i\} \cup B \times 0 \cup p(e) \times I \subset B \times I.$$

Let $q\colon C\to D$ be defined by $q(e,t)=(p(e),t)$. From the construction q is a closed submap of $(p\times 1_I)$. Let $f_1\colon C\to E$ be defined by $f_1(x,t)=x$ if $t>0$ and $f_1(x,0)=e$. Let $f_2\colon D\to B$ be defined by $f_2(y,t)=y$ if $t>0$ and $f_2(y,0)=p(e)$. Then $(f_1,f_2)\colon q\to p$. Therefore it has an extension (F_1,F_2) from an open submap $l\colon S\to T$ of $(p\times 1_I)$ into p. Since $F_1(e,0)=e$ there exists an open set Z in S such that $(e,0)\in Z$ and $F_1(Z)\subset U$. Finally there exists an i such that $\overline{V}_i\times[0,1/i]$ is contained in Z. Hence we can define $g\colon V_i\times[0,1]\to U$ by $g(v,t)=F_1(v,(1/i)t)$ and $h\colon p(V_i)\times[0,1]\to U$ by $h(b,t)=F_2(b,(1/i)t)$. It is easily seen that g and h are maps of the forbidden type hence we have a contradiction.

(4.15) THEOREM. *Let $p\colon E\twoheadrightarrow B$ and assume E is first countable. If p, E and B satisfy any of the three sets of conditions of* (4.14) *and if $p\times 1_I\tau_v p$ then p is locally contractible.*

Proof. Assume p is not locally contractible. Then there exists a neighborhood U of p such that every neighborhood V of p has a point e_V for which there exist no maps g and h satisfying the definition.

Let $\{V_i\}_{i=1}^{\infty}$ be a countable nested base at e such that each $\overline{V}_i\subset U$ and $p(\overline{V}_i)$ is closed, and let v_i be a point in V_i such that the maps defining local contractibility do not exist.

By the previous theorem there exist neighborhoods W_i such that $v_i\in W_i\subset V_i$ and $p(\overline{W}_i)$ is closed, and maps $g_i\colon \overline{W}_i\times I\to V_i$, $h_i\colon p(\overline{W}_i)\times I\to p(V_i)$ such that $g_i(w,0)=v_i$, $g_i(w,1)=w$, $h_i(p(v_i),t)=p(v_i)$ and $pg_i=h_i(p|V_i,1_I)$. Let C be the following subset of $E\times I$

$$C=E\times 1\cup\bigcup_{i=1}^{\infty}\Big\{\overline{V}_i\times\frac{1}{2^i}\Big\}\cup\bigcup_{i=1}^{\infty}\Big\{\overline{V}_i\times\frac{3}{2^{i+2}}\Big\}\cup\bigcup_{i=1}^{\infty}\Big\{\overline{W}_i\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big]\Big\}\cup E\times 0\,.$$

Note C is closed in $E\times I$ and $q'\colon C\to(p\times 1_I)(C)$ defined by $q'(e,t)=(p(e),t)$ is a closed submap of $p\times I_I$. We may assume that g_i is defined on $\overline{W}_i\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big]$.

Define $g\colon C\to E$ by

$$g|\overline{W}_i\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big]=g_i\,,\qquad g|\overline{V}_i\times\frac{1}{2^i}=\pi_i\,,\qquad g|\overline{V}_i\times\frac{3}{2^{i+2}}=v_i\,,$$

$$g|E\times 1=\pi_1\quad\text{and}\quad g|E\times 0=e.$$

It is clear that g is continuous. Define $h\colon q'(c)\to B$ by

$$h|p(\overline{W}_i)\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big]=h_i\,,\qquad h|p(\overline{V}_i)\times\frac{1}{2^i}=\pi_1\,,$$

$$h|p(\overline{V}_i)\times\frac{3}{2^{i+2}}=p(v_i)\quad\text{and}\quad h|E\times 1=\pi_1\,.$$

Again it is clear that h is continuous and $(g,h)\colon q'\to p$. Hence there exists neighborhoods S and T of C and $q(C)$ and maps $G,H\colon k\to p$ where $k\colon S\to T$ is a submap of g. Now there exists a neighborhood Z of $e\times 0$ such that $G(Z)\subset U$. Note Z contains

$$V_i\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big]\quad\text{for some } i$$

and

$$G|V_i\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big],\qquad H|p(V_i)\times\Big[\frac{3}{2^{i+2}},\frac{1}{2^i}\Big]$$

are maps of the forbidden kind.

(4.16) COROLLARY. *Let p be a map from a metric space E onto a metric space B. If* $\dim p\leqslant n$ *and p is an $(n+1)$-*NES$(M(M))$ *then p is locally contractible (hence k regular for all k).*

Proof. This follows from (4.15) since $\dim p\times 1_I\leqslant n+1$, and since p is open by (3.2).

(4.17) COROLLARY. *Let p be an n regular map from a metric space E onto an* LC^n *metric space B. If* $\dim p\leqslant n$ *and p has complete fibers then p is locally contractible (hence k regular for all k).*

Proof. By (3.4) p is an $(n+1)$-NES$(M(M))$ hence this result follows from (4.16).

(4.18) THEOREM. *If E and B are* NES(M)*'s then projection $\pi_2\colon E\times B\to B$ is an* NES$(M(M))$. *(And hence by* (3.10) *of* [13] *an* ANR$(M(M))$.

Proof. Let $f\colon X\to Y$ and let $f'\colon C\to D$ be a closed submap of f. Let $(i,j)\colon f'\to\pi_2$. Since B is an NES(M) there exists $k\colon V\to B$ which is an extension of j. Also since E is an NES(M) there exists $l\colon W\to E$ which is an extension of $\pi_1 i$. Let $U=W\cap f^{-1}(V)$, let $g\colon U\to V$ be defined by $g(u)=f(u)$ let $m\colon U\to E\times B$ be defined by $m(u)=(l(u),\pi_2 k(u))$. It is easily seen that $(m,k)\colon g\to\pi_2$ is an extension of (i,j) to an open submap which contains f'.

(4.19) THEOREM. *A map $p\colon E\to B$ is an* ANR$(M(M))$ *iff it is an* NES$(M(M))$.

Proof. From (3.10) of [13] we know that an NES$(M(M))$ is an ANR$(M(M))$. In order to prove the converse, let $p\colon E\to B$ be an ANR$(M(M))$. Then by (4.2) E and B are ANR(M)'s and hence they are NES(M)'s. Therefore $\pi_2\colon E\times B\to B$ is an NES$(M(M))$ by (4.18). If we let $i\colon E\to E\times B$ be defined by $i(e)=(e,p(e))$ then it is easily seen that $(i,1_B)\colon p\to\pi_2$ is an embedding of p onto a closed submap of π_2. Since p is an ANR$(M(M))$ it is a retract of an open subobject of π_2 and hence by (3.14) of [13] p is an NES$(M(M))$.

(4.20) THEOREM. *If p is a map from a complete metric space E onto a metric space B such that* $\dim p\leqslant n$, *then the following are equivalent.*

(1) *p is an* NES$(M(M))$.

(2) *p is an* ANR$(M(M))$.

(3) *p is open and* $p\times 1_I\tau_v p$.

(4) *p is open and locally contractible.*

(5) *p is n-regular and B is* LC^n.

Proof. (1)$\Leftrightarrow$(2) follows from (4.19). (1)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(4) follows from (4.15).

(4)$\Rightarrow$(5) follows from (4.7), (4.8) and (3.8).

(5)$\Rightarrow$(4) follows from (4.17). To complete the proof we will show that (4)$\Rightarrow$(1).

The idea of the proof is almost the same as in (4.19). Let $i\colon E\to E\times B$ be defined by $i(e) = (e, p(e))$ and note as before that $(i, 1_B)\colon p\to\pi_2$ is a closed submap. Since p is locally contractible (4.7) implies that B is locally contractible and since it is finite dimensional B is an $\mathrm{NES}(M)$. Also by (4.8) p is k-regular for all k. Therefore, by (3.4) p is a k-$\mathrm{NES}(M(M))$ and hence by (3.10) of [13] a k-$\mathrm{ANR}(M(M))$ for all k and in particular a $2n$-$\mathrm{ANR}(M(M))$. From (3.7) E is LC^k for all k and since it is finite dimensional E is an $\mathrm{NES}(M)$. By (4.18) $\pi_2\colon E\times B\to B$ is an $\mathrm{NES}(M)$ and $\dim\pi_2\leqslant 2n$. Since by the above p is a $2n$-$\mathrm{ANR}(M(M))$ and $(i, 1_B)\colon p\to\pi_2$ and $\dim\pi_2\leqslant 2n$ we get that p is a retract of an open submap of π_2 and then by (3.14) of [13] p is an $\mathrm{NES}(M(M))$ as desired.

(4.21) Theorem. *If E is compact then p is an $\mathrm{NES}(M(M))$ iff it is a retract of an open subobject of $\pi_2\colon I^w\times I^w\to I^w$ (I^w is the Hilbert cube).*

Proof. The sufficiency of this condition follows from (3.14) of [13] (4.18) and the fact that I^w is an ANR(NORMAL). For the necessity assume that p is an $\mathrm{NES}(M(M))$ and embed p in $\pi_2\colon I^w\times I^w\to I^w$ as follows. Let $f\colon E\to I^w$ and $g\colon B\to I^w$ be embeddings which exist since both E and B are compact metric. Let $\alpha\colon E\to I^w\times\times I^w$ be defined by $\alpha(e) = (f(e), gp(e))$. Then $(\alpha, g)\colon p\to\pi_2$ is an embedding of p onto a closed submap of π_2. Finally since p is an $\mathrm{NES}(M(M))$ it is an $\mathrm{ANR}(M(M))$, hence p is a retract of an open submap of π_2 as desired.

5. Unions of n-NES's, NES's and n-regular maps.

In this section we will assume that $p\colon E\to B$ and B is the union of two closed sets B_1 and B_2 whose intersection will be denoted by B_0. We will also let $p_i\colon p^{-1}(B_i)\to B_i$ be the restriction of p to $p^{-1}(B_i)$. The question we will attempt to answer is if each p_i has one of the properties mentioned in the title of this section does p have the same property? For the most part the answer will be yes. In Section 6 these results will be applied to fiber maps.

A trivial lemma which will be used several times is

(5.1) Lemma. *Let X be the union of two closed sets X_1 and X_2 and let $x\in X_1\cap\cap X_2$. If U_i $(i = 1, 2)$ is a neighborhood of x in X_i then there is a neighborhood V of x in X such that $V\subset U_1\cup U_2$.*

(5.2) Theorem. *If each p_i, $i = 1, 2$ is an open map, then p is an open map.*

Proof. Let U be open in E and let $e\in E$. If $e\in p^{-1}(B_i)-p^{-1}(B_{3-i})$ for $i = 1$ or 2, then $V = U\cap[p^{-1}(B_i)-p^{-1}(B_{3-i})]$ is an open set in $p^{-1}(B_i)$ which contains e and $p(V) = p_i(V)$ is open in B_i. However, $p(V)\cap B_{3-i}$ is empty, so $p(V)$ is open in B. Hence $p(e)$ is an interior point of $p(U)$ as desired. Therefore, assume that $e\in p^{-1}(B_1)\cap p^{-1}(B_2)$. Then $U\cap p^{-1}(B_i)$ is open in $p^{-1}(B_i)$ for $i = 1, 2$ and hence $p_i(U\cap p^{-1}(B_i))$ is open in B_i for $i = 1, 2$. By (5.1) there exists an open set V in B such that $p(e)\in V\subset p_i(U\cap p^{-1}(B_1))\cup p_i(U\cap p^{-1}(B_2)) = p(U)$. Hence again $p(e)$ is an interior point of $p(U)$ and the proof is complete.

(5.3) Theorem. *If p_i, $i = 1, 2$ is n-regular, then p is n-regular.*

Proof. That p is open follows from (5.2). Therefore let $e\in E$ and let U be a neighborhood of e. Again if $e\in p^{-1}(B_i)-p^{-1}(B_{3-i})$ for $i = 1$ or 2 the proof is trivial. Hence assume that $e\in p^{-1}(B_1)\cap p^{-1}(B_2)$. By the n-regularity of p_i, $i = 1, 2$ there exists neighborhoods V_i of e in $p^{-1}(B_i)$ such that if $k\leqslant n$ and $f\colon S^k\to p_i^{-1}(b)\cap V_i$, then there exists an extension $F\colon B^{k+1}\to p_i^{-1}(b)\cap(U\cap p^{-1}(B_i)) = p_i^{-1}(b)\cap U = p^{-1}(b)\cap U$ of f. Now let V be a neighborhood in E of e such that $V\subset V_1\cup V_2$. It is easily seen that any map $f\colon S^k\to p^{-1}(b)\cap V$ has an extension of the desired type and hence p is n-regular.

(5.4) Corollary. *If E is a complete metric space, B metric, p_i an $(n+1)$-$\mathrm{NES}(M(M))$, $i = 1, 2$ and p_0 an n-$\mathrm{NES}(M(M))$, then p is an $(n+1)$-$\mathrm{NES}(M(M))$.*

Proof. By (5.3) p is n-regular and hence by (3.4) all that need be shown is that B is LC^n. This follows from (3.3) which implies B_1 and B_2 are LC^n and B_0 is LC^{n-1} which implies that B is LC^n.

The next corollary is included only because of its simple proof. It is immediately superceded by (5.6).

(5.5) Corollary. *If E and B are finite dimensional metric spaces, E complete and each p_i is an $\mathrm{NES}(M(M))$, $i = 0, 1, 2$, then p is an $\mathrm{NES}(M(M))$.*

Proof. This follows directly from (5.4) and (4.20).

(5.6) Theorem. *Suppose that E and B are metric spaces. Then*

(1) *If each p_i is an $\mathrm{ES}(M(M))$, then p is an $\mathrm{ES}(M(M))$.*

(2) *If each p_i is an $\mathrm{NES}(M(M))$, then p is an $\mathrm{NES}(M(M))$.*

Proof. The proof is modeled after (6.1) [1]. We only prove (2). The proof of (1) will then be clear. By (4.19) each p_i is an $\mathrm{ANR}(M(M))$, hence we need only prove that p is an $\mathrm{ANR}(M(M))$. To do this let p be embedded as a closed submap of $q\colon X\to Y$, so that we have the following commutative diagram:

$$\begin{array}{ccc} E & \subset & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \subset & Y \end{array}$$

Set

$$Y_0 = \{y\in Y|\ d(y, B_1) = d(y, B_2)\},$$
$$Y_1 = \{y\in Y|\ d(y, B_1) < d(y, B_2)\},$$
$$Y_2 = \{y\in Y|\ d(y, B_1) > d(y, B_2)\}.$$

Clearly $Y = Y_0 \cup Y_1 \cup Y_2$ and $B_0 \subset Y_0$. Let $X_i = q^{-1}(Y_i)$ and let $q_i\colon X_i \to Y_i$ be the restriction of q to X_i. It is easily seen that p_0 is a closed submap of q_0 and since p_0 is an ANR$(M(M))$, there exists a retraction (r_0, r_0') from an open submap $q_0'\colon U_0 \to V_0$ of q_0 to p_0. Note U_0 is open in X_0, V_0 is open in Y_0, $p^{-1}(B_0) \subset U_0$ and $B_0 \subset V_0$. By the normality of X and Y we could find closed neighborhoods S_0 of $p^{-1}(B_0)$ in X_0, and T_0 of B_0 in Y_0 such that $S_0 \subset U_0$, $T_0 \subset V_0$ and $q_0'(S_0) \subset T_0$. Therefore let $q_0''\colon S_0 \to T_0$ be the restriction of q_0' to U_0 and consider (r_0, r_0') as a retraction from q_0'' to p_0.

For $i = 1, 2$ define $(r_i, r_i')\colon p_i \to q_0''\ p_i$ as follows:

$r_i\colon p^{-1}(B_i) \cup S_0 \to p^{-1}(B_1)$ is defined by

$$r_i(x) = \begin{cases} x & \text{if} \quad x \in p^{-1}(B_i)\,, \\ r_0(x) & \text{if} \quad x \in S_0 \end{cases}$$

and $r_i'\colon p^{-1}(B_i) \cup T_0 \to B_i$ is defined by

$$r_i'(x) = \begin{cases} x & \text{if} \quad x \in B_i\,, \\ r_0'(x) & \text{if} \quad x \in V_0\,. \end{cases}$$

Both $r_i(x)$ and $r_i'(x)$ are defined to be continuous on two closed sets and since the definitions agree on the intersection r_i and r_i' are well defined continuous functions. Now since $p_i \cup q_0''$ is a closed submap of q_i and since each p_i is an NES$(M(M))$, there exists an extension $(f_i, g_i)\colon q_i' \to p_i$ of (r_i, r_i') to an open submap $q_i'\colon S_i \to T_i$ of q_i.

It is now easily seen that there exists open sets U in X, V in Y such that $E \subset U \subset S_0 \cup S_1 \cup S_2$, $B \subset V \subset T_0 \cup T_1 \cup T_2$, $U \cap X_0 \subset S_0$, $V \cap Y_0 \subset T_0$ and $q(U) \subset V$. Let $q'\colon U \to V$ be the restriction of q to U. Define $(r, r')\colon q' \to p$ as follows:

$r\colon U \to E$ is defined by

$$r(x) = f_i(x) \quad \text{if} \quad x \in S_i$$

and $r'\colon V \to B$ is defined by

$$r'(x) = g_i(x) \quad \text{if} \quad x \in T_i\,.$$

From the construction and the choice of U and V it is again easily seen that both r and r' are well defined and continuous and (r, r') is a retraction of q' (which is an open submap of q) onto p. Hence p is an ANR$(M(M))$ as desired.

6. NES$(M(M))$'s and the slicing structure property. In this section we show that under certain conditions NES's are almost like maps with the slicing structure property.

(6.1) THEOREM. *If $p\colon E \to B$ is a closed* NES$(M(M))$ *then p has the* SSP.

Proof. Let $\pi_2\colon E \times B \to B$, $f\colon \mathrm{Gr}p \to B$ ($\mathrm{Gr}p$ is the graph of p) be defined $f(e, b) = p(e) = b$, let $g\colon \mathrm{Gr}p \to E$ be defined by $g(e, b) = e$. Note that f is a closed submap of π_2 and $(g, 1_B)\colon f \to p$. Hence there exists a neighborhood U of $\mathrm{Gr}p$ and a map $F\colon U \to E$ which extends f and such that $pF = \pi_2$.

Let b be in B. By Stone's theorem $p^{-1}(b)$ is open or compact. If $p^{-1}(b)$ is open, let $Q = b$. If $p^{-1}(b)$ is compact (and not open) cover $p^{-1}(b)$ with finitely many open sets $\{V_i\}_{i=1}^n$ such that $V_i \times p(V_i) \subset U$. In this case let

$$Q = \bigcap_{i=1}^{n} p(V_i) \cap \{y \in B \mid p^{-1}(y) \subset \bigcup_{i=1}^{n} V_i\}\,.$$

In either case note that $p^{-1}(Q) \times Q \subset U$. Finally define $\psi\colon p^{-1}(Q) \times Q \to p^{-1}(Q)$ by $\psi(x, y) = F(x, y)$ and note from the construction that ψ is a slicing function.

(6.2) COROLLARY. *If $p\colon E \to B$ is a closed n-regular map from a complete metric space E onto an* LC^n *metric space B and if* $\dim p \leqslant n$, *then p has the* SSP.

Proof. This follows from (4.20) and (6.1).

(6.3) Note. If an NES$(M(M))$ is not closed it need not be a fibration of any type as seen by Examples (6.6) and (6.7). As a matter of fact (6.7) shows that it need not even have the property that paths could be lifted. However, the following conjecture seems plausible. If $p\colon E \to B$ is an NES$(M(M))$, then p is the union of open subobjects each of which has the SSP.

The next theorem is a special case of the conjecture. First we need the following:

(6.4) DEFINITION. A locally compact ANR(M), E is *locally free of the singularity of Mazurkiewicz* if for each e in E and each neighborhood U of e there exists a compact neighborhood V of e which can be expressed as the union of relatively open AR(M) sets of arbitrarily small diameter.

(6.5) THEOREM. *If p is a finite dimensional* NES$(M(M))$ *from a locally compact space E which is locally free of the singularity of Mazurkiewicz then each point of E has a neighborhood W such that $p|W\colon W \to p(W)$ has the* SSP.

Proof. Starting as in (6.1) we obtain a neighborhood U of $\mathrm{Gr}p$ in $E \times B$ and a map $F\colon U \to E$ such that $F(e, p(e)) = e$ and $pF = \pi_2$. Let e be a point of E and let V be a compact neighborhood of e such that V is an ANR(M), V is the union of arbitrarily small relatively open AR(M) sets and $V \times p(V) \subset U$. Let $W = F(V \times p(V))$ and let $\tilde{p}\colon W \to p(W) = p(V)$ be $p|W$. It will be shown using (4.20) that $\tilde{p}$ is an NES$(M(M))$ and since W is compact (6.1) will imply that $\tilde{p}$ has the SSP.

That p is open follows from the fact that if Q is open in $F(V \times p(V))$ then $\tilde{p}(Q) = \pi_2 F^{-1}(Q)$ which is open in $p(V)$. In order to show that $\tilde{p}$ is n regular, let e be in $F(V \times p(V))$ and let Q be a neighborhood of e. Then there exists a relatively open AR(M) set T in V such that e is in T and $F(T \times p(T)) \subset Q$. Note that since $F(x, p(x)) = x$ we have $T \subset Q$. Let $g\colon S^n \to T \cap \tilde{p}^{-1}(b)$. Since T is C^n there exists a map $H\colon B^{n+1} \to T$ which extends g. Define $G\colon B^{n+1} \to Q$ by $G(x) = F(H(x), b)$. Then G is easily seen to be an extension of g and the image of G is contained in $p^{-1}(b) \cap Q$ since $H(T \times p(T)) \subset Q$ and $\tilde{p}G(x) = \tilde{p}F(G(x), b) = pF(G(x), b) = \pi_2(G(x), b) = b$. Finally $p(V)$ is LC^n follows from (2.3), [10]. Therefore by

(4.20) p is an NES$(M(M))$ and since W is compact p is closed. Hence by (6.2) p has the SSP.

(6.6) EXAMPLE. Let $E = I^2 - \{(x,y)|\ y = \frac{1}{2} \text{ and } x \geqslant \frac{1}{2}\}$. Let $B = I$ and let $p\colon E \to B$ be defined by $p(x,y) = x$. By the previous theorems p is an NES$(M(M))$ and it is easily seen that p is not even a Serre fibration.

(6.7) EXAMPLE. Let $E = \{(x,y) \in R^2|\ x^2+y^2 = 1\} \cup \{(x,y) \in R^2|\ y = \frac{1}{3}x + \frac{1}{3}$ and $-1 \leqslant x < \frac{1}{2}\} \cup \{(x,y) \in R^2|\ -\frac{1}{2} < x \leqslant 1$ and $y = 0\}$. Let

$$B = \{(x,y) \in R^2|\ x^2+y^2 = 1 \text{ or } y = 0 \text{ and } -1 \leqslant x \leqslant 1\}.$$

Define $p\colon E \to B$ by

$$p(x,y) = \begin{cases} (x,y) & \text{if} \quad x^2+y^2 = 1 \text{ or } y = 0, \\ (x,0) & \text{otherwise}. \end{cases}$$

Again the previous theorems show that p is an NES$(M(M))$ but p does not even have the property that paths could be lifted.

(6.8) Note 1. From the proof of (6.1) it is easily seen that an ES$(M(M))$ has the slicing structure property.

2. From (3.11) it follows that an $(n+1)$-ES$(M(M))$ from a complete metric space has the covering homotopy property for spaces of dimension $\leqslant n$.

The question now is if p is a nice fiber map is p an NES$(M(M))$. There are some partial answers to this.

(6.9) THEOREM (McAuley and Tulley [4]). *If p is a map from an LC^n metric space onto an LC^{n+1} metric space and p has the covering homotopy property for $(n+1)$-cells then p is n-regular.*

(6.10) COROLLARY. *If p is a Serre fibration between ANR(M)'s, then p is n-regular for all n.*

(6.11) COROLLARY. *If p is as in (6.10) and E and B are finite dimensional, then p is an NES$(M(M))$.*

This follows from (6.10) and (4.20).

(6.12) COROLLARY. *If p is as in (6.11) and p is closed, then p has the SSP.*

Proof. This follows from (6.1).

(6.13) COROLLARY. *Let $p\colon E \to B$, assume that B is the union of 2 closed subspaces B_1 and B_2 with $B_0 = B_1 \cap B_2$. Further assume that $p_i\colon p^{-1}(B_i) \to B_i$ is a closed Serre fibration for each i and that $p^{-1}(B_i)$ and B_i are finite dimensional ANR(M)'s. Then p has the SSP.*

Proof. This follows from (6.12) and (5.6).

One could derive other corollaries similar to (6.13). However, they are more complicated than needed as seen by Theorem (4.2) [1]. Part of the difficulty is that the methods of this paper lean too heavily on the local n-connectivity of the base

spaces. I am trying to but have as yet not developed this material so that if p is an NES$(M(M))$, it would not imply that E and B are NES(M)'s.

(6.14) Concluding remarks. I think that ANR$(M(M))$'s will play a role in the study of mappings as ANR's have in the study of spaces. In particular I think that almost any theorem about ANR's has an analogue to the mapping case. As an example, the theorem that compact ANR's are dominated by polyhedra might possibly turn out to be that a compact ANR$(M(M))$ is dominated by a piecewise linear map between polyhedra. If a complete study is made of the mapping case it will include the theory of ANR's by studying identity mappings, since one has that B is an ANR(M) if 1_B is an ANR$(M(M))$.

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Accepté par la Rédaction le 11. 4. 1975