Some functions on hyperspaces of hereditarily unicoherent continua *

by

Jack T. Goodykoontz, Jr. (Morgantown, W. Va.)

Abstract. Several natural functions defined on the hyperspaces of hereditarily unicoherent continua are studied in this paper. The continuity of these functions is investigated and point-wise continuity is related to the internal connectivity structure of the base space and the hyperspace. Four results which characterize smooth dendroids within the class of dendroids are obtained and some relevant examples are given.

1. Introduction. In this paper several natural functions defined on the hyperspaces of hereditarily unicoherent continua are studied. Let X be an hereditarily unicoherent continuum and $A \in 2^X$. Define $f, g\colon 2^X \to C(X)$ by $f(A) = \bigcap\{M \in C(X) |\ A \subset M\}$ and $g(A) = \bigcap\{M \in C(X) |\ A \subset \operatorname{Int} M\}$. In Section 3 we investigate the continuity of these functions and relationships between the two functions. In particular, we investigate how point-wise continuity is related to the internal connectivity structure of X and 2^X. We determine the class of spaces for which f is continuous and obtain two characterizations of the points of $C(X)$ at which 2^X is connected im kleinen. Three relevant examples are given. In Section 4 we obtain four results which characterize smooth dendroids within the class of dendroids in terms of continuity of functions defined on (or to) their hyperspaces.

2. Preliminaries. A *continuum* X will be a compact connected metric space. X is *unicoherent* if whenever A and B are proper subcontinua of X such that $X = A \cup B$, then $A \cap B$ is connected. If each subcontinuum of X has this property, then X is *hereditarily unicoherent*. In this paper X will always denote an hereditarily unicoherent continuum.

A *dendroid* is an hereditarily unicoherent continuum which is arcwise connected. A *dendrite* is an hereditarily unicoherent continuum which is locally connected. It follows from [11] (Theorem 5.2, page 38) that a dendrite is arcwise connected. Hence a dendrite is a locally connected dendroid.

---

* Most of the results in this paper are part of the author's doctoral dissertation written under the direction of Professor Carl Eberhart at the University of Kentucky, 1971.

$2^X(C(X))$ denotes the hyperspace of closed subsets (subcontinua) of X, each with the finite (Vietoris) topology, and since X is a continuum, each of 2^X and $C(X)$ is also a continuum (see [8]).

For notational purposes, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters will denote subsets of 2^X. If $A \subset X$, then A^* ($\operatorname{Int} A$) will denote the closure (interior) of A in X.

If $A_1, \ldots, A_n$ are subsets of X, then

$$N(A_1, \ldots, A_n) = \{B \in 2^X | \text{ for each } i = 1, \ldots, n,\ B \cap A_i \neq \emptyset, \text{ and } B \subset \bigcup_{i=1}^{n} A_i\}.$$

The collection of all sets of the form $N(U_1, \ldots, U_n)$, with $U_1, \ldots, U_n$ open in X, is a base for the finite topology. It is easy to establish that $N(U_1, \ldots, U_n)^* = N(U_1^*, \ldots, U_n^*)$ and that $N(V_1, \ldots, V_m) \subset N(U_1, \ldots, U_n)$ if and only if $\bigcup_{j=1}^{m} V_j \subset \bigcup_{i=1}^{n} U_i$ and for each U_i there exists a V_j such that $V_j \subset U_i$ (see [8]).

If $\mathscr{A} \subset 2^X$, then $\bigcup \{A | A \in \mathscr{A}\}$ is open (closed) in X whenever $\mathscr{A}$ is open (closed) in 2^X (see [8]). Furthermore, if $\mathscr{A} \cap C(X) \neq \emptyset$ and $\mathscr{A}$ is connected, then $\bigcup \{A | A \in \mathscr{A}\}$ is connected (Lemma 1.2 of [5]).

An *order arc* in $2^X(C(X))$ is an arc which is also a chain with respect to the partial order on $2^X(C(X))$ induced by set inclusion. If $A, B \in 2^X$, then there exists an order arc from A to B if and only if $A \subset B$ and each component of B meets A (Lemma 2.3 of [5]). It follows (Lemma 2.6 of [5]) that every order arc whose initial point is an element of $C(X)$ is entirely contained within $C(X)$.

Let $f\colon X \to 2^Y$ be a function. Then f is *upper* (*lower*) *semi-continuous* if for each $x \in X$ and each open set U in Y such that $f(x) \subset U$ ($f(x) \cap U \neq \emptyset$) there exists an open set V in X containing x such that if $z \in V$, then $f(z) \subset U$ ($f(z) \cap U \neq \emptyset$). It is easy to verify that f is continuous if and only if f is both upper and lower semi-continuous.

Let $x \in X$. Then X is *connected im kleinen* at x if for each open set U containing x there exists an open set V containing x such that if $y \in V$, then U contains a connected subset containing x and y.

3. The functions f and g. X is hereditarily unicoherent if and only if for each closed subset A of X there exists a unique continuum M_A such that M_A is irreducible about A (see [1]). Obviously, $M_A = \bigcap \{M \in C(X) | A \subset M\}$. This characterization of hereditarily unicoherent continua induces a natural function $f\colon 2^X \to C(X)$ defined by $f(A) = M_A$.

LEMMA 1. *The function f is lower semi-continuous.*

Proof. Let $A \in 2^X$ and V be an open set which meets $f(A)$. Suppose $V \cap A \neq \emptyset$. Let W be an open set containing $A - V$. Then $A \in N(V, W)$, and if $B \in N(V, W)$, then $f(B) \cap V \neq \emptyset$, because $B \cap V \neq \emptyset$ and $B \subset f(B)$.

Suppose $V \cap A = \emptyset$. Let $a_1 \in A$. Observe that $A \subset f(A) \subset (\bigcup \{f(\{a_1, a\}) | a \in A\})^*$. Hence there exists $a_2 \in A$ such that $f(\{a_1, a_2\}) \cap V \neq \emptyset$.

We now claim that there exist open sets O_1 and O_2 containing a_1 and a_2 respectively such that if $x \in O_1$ and $y \in O_2$, then $f(\{x, y\}) \cap V \neq \emptyset$. For if not, there exists a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that $x_n \to a_1$ and $y_n \to a_2$ and such that for each positive integer n, $f(\{x_n, y_n\}) \cap V = \emptyset$. Now $C(X)$ is closed in 2^X, so the sequence $\{f(\{x_n, y_n\})\}_{n=1}^{\infty}$ has a limit point M in $C(X)$. Observe that $a_1, a_2 \in M$. Furthermore, $M \cap V = \emptyset$, because each point of M is a limit point of $\bigcup_{n=1}^{\infty} f(\{x_n, y_n\})$ and $\bigcup_{n=1}^{\infty} f(\{x_n, y_n\}) \subset X - V$. Since $f(\{a_1, a_2\}) \subset M$, it follows that $f(\{a_1, a_2\}) \cap V = \emptyset$, which is a contradiction. This establishes the existence of the sets O_1 and O_2.

Now let U be an open set containing $A - (O_1 \cup O_2)$ such that $a_1, a_2 \notin U$. Then $A \in N(O_1, O_2, U)$. Let $B \in N(O_1, O_2, U)$, $b_1 \in B \cap O_1$, and $b_2 \in B \cap O_2$. Then $f(\{b_1, b_2\}) \cap V \neq \emptyset$, so $f(B) \cap V \neq \emptyset$.

In each case we have found an open set containing A with the property that the image of each element in that open set meets V. Hence f is lower semi-continuous.

THEOREM 1. *The function f is continuous if and only if X is a dendrite.*

Proof. Suppose f is continuous. Let $x \in X$ and U be an open set containing x. Then $\{x\} \in N(U)$. Since $f(\{x\}) = \{x\}$ and f is continuous at $\{x\}$, there exists an open set $N(V)$ containing $\{x\}$ such that if $B \in N(V)$, then $f(B) \in N(U)$. Observe that $V \subset U$. Let $y \in V$. Then $\{x, y\} \in N(V)$, so $f(\{x, y\}) \in N(U)$. So U contains a continuum which contains x and y. Hence X is connected im kleinen at x. It follows that X is connected im kleinen at each of its points. Hence X is locally connected.

Suppose that X is a dendrite. In view of Lemma 1, it will suffice to show that f is upper semi-continuous. Let $A \in 2^X$ and U be an open set containing $f(A)$. For each $x \in A$ choose a connected open set V_x such that $V_x^* \subset U$. Since A is compact, there exist $x_1, \ldots, x_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} V_{x_i} \subset \bigcup_{i=1}^{n} V_{x_i}^* \subset U.$$

Then $A \in N(V_{x_1}, \ldots, V_{x_n})$. Let $B \in N(V_{x_1}, \ldots, V_{x_n})$. Then $f(A) \cup (\bigcup_{i=1}^{n} V_{x_i}^*)$ is a continuum containing B. So

$$f(B) \subset f(A) \cup (\bigcup_{i=1}^{n} V_{x_i}^*) \subset U.$$

Hence f is upper semi-continuous.

If X is a locally connected continuum, then $C(X)$ is a retract of 2^X (see [5], Theorem 4.4). Theorem 1 shows that f defines a retraction of 2^X onto $C(X)$ when X is a dendrite.

THEOREM 2. *Let $M \in C(X)$. Then f is continuous at M if and only if 2^X is connected im kleinen at M.*

Proof. Suppose that f is continuous at M. Let $N(U_1, \ldots, U_n)$ be an open set containing $M = f(M)$. Then there exists an open set $N(V_1, \ldots, V_m)$ such that $M \in N(V_1, \ldots, V_m) \subset N(U_1, \ldots, U_n)$ and such that $B \in N(V_1, \ldots, V_m)$ implies $f(B) \in N(U_1, \ldots, U_n)$. Let $B \in N(V_1, \ldots, V_m)$. Then $B \cup M \in N(V_1, \ldots, V_m)$, so $f(B \cup M) \in N(U_1, \ldots, U_n)$. Now $B, M \subset f(B \cup M)$ so there exist order arcs $\mathscr{L}_B$ and $\mathscr{L}_M$ from B to $f(B \cup M)$ and from M to $f(B \cup M)$. If $L \in \mathscr{L}_B$, then $B \subset L \subset f(B \cup M)$, so $L \in N(U_1, \ldots, U_n)$. Hence $\mathscr{L}_B \subset N(U_1, \ldots, U_n)$. Similarly, $\mathscr{L}_M \subset N(U_1, \ldots, U_n)$. So $\mathscr{L}_B \cup \mathscr{L}_M$ is a continuum in $N(U_1, \ldots, U_n)$ containing B and M. It follows that 2^X is connected im kleinen at M.

Suppose that 2^X is connected im kleinen at M. Let $N(U_1, \ldots, U_n)$ be an open set containing $f(M) = M$. Then there exist an open set $N(V_1, \ldots, V_m)$ and a continuum $\mathscr{M}$ such that $M \in N(V_1, \ldots, V_m) \subset \mathscr{M} \subset N(U_1, \ldots, U_n)$. Since $M \in \mathscr{M}$, $\bigcup \{A|\ A \in \mathscr{M}\}$ is a continuum in X. Moreover, since each element of $\mathscr{M}$ is an element of $N(U_1, \ldots, U_n)$, $\bigcup \{A|\ A \in \mathscr{M}\} \in N(U_1, \ldots, U_n)$. If $B \in N(V_1, \ldots, V_m)$, then $B \subset f(B) \subset \bigcup \{A|\ A \in \mathscr{M}\}$, so $f(B) \in N(U_1, \ldots, U_n)$. Hence f is continuous at M.

COROLLARY 1. *Let $x \in X$. Then f is continuous at $\{x\}$ if and only if X is connected im kleinen at x.*

Proof. This corollary follows from Corollary 1 of [4] and Theorem 2.

THEOREM 3. *Let $A \in 2^X$. If f is continuous at $f(A)$, then f is continuous at A.*

Proof. By Lemma 1 it will suffice to show that f is upper semi-continuous at A. Let V be an open set containing $f(A)$. Since f is upper semi-continuous at $f(A)$ there exists an open set $N(U_1, \ldots, U_n)$ containing $f(A)$ such that $B \in N(U_1, \ldots, U_n)$ implies $f(B) \subset V$. Since $A \subset f(A)$ there exists a subset $\{U_{i_1}, \ldots, U_{i_j}\}$ of $\{U_1, \ldots, U_n\}$ such that $A \in N(U_{i_1}, \ldots, U_{i_j})$. Let $C \in N(U_{i_1}, \ldots, U_{i_j})$. For each $i = 1, \ldots, n$, let $x_i \in U_i$. Then $C \cup \{x_1, \ldots, x_n\} \in N(U_1, \ldots, U_n)$, so $f(C \cup \{x_1, \ldots, x_n\}) \subset V$. Hence $f(C) \subset V$. It follows that f is upper semi-continuous at A.

The following example shows that the converse of Theorem 3 is false.

EXAMPLE 1. For each positive integer n let L_n denote the line segment in the plane joining the points $(0, 1)$ and $(1/n, 0)$. Let Y be the line segment joining $(-1, 1)$ and $(1, -1)$ and let

$$X = \Big(\bigcup_{n=1}^{\infty} L_n\Big)^* \cup Y.$$

Let $A = \{(-1, 1), (1, -1)\}$. Then $f(A) = Y$. Applying Theorem 1 to the subspace Y, we can show that f is continuous at A, but 2^X is not connected im kleinen at $f(A)$ (see Theorem 1 of [4]). Consequently, f is not continuous at $f(A)$.

THEOREM 4. *Let $A \in 2^X$. If 2^X is connected im kleinen at A, then f is continuous at A.*

Proof. By Lemma 1 it will suffice to show that f is upper semi-continuous at A. Let U be an open set containing $f(A)$. Since 2^X is connected im kleinen at A, there exist an open set $N(V_1, \ldots, V_n)$ and a continuum $\mathscr{M}$ such that $A \in N(V_1, \ldots, V_n) \subset \mathscr{M} \subset N(U)$. Now $\{f(A) \cup B|\ B \in \mathscr{M}\}$ is the continuous image of $\mathscr{M}$, and since $A \in \mathscr{M}$, $f(A) \in \{f(A) \cup B|\ B \in \mathscr{M}\}$. It follows that $\bigcup \{f(A) \cup B|\ B \in \mathscr{M}\} \in C(X)$. For each $B \in \mathscr{M}$, $f(A) \cup B \in N(U)$, so $\bigcup \{f(A) \cup B|\ B \in \mathscr{M}\} \in N(U)$. If $C \in N(V_1, \ldots, V_n)$, then $\bigcup \{f(A) \cup B|\ B \in \mathscr{M}\}$ is a continuum containing C. It follows that $f(C) \in N(U)$, so $f(C) \subset U$. Hence f is upper semi-continuous at A.

The next example shows that the converse of Theorem 4 is false.

EXAMPLE 2. For each positive integer n let L_n denote the line segment in the plane joining the points $(-1, 1/n)$ and $(1, 1/n)$. Let Y denote the unit interval on the y-axis and let

$$X = \Big(\bigcup_{n=1}^{\infty} L_n\Big)^* \cup Y.$$

Let $p = (-1, 0)$ and $q = (1, 0)$. Then 2^X is not connected im kleinen at $\{p, q\}$ (see Theorems 1 and 3 of [4]), but it is easy to verify that f is continuous at $\{p, q\}$.

Define $g\colon 2^X \to C(X)$ by $g(A) = \bigcap \{M \in C(X)|\ A \subset \operatorname{Int} M\}$.

LEMMA 2. *The function g is upper semi-continuous.*

Proof. Let $A \in 2^X$ and U be an open set containing $g(A)$. By [3] (Theorem 1.6, page 225), U contains a continuum M such that $A \subset \operatorname{Int} M$. Then $A \in N(\operatorname{Int} M)$. Suppose $B \in N(\operatorname{Int} M)$. Then M is a continuum containing B in its interior. Hence $g(B) \subset M \subset U$. So g is upper semi-continuous.

THEOREM 5. *Let $A \in 2^X$. Then f is continuous at A if and only if $f(A) = g(A)$.*

Proof. Suppose that f is continuous at A. Let U be an open set containing $f(A)$. Let $N(V_1, \ldots, V_n)$ be an open set such that $f(A) \in N(V_1, \ldots, V_n) \subset N(V_1, \ldots, V_n)^* \subset N(U)$. Since f is continuous at A, there exists an open set $N(W_1, \ldots, W_m)$ containing A such that $B \in N(W_1, \ldots, W_m)$ implies $f(B) \in N(V_1, \ldots, V_n)$. Let

$$M = \big(\bigcup \{f(A \cup B)|\ B \in N(W_1, \ldots, W_m)\}\big)^*.$$

Then $M \in C(X)$, because M is the closure of a union of connected sets each containing A, and $M \in N(V_1, \ldots, V_n)^*$, because for each $B \in N(W_1, \ldots, W_m)$, $f(A \cup B) \in N(V_1, \ldots, V_n)$. Furthermore, $A \subset \operatorname{Int} M$, since $A \subset \bigcup_{i=1}^{m} W_i$. It follows that $g(A) \subset M \subset U$. So every open set containing $f(A)$ also contains $g(A)$. Hence $f(A) = g(A)$.

Suppose that $f(A) = g(A)$. By Lemma 1, it will suffice to show that f is upper semi-continuous at A. Let U be an open set containing $f(A) = g(A)$. By Lemma 2, g is upper semi-continuous at A, so there exists an open set $N(V_1, \ldots, V_n)$ containing A such that $B \in N(V_1, \ldots, V_n)$ implies $g(B) \subset U$. Since for each $B \in 2^X$, $f(B) \subset g(B)$, it follows that f is upper semi-continuous at A.

COROLLARY 2. *Let $M \in C(X)$. Then 2^X is connected im kleinen at M if and only if $M = g(M)$.*

THEOREM 6. *Let $A \in 2^X$. If f is continuous at A, then g is continuous at A.*

Proof. By Lemma 2, it will suffice to show that g is lower semi-continuous at A. Let U be an open set such that $g(A) \cap U \neq \emptyset$. By Theorem 5, $f(A) = g(A)$. Since f is lower semi-continuous at A, there exists an open set $N(V_1, \ldots, V_n)$ containing A such that $B \in N(V_1, \ldots, V_n)$ implies $f(B) \cap U \neq \emptyset$. Since $f(B) \subset g(B)$, it follows that g is lower semi-continuous.

Corollary 3. *If 2^X is connected im kleinen at $f(A)$, then g is continuous at A.*

Corollary 4. *If 2^X is connected im kleinen at A, then g is continuous at A.*

Corollary 5. *If $M \in C(X)$ and $g(M) = M$, then g is continuous at M.*

The converses of Theorem 6, Corollary 3, Corollary 4, and Corollary 5 are all false. In Example 1 let Z denote the line segment joining $(0, 0)$ and $(0, 1)$. Observe that $f(Y) = Y$ and $g(Y) = Y \cup Z$. It is easy to verify that g is lower semi-continuous at Y and hence continuous at Y, but $g(Y) \neq Y$, 2^X is not connected im kleinen at $Y = f(Y)$ (see Theorem 1 of [4]), and f is not continuous at Y.

Theorem 7. *If for each $x \in X$, g is continuous at $\{x\}$, then g is continuous.*

Proof. Let $A \in 2^X$. It will suffice to show that g is lower semi-continuous at A. Let V be an open set such that $V \cap g(A) \neq \emptyset$.

Suppose that $V \cap (\bigcup \{g(\{a\})|\ a \in A\})^* \neq \emptyset$. Then for some $a \in A$, $V \cap g(\{a\}) \neq \emptyset$. Since g is continuous at $\{a\}$ there exists an open set U containing a such that $B \in N(U)$ implies $g(B) \cap V \neq \emptyset$. Let W be an open set containing $A - U$ such that $a \notin W$. Then $A \in N(U, W)$ and if $C \in N(U, W)$, $g(C) \cap V \neq \emptyset$, since $C \cap U \neq \emptyset$.

Suppose that $V \cap (\bigcup \{g(\{a\})|\ a \in A\})^* = \emptyset$. Let $p \in g(A) \cap V$ and let 0 and U be disjoint open sets such that $p \in 0 \subset V$ and $(\bigcup \{g(\{a\})|\ a \in A\})^* \subset U$. Let W be an open set such that $(\bigcup \{g(\{a\})|\ a \in A\})^* \subset W \subset W^* \subset U$. For each $a \in A$ there exists an open set W_a containing a such that $B \in N(W_a)$ implies $g(B) \subset W$. Since A is compact there exist $a_1, \ldots, a_n$ such that $A \in N(W_{a_1}, \ldots, W_{a_n})$. Let $C \in N(W_{a_1}, \ldots, W_{a_n})$. For each $i = 1, \ldots, n$, let $c_i \in W_{a_i}$. Let

$$D_i = \bigcup \{g(\{c_i, d\})|\ d \in W_{a_i}\}.$$

Then $D_i \subset W$ and D_i is a union of continua, each of which meets $g(C)$. So $\bigcup_{i=1}^{n} D_i^* \subset U$ and $(\bigcup_{i=1}^{n} D_i^*) \cup g(C)$ is a continuum containing A in its interior. Hence

$$((\bigcup_{i=1}^{n} D_i^*) \cup g(C)) \cap 0 \neq \emptyset.$$

It follows that $g(C) \cap 0 \neq \emptyset$, so $g(C) \cap V \neq \emptyset$. Hence g is lower semi-continuous at A.

It follows from Theorems 1 and 5 that if X is a dendrite, then g is continuous. The converse is not true. We give an example below of a smooth dendroid (see Section 4) which is not locally connected and for which g is continuous.

Example 3. Let I denote the unit interval on the x-axis in the plane and let C denote the Cantor set embedded in I in the natural manner. For each $(x, 0) \in C$

let L_x denote the line segment joining the points $(x, 0)$ and $(x, 1)$ and let $X = I \cup (\bigcup \{L_x|\ (x, 0) \in C\})$. Then for each $(x, y) \in X$, g is continuous at $\{(x, y)\}$, so by Theorem 7, g is continuous. If $y > 0$, $g(\{(x, y)\}) \neq \{(x, y)\}$, so g is not a retraction of 2^X onto $C(X)$.

We remark also that there exist smooth dendroids for which g is not continuous and that the analogue of Theorem 3 for g does not hold. The continuum in Example 2 is a smooth dendroid. Let Z denote the line segment joining $(0, 0)$ and $q = (1, 0)$. Then $g(\{q\}) = Z$ and 2^X is connected im kleinen at $g(\{q\})$ (see Theorem 1 of [4]), but g is not continuous at $\{q\}$.

4. Some characterizations of smooth dendroids. If X is a dendroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y. Let $p \in X$. Define a relation $\leqslant_p$ on X by $x \leqslant_p y$ if and only if $x \in [p, y]$. It is easy to verify that $\leqslant_p$ is a partial order, called the *weak cut point order with respect to* p ([6]). X is *smooth* if and only if there exists $p \in X$ such that $\leqslant_p$ is closed ($\{(x, y|\ x \leqslant y\}$ is a closed subset of $X \times X$). If $\leqslant_p$ is closed, then p is said to be an *initial point* of X. If p is an initial point, then p is a point of local connectivity ([6]).

Let X be a dendroid and $p \in X$. Define f_p: $X \to C(X)$ by $f_p(x) = [p, x]$ and F_p: $2^X \to C(X)$ by $F_p(A) = (\bigcup \{[p, a]|\ a \in A\})^*$. The function f_p is the same as the function η_p defined on page 112 of [7]. The equivalence of statements (1) and (3) in Theorem 8 below was proved in Theorem 1 of [7] (1).

Lemma 3. *The function f_p is lower semi-continuous.*

Proof. Let $F_2(X, p) = \{\{p, x\}|\ x \in X\}$ and h be the natural homeomorphism from X to $F_2(X, p)$. Then $f_p(x) = F|_{F_2(X, p)}(h(x))$. Since the restriction of a lower semi-continuous function is lower semi-continuous, it follows from Lemma 1 that f_p is lower semi-continuous.

Lemma 4. *The function F_p is lower semi-continuous.*

Proof. Let $A \in 2^X$ and V be an open set which meets $F_p(A)$. Then there exists $a \in A$ such that $V \cap [p, a] = V \cap f_p(a) \neq \emptyset$. By Lemma 3, f_p is lower semi-continuous, so there exists an open set U containing a such that $t \in U$ implies $[p, t] \cap V \neq \emptyset$. Let W be an open set containing $A - U$ such that $a \notin W$. Then $A \in N(U, W)$. Furthermore, if $B \in N(U, W)$, then $F_p(B) \cap V \neq \emptyset$, because $B \cap U \neq \emptyset$. Hence F_p is lower semi-continuous at A.

If X is a dendroid and $p \in X$, then each subcontinuum of X has a zero with respect to $\leqslant_p$ (see Ward [10]). Define g_p: $C(X) \to X$ by $g_p(M) =$ zero of M (with respect to $\leqslant_p$). This is the function (with p chosen as an initial point) used by Nadler and Ward [9] in proving that there exists a selection on $C(X)$ when X is a smooth dendroid.

Theorem 8. *Let X be a dendroid and $p \in X$. Then the following statements are equivalent*:

(1) *The point p is an initial point of X.*

(1) The author wishes to thank the referee for this reference.

(2) *For each* $x \in X$, f *is continuous at* $\{p, x\}$.
(3) *The function* f_p *is continuous.*
(4) *The function* F_p *is continuous.*
(5) *The function* g_p *is continuous.*

Proof. (1)⇒(2). Let p be an initial point of X and $x \in X$. Since $[p, x]$ is irreducible between p and x and since $[p, x]$ is unique, $f(\{p, x\}) = [p, x]$. By Lemma 1, it will suffice to show that f is upper semi-continuous at $\{p, x\}$. Let U be an open set containing $[p, x]$ and V be an open set such that $[p, x] \subset V \subset V^* \subset U$. Let $y \in X-V$. Then $y \notin [p, x]$, so $y \not\leqslant x$. Since $\leqslant_p$ is closed, for each $y \in X-V$ there exist disjoint open sets W_y and V_y such that $y \in W_y$ and $x \in V_y \subset V$ and such that if $s \in W_y$ and $t \in V_y$, then $s \not\leqslant t$. Since $X-V$ is compact, there exist $y_1, \ldots, y_n$ such that $X-V \subset \bigcup_{i=1}^{n} W_{y_i}$. Now $x \in \bigcap_{i=1}^{n} V_{y_i} = V_x$. If $t \in V_x$ and $s \in \bigcup_{i=1}^{n} W_{y_i}$, then $s \not\leqslant t$, so

$$[p, t] \cap \bigcap_{i=1}^{n} W_{y_i} = \emptyset .$$

Hence

$$f(\{p, t\}) = [p, t] \subset V .$$

Now p is a point of local connectivity. Let V_p be a connected open set containing p such that $V_p \subset V$. Then $\{p, x\} \in N(V_p, V_x)$. Let $A \in N(V_p, V_x)$. For each $a \in A \cap V_x$, $[p, a] \subset V$. So $(\bigcup \{[p, a] |\ a \in A \cap V_x\})^*$ is a continuum containing p and $A \cap V_x$ which lies in V^*. Also V_p^* is a continuum containing p and $A \cap V_p$ which lies in V^*. So $V_p^* \cup (\bigcup \{[p, a] |\ a \in A \cap V_x\})^*$ is a continuum containing A which lies in V^*. Since $V^* \subset U$, it follows that $f(A) \subset U$. Hence f is upper semi-continuous at $\{p, x\}$.

(2)⇒(3). Let $p \in X$ such that for each $x \in X$, f is continuous at $\{p, x\}$. Let $F_2(X, p) = \{\{p, x\} |\ x \in X\}$ and h be the natural homeomorphism from X to $F_2(X, p)$. Then $f_p(x) = f|_{F_2(X,p)}(h(x))$. Since $f|_{F_2(X,p)}$ is a continuous function, it follows that f_p is continuous.

(3)⇒(4). Let $p \in X$ such that f_p is continuous. By Lemma 4 it will suffice to show that F_p is upper semi-continuous. Let $A \in 2^X$ and V be an open set containing $F_p(A)$. Let W be an open set such that $F_p(A) \subset W \subset W^* \subset V$. Since f_p is upper semi-continuous, for each $a \in A$ there exists an open set U_a containing a such that $t \in U_a$ implies $f_p(a_t) = [p, a_t] \subset W$. Since A is compact, there exist $a_1, \ldots, a_n$ such that $A \subset \bigcup_{i=1}^{n} U_{a_i}$. Then $A \in N(U_{a_1}, \ldots, U_{a_n})$. Let $B \in N(U_{a_1}, \ldots, U_{a_n})$. If $b \in B$, then for some i, $b \in U_{a_i}$, so $f_p(b) = [p, b] \subset W$. Hence $F_p(B) = (\bigcup \{[p, b] |\ b \in B\})^* \subset W^* \subset V$. It follows that F_p is upper semi-continuous.

(4)⇒(1). Let $p \in X$ such that F_p is continuous. Let $F_1(X) = \{\{x\} |\ x \in X\}$. Then $F_p|_{F_1(X)}$ is continuous. Let h be the natural homeomorphism from X to $F_1(X)$. Then $f_p(x) = F_p|_{F_1(X)}(h(x))$, so f_p is continuous. Let $x, y \in X$ such that $x \not\leqslant y$. Then $x \notin [p, y]$. Let U and V be disjoint open sets such that $x \in U$ and $[p, y] \subset V$. Since f_p is upper semi-continuous, there exists an open set W containing y such that $Z \in W$ implies $[p, z] \subset V$. So if $s \in U$ and $t \in W$, $s \not\leqslant t$, because $[p, t] \subset V$ and $U \cap V = \emptyset$. It follows that $\leqslant_p$ is closed. Hence p is an initial point of X.

(1)⇒(5). Let p be an initial point of X. Let $M \in C(X)$ and $g_p(M) = m$. Let U be an open set containi g m. If $x \in M-U$, $x \not\leqslant m$, and since $\leqslant_p$ is closed, for each $x \in M-U$ there exist disjoint open sets U_x and V_x such that $m \in U_x \subset U$ and $x \in V_x$ and such that if $s \in V_x$ and $t \in U_x$, then $s \not\leqslant t$. Since $M-U$ is compact, there exist $x_1, \ldots, x_n$ such that $M-U \subset \bigcup_{i=1}^{n} V_{x_i}$. Then $m \in \bigcap_{i=1}^{n} U_{x_i} = W$. Note that $M \in N(U, W, V_{x_1}, \ldots, V_{x_n})$. Let $K \in N(U, W, V_{x_1}, \ldots, V_{x_n})$ and $y \in K \cap W$. If $s \in \bigcup_{i=1}^{n} V_{x_i}$, $s \not\leqslant y$, so $g_p(K) \notin \bigcup_{i=1}^{n} V_{x_i}$. Since $g_p(K) \in K$ and $K \subset U \cup W \cup (\bigcup_{i=1}^{n} V_{x_i})$, it follows that $g_p(K) \in U \cup W$. But $W \subset U$. Hence $g_p(K) \in U$, so g_p is continuous at M.

(5)⇒(1). Suppose p is not an initial point of X. Then $\leqslant_p$ is not closed, so there exists a sequence $(x_n, y_n) \in X \times X$ converging to (x, y) such that for each positive integer n, $x_n \leqslant y_n$, but $x \not\leqslant y$. Since $C(X)$ is closed in 2^X, the sequence $\{[x_n, y_n]\}_{n=1}^{\infty}$ in $C(X)$ has a subsequence $\{[x_{n_k}, y_{n_k}]\}_{k=1}^{\infty}$ which converges to some point $M \in C(X)$. Then $x, y \in M$. Let $g_p(M) = z$. Then $z \neq x$, since $x \not\leqslant y$. Let U be an open set containing z such that $x \notin U^*$. Then there exists a positive integer N_1 such that $k \geqslant N_1$ implies $x_{n_k} \notin U$. Let $N(V_1, \ldots, V_m)$ be an open set containing M. Then there exists a positive integer N_2 such that $k \geqslant N_2$ implies $[x_{n_k}, y_{n_k}] \in N(V_1, \ldots, V_m)$. Let $N = \max\{N_1, N_2\}$. Then for $k \geqslant N$, $[x_{n_k}, y_{n_k}] \in N(V_1, \ldots, V_m)$ but $g_p([x_{n_k}, y_{n_k}]) = x_{n_k} \notin U$. Hence g_p is not continuous at M.

Charatonik and Eberhart [2] have shown that if X is a dendrite, then each point of X is an initial point. Let X be a dendrite. Then for each $p \in X$ we have the following diagram:

$$2^X \overset{f}{\to} C(X) \overset{g_p}{\to} X .$$

The composition $g_p \circ f = h_p$: $2^X \to X$ is continuous and has the property that $h_p(A)$ is the least element with respect to p of the minimal continuum containing A. So h_p is "close" to being a selection. Furthermore, the restriction of h_p to $C(X)$ is a selection.

References

[1] J. J. Charatonik, *Two invariants under continuity and the incomparability of fans*, Fund. Math. 53 (1964), pp. 187–204.
[2] — and C. A. Eberhart, *On smooth dendroids*, Fund. Math. 67 (1970), pp. 297–322.
[3] J. Dugundji, *Topology*, Boston 1966.
[4] J. T. Goodykoontz, Jr., *Connectedness im kleinen and local connectedness in* 2^X *and* $C(X)$, Pacific J. Math. 53 (1974), pp. 387–397.
[5] J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. 52 (1942), pp. 23–36.

[6] R. J. Koch and I. S. Krule, *Weak cutpoint ordering on hereditarily unicoherent continua*, Proc. Amer. Math. Soc. 11 (1960), pp. 679–681.
[7] L. Lum, *Weakly smooth dendroids*, Fund. Math. 83 (1974), pp. 111–120.
[8] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), pp. 152-182.
[9] S. B. Nadler, Jr. and L. E. Ward, Jr., *Concerning continuous selections*, Proc. Amer. Math. Soc. 25 (1970), pp. 369–374.
[10] L. E. Ward, Jr., *A fixed point theorem for multi-valued functions*, Pacific J. Math. 8 (1958), pp. 921–927.
[11] G. T. Whyburn, *Analytical Topology*, Amer. Math. Soc. Colloquium Publications, Vol. 28 (1942), New York.

UNIVERSITY OF KENTUCKY
and
WEST VIRGINIA UNIVERSITY

Accepté par la Rédaction le 28. 10. 1974

Emploi des filtres sur N dans l'étude descriptive des fonctions

par

Maryvonne Daguenet (Paris)

Résumé. Etude des premières propriétés du préordre entre les filtres défini par $\mathcal{F} \leqslant \mathcal{G}$ si — $\mathcal{F}$ étant un filtre sur I et $\mathcal{G}$ un filtre sur J — il existe une application $h: J \to I$ telle que $\mathcal{F} \subset h(\mathcal{G})$, et étude des positions relatives de certains filtres simples: $\mathcal{N}^\alpha$, $\mathcal{M}$. Ce préordre çi dessus est une extension du préordre entre les ultrafiltres appelé préordre de Rudin-Keisler. Il a été introduit par M. Katětov afin de classifier les fonctions discontinues [7], [8].

Nous retrouvons, au moyen de comparaison avec ces filtres simples $\mathcal{M}$ et $\mathcal{N}^\alpha$, des classes d'ultrafiltres aux propriétés combinatoires intéressantes déjà étudiées, voir M. Choquet [4]. Il semble que l'extension aux filtres de qualités définies pour les ultrafiltres, obtenue de cette façon soit significative pour l'étude des fonctions, d'après la forme des résultats qu'elle permet d'obtenir, et d'après les questions qu'elle permet de poser.

Le présent article est principalement destiné à répondre à trois questions de M. Katětov posées en [8]; mais nous avons nettement séparé la partie combinatoire concernant l'étude du préordre afin de rendre sa lecture indépendante de l'application à l'étude des fonctions, objet de [8], et içi des §§ 0 et 4.

Introduction. M. Katětov donne en [7] et [8] une classification des fonctions et une caractérisation de certaines classes de fonctions au moyen des filtres sur l'ensemble des entiers. Nous répondons ici à trois des problèmes combinatoires posés par lui en [8], problèmes approchant la question centrale: "De quelle façon cette classification à l'aide des filtres, simplifie, étend et précise la classification partielle en les classes de Baire d'ordre α".

Pour simplifier le travail du lecteur, nous donnons d'abord un résumé de la partie du travail de Katětov utilisée ici, et ce faisant, nous introduisons les notations et définitions dont nous avons besoin. Ceci sera l'objet du § 0 de cet article.

La suite traite des filtres. Ce travail permet de répondre à des questions posées en [8], mais présente aussi un intérêt en lui-même: le préordre combinatoire entre les filtres — que nous appelons préordre de Katětov — qui y est étudié, est une extension de l'ordre combinatoire entre les ultrafiltres connu sous le nom de préordre de Rudin–Keisler. Nous démontrons (i) que l'ordre associé au préordre de Katětov n'est pas linéaire, ce indépendamment de l'axiome du choix et de l'hypothèse du continu; (ii) que les filtres $\mathcal{N}^\alpha$ — que nous définirons plus loin; ils sont essentiels dans l'étude de la classe des fonctions de Baire d'ordre α — possèdent