In particular, we have now established

THEOREM 3.1. *Suppose* $\mathfrak{M}$ *is* $(P^+)^+$-*generic. Then* $\mathfrak{M}$ *has Scott height at most* $o(P^+)$.

An absoluteness argument shows that it is not really necessary to assume that A is countable.

The example of the previous section shows that a $(P^+)^+$-generic structure need not be P^+-^{s}generic, nor even of height $\leqslant o(P^+)$. If we assume that our original theory T is complete, it is then clear that all $(P^+)^+$-generic structures have the same Scott height.

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Homogeneity, universality and saturatedness of limit reduced powers III

by

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Abstract. Let $\mathscr{F}$ be an ultrafilter on I and $\mathscr{G}$ a filter over $I\times I$. The paper gives a characterization of those pairs $(\mathscr{F}, \mathscr{G})$ which have the property that for every relational structure $\mathfrak{A}$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}$ is $\varkappa^+$-saturated. The notion used to obtain this characterization is a natural extension of Keisler's notion of a $\varkappa$-good filter.

A property **P** of a relational structure $\mathfrak{A}$ is a compactness type property if there is a definition of **P** which is of the form: for every set Σ of formulae (of some language connected with $\mathfrak{A}$), Σ can be satisfied in $\mathfrak{A}$ if and only if every finite subset of Σ can be satisfied in $\mathfrak{A}$. The saturatedness, universality and homogeneity of relational structures can be considered as properties of the compactness type. Various other properties of the compactness type have been investigated by several authors (e.g. atomic compactness [6], [11], positive compactness [11]). Here we restrict ourselves to saturatedness, homogeneity and universality.

By the classical results of Keisler ([3], [4]) ultraproducts can be used to obtain structures with a given compactness type property. For example, if a filter $\mathscr{F}$ is $(\omega, \varkappa)$-regular, then for every relational structure $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\varkappa$ the ultrapower $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa^+$-universal. If $\mathscr{F}$ is $\varkappa$-good, then for every family $\{\mathfrak{A}_i\colon i\in I\}$ of similar relational structures with $|L(\mathfrak{A}_i)|\leqslant\varkappa$ the ultraproduct $\prod_{i\in I}\mathfrak{A}_i/\mathscr{F}$ is $\varkappa^+$-saturated. The results of Keisler have been extended by Shelah and the present author to the case of products which are not necessarily maximal (see [7] and [10]). Another application of reduced products to compactness can be found in [8]. For the generalization of Keisler's results to Boolean ultrapowers see [5].

The problem of homogeneity of reduced products had not been extensively investigated. By a recent result of Wierzejewski [13] if the ultrapower $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa^+$-homogeneous for every structure $\mathfrak{A}$, then for every $\mathfrak{A}$ the ultrapower $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa^+$-saturated.

In the present paper we investigate the problem of compactness of limit ultrapowers. We give a characterization of pairs $(\mathscr{F}, \mathscr{G})$ which have the property that for every relational structure $\mathfrak{A}$ such that $|L(\mathfrak{A})|\leqslant\varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}$ is

$\varkappa^+$-saturated ($\mathcal{F}$ is an ultrafilter on I and $\mathcal{G}$ is a filter over $I\times I$). We also deal with limit ultrapowers which are $\varkappa^+$-universal.

This paper is a by product of an attempt to answer the question of J. Wierzejewski (see the introduction in [13]) whether it is possible to give for homogeneity a characterization similar to that given in [7] (cf. also [10]) for saturatedness.

I would like to mention that the results below would not have been obtained without the encouragement of all the members of the seminar on model theory in Wrocław, especially by B. Węglorz.

An extension of the results below to the case of filters which are not necessarily maximal will be published in [9].

0. Our terminology is standard and coincides with the terminology of [1]. Let I be a non-empty set. Then by $E(I)$ we denote the set of all equivalence relations on I. Let $f\colon I\to A$ be a function. Then $eq(f) = \{(i,j)\colon f(i) = f(j)\}$. Of course $eq(f)\in E(I)$. Let $\mathcal{G}$ be a filter in $E(I)$ (i.e., $\mathcal{G}\subseteq E(I)$; if $\varrho_1, \varrho_2\in\mathcal{G}$, then $\varrho_1\cap\varrho_2\in\mathcal{G}$; if $\varrho_1\subseteq\varrho_2\in E(I)$, $\varrho_1\in\mathcal{G}$, then $\varrho_2\in\mathcal{G}$). If $A\neq 0$, then by $A^I|\mathcal{G}$ we denote $\{f\in A^I\colon eq(f)\in\mathcal{G}\}$. In particular, $2^I|\mathcal{G}$ is the algebra of subsets of I which can be composed of the equivalence classes of a relation in $\mathcal{G}$. If $\varrho\in E(I)$, then $2^I|\varrho$ denotes the algebra of subsets which are unions of equivalence classes of ϱ. Let $\mathcal{G}$ and A be as above and let $\mathcal{F}$ be a filter over I. If $f\in A^I|\mathcal{G}$, then

$$f/\mathcal{F} = \{g\in A^I\colon \{i\in I\colon f(i) = g(i)\}\in\mathcal{F}\}\,.$$

We put $A^I_{\mathcal{F}}|\mathcal{G} = \{f/\mathcal{F}\colon f\in A^I|\mathcal{G}\}$. If $\mathfrak{A}$ is a relational structure, then $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is a substructure of $\mathfrak{A}^I_{\mathcal{F}}$ with the universe $A^I_{\mathcal{F}}|\mathcal{G}$ (see [2]). If I is a set, $i\in I$ and $\varrho\in E(I)$, then

$$i/\varrho = \{j\in I\colon (i,j)\in\varrho\} \quad\text{ and }\quad I/\varrho = \{i/\varrho\colon i\in I\}\,.$$

Let X be a set; then $S(X)$ denotes the set of all subsets of X and $S_\omega(X)$ is the set of all finite subsets of X. Let $f\colon S_\omega(X)\to S(I)$. We say that f is *monotonic* if $s\subseteq t$ implies $f(s)\supseteq f(t)$. A function f is *additive* if $f(s\cup t) = f(s)\cap f(t)$ for every $s, t\in S_\omega(X)$. Let $g\colon S_\omega(X)\to S(I)$. We write $f\leqslant g$ to denote that $f(s)\subseteq g(s)$ for every $s\in S_\omega(X)$. The image of X by f is denoted by f^*X. If A is a set, then $|A|$ is the cardinality of A, and $\varkappa$ is always an infinite cardinal. By $L(\mathfrak{A})$ we denote the language of $\mathfrak{A}$. For other definitions consult [1].

1. Recall that a filter $\mathcal{F}$ is $(\omega,\varkappa)$-regular if and only if there is an $\mathcal{F}_0\subseteq\mathcal{F}$ such that $|\mathcal{F}_0| = \varkappa$ and $\bigcap\mathcal{F}_1 = 0$ for every infinite $\mathcal{F}_1\subseteq\mathcal{F}_0$. $\mathcal{F}$ is $\varkappa$-good if and only if $\mathcal{F}$ is (ω,ω)-regular and for every monotonic function $f\colon S_\omega(\varkappa)\to\mathcal{F}$, there is an additive function $g\colon S_\omega(\varkappa)\to\mathcal{F}$ such that $g\leqslant f$. If $\mathcal{F}$ is a $\varkappa$-good filter, then $\mathcal{F}$ is $(\omega,\varkappa)$-regular (see [4]). It is possible to give a definition of $\varkappa$-goodness in which $(\omega,\varkappa)$-regularity is explicitly stated.

DEFINITION 1.1.

1. Let $h\colon S_\omega(\varkappa)\to S(I)$. We say that h is a *partition function* if $s\neq t$ implies $h(s)\cap h(t) = 0$ and moreover $\bigcup_{s\in S_\omega(\varkappa)} h(s) = I$.

2. Let $h\colon S_\omega(\varkappa)\to S(I)$. Then g is a *union function* of h if $g(s) = \bigcup_{t\supseteq s} h(t)$. The union function of h is denoted by u_h.

3. $\mathcal{F}$ is $(\omega,\varkappa)$-*regular** if there is a partition function $h\colon S_\omega(\varkappa)\to S(I)$ such that for $s\in S_\omega(\varkappa)$, $u_h(s)\in\mathcal{F}$.

4. $\mathcal{F}$ is $\varkappa$-*good** if for every monotonic function $f\colon S_\omega(\varkappa)\to\mathcal{F}$ there is a partition function $h\colon S_\omega(\varkappa)\to S(I)$ such that $u_h(s)\in\mathcal{F}$ for $s\in S_\omega(\varkappa)$ and $u_h\leqslant f$.

It is obvious that if $\mathcal{F}$ is $\varkappa$-good*, then it is also $(\omega,\varkappa)$-regular*.

PROPOSITION 1.2.

1. *$\mathcal{F}$ is $(\omega,\varkappa)$-regular if and only if $\mathcal{F}$ is $(\omega,\varkappa)$-regular*.*

2. *$\mathcal{F}$ is $\varkappa$-good if and only if $\mathcal{F}$ is $\varkappa$-good*.*

Proof. 1. Assume that $\mathcal{F}$ is $(\omega,\varkappa)$-regular. Then there is a family $\{I_\alpha\colon \alpha<\varkappa\}$ of distinct elements of $\mathcal{F}$ such that for every infinite subset X of $\varkappa$ we have $\bigcap_{\alpha\in X} I_\alpha = 0$. We put $h(s) = \{i\colon i\in I_\alpha\leftrightarrow\alpha\in s\}$. It is obvious that h is a partition function. Moreover $u_h(s) = \bigcap_{\alpha\in s} I_\alpha$, whence $u_h(s)\in\mathcal{F}$.

Now assume that $\mathcal{F}$ is $(\omega,\varkappa)$-regular*. Since $u_h(s)\in\mathcal{F}$ for every $s\in S_\omega(\varkappa)$, it remains to prove that if X is an infinite subset of $S_\omega(\varkappa)$, then $\bigcap_{s\in X} u_h(s) = 0$. In fact, assume that $i\in\bigcap_{s\in X} u_h(s)$; then there is a $t\in S_\omega(\varkappa)$ such that $i\in h(t)$ and $t\supseteq s$ for all $s\in X$. But $\bigcup_{s\in X} s$ is infinite, whence t is infinite, which is impossible since $t\in S_\omega(\varkappa)$.

2. Assume that $\mathcal{F}$ is $\varkappa$-good. Then by a theorem of Keisler [3] $\mathcal{F}$ is $(\omega,\varkappa)$-regular. Let $\{I_s\colon s\in S_\omega(\varkappa)\}$ be a family of distinct elements of $\mathcal{F}$ such that $\bigcap_{s\in X} I_s$ is empty for every infinite X, $X\subseteq S_\omega(\varkappa)$. Let $f\colon S_\omega(\varkappa)\to\mathcal{F}$ and let $g\colon S_\omega(\varkappa)\to\mathcal{F}$ be such that $g\leqslant f$. We put $u(s) = g(s)\cap I_s$ and $h(s) = \{i\colon i\in u(s)\leftrightarrow\alpha\in s\}$. It is a matter of simple computation to check that $u = u_h$ and consequently $u_h\leqslant f$ and $u_h(s)\in\mathcal{F}$ for $s\in S_\omega(\varkappa)$.

DEFINITION 1.3. Let $\mathcal{F}\subseteq S(I)$ and $\mathcal{G}\subseteq E(I)$ be arbitrary filters.

1. We say that the pair $(\mathcal{F},\mathcal{G})$ is $(\omega,\varkappa)$-*regular* if there is a $\varrho\in\mathcal{G}$ and a function $h\colon S_\omega(\varkappa)\to I/\varrho$ such that h is a partition function and $u_h(s)\in\mathcal{F}$ for every $s\in S_\omega(\varkappa)$.

2. Let $h\colon S_\omega(\varkappa)\to S(I)$ be a partition function and let $F\colon S_\omega(\varkappa)\to E(I)$. Then $h*F$ is an equivalence relation on I such that $(i,j)\in h*F$ if and only if $i\in h(s)\leftrightarrow j\in h(s)$ holds for every $s\in S_\omega(\varkappa)$ and moreover if $i\in h(t)$ for some $t\in S_\omega(\varkappa)$, then $(i,j)\in F(t)$.

3. The pair $(\mathcal{F},\mathcal{G})$ is $\varkappa$-*good** if and only if for every additive function $F\colon S_\omega(\varkappa)\to\mathcal{G}$ and every monotonic function $f\colon S_\omega(\varkappa)\to\mathcal{F}$ such that

$$f^*S_\omega(\varkappa)\subseteq 2^I|\mathcal{G} \tag{g.0}$$

there is a partition function $h\colon S_\omega(\varkappa)\to S(I)$ such that

(*.1) $$u_h \leqslant f,$$

(*.2) $$u_h(s)\in\mathscr{F} \quad \text{for every } s\in S_\omega(\varkappa)$$

and

(*.3) $$h*F\in\mathscr{G}.$$

It follows from the definition that if a pair $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good*, then it is $(\omega,\varkappa)$-regular. To check this take $F(s) = I\times I$ and $f(s) = I$.

It is possible to give a definition of $\varkappa$-goodness of a pair of filters which is more similar to the original definition of $\varkappa$-goodness of a filter.

DEFINITION 1.4. The pair $(\mathscr{F},\mathscr{G})$ is *$\varkappa$-good* if and only if it is (ω,ω)-regular and for every monotonic function $f\colon S_\omega(\varkappa)\to\mathscr{F}$ and every additive function $F\colon S_\omega(\varkappa)\to\mathscr{G}$ such that $f^*S_\omega(\varkappa)\subseteq 2^I|\mathscr{G}$ there is a $\varrho\in\mathscr{G}$ and an additive function $g\colon S_\omega(\varkappa)\to\mathscr{F}$ such that

(g.1) $$g\leqslant f$$

(g.2) $$g^*S_\omega(\varkappa)\subseteq 2^I|\varrho$$

and

(g.3) for every $x\in I/\varrho$ and every $i,j\in x$ if $x\subseteq g(s)$, then $(i,j\in F(s)$ (i.e., relation ϱ is on $g(s)$ finer than $F(s)$).

LEMMA 1.5. *If $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good, then $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good*.*

Proof. Let $h_0\colon S_\omega(\omega)\to S(I)$ be a function whose existence follows from the (ω,ω)-regularity of $(\mathscr{F},\mathscr{G})$. Let $\varrho_0\in\mathscr{G}$ be such that $h_0^*(S_\omega(\omega)) = I/\varrho_0$. Let F and f be functions as in the definition of $\varkappa$-goodness*. For $s\in S_\omega(\varkappa)$ we put $F_0(s) = F(s)\cap\varrho_0$ and $f_0(s) = f(s)\cap u_{h_0}(\bar{s})$, where $\bar{s} = \{0,1,2,\ldots,|s|-1\}$. Since f_0 and F_0 satisfy the hypotheses of Definition 1.4, there are $\varrho\in\mathscr{G}$ and $g\colon S_\omega(\varkappa)\to F_0$ such that (g.1)-(g.3) hold (with f, F replaced by f_0, F_0). Let

$$h(s) = \{i\in I\colon i\in g(\{\alpha\})\leftrightarrow\alpha\in s\}.$$

Of course h is a partition function. Now let $i\in g(s)$. We claim that there is a $t\supseteq s$ such that $i\in g(t)$ and $i\notin g(t')$ for every $t'\supsetneq t$. Suppose not; then there is an infinite sequence $\{t_n\}_{n<\omega}$ such that $i\in g(t_n)$ and $t_n\subseteq t_{n+1}$ for every $n<\omega$. Since $g(s)\subseteq f_0(s)$, we have $i\in\bigcap_{n<\omega} f_0(t_n)$, whence $i\in\bigcap_{n<\omega} u_{h_0}(t_n)$, which is impossible. We have just proved that for every $i\in g(s)$ there is a $t\supseteq s$ such that $i\in g(\{\alpha\})\leftrightarrow\alpha\in t$, whence $i\in h(t)$. This proves that $g(s) = \bigcup_{t\supseteq s} h(t) = u_h(s)$. But $g(s)\in\mathscr{F}$, and consequently $u_h(s)\in\mathscr{F}$. Moreover, since $g(s) = u_h(s)$, we have $u_h(s)\subseteq f_0(s)\subseteq f(s)$.

It remains to prove that $h*F\in\mathscr{G}$. To do this we shall check that $h*F\supseteq\varrho$. In fact, let $(i,j)\in\varrho$; then for every $s\in S_\omega(\varkappa)$ we have $i\in h(s)\leftrightarrow j\in h(s)$, whence for arbitrary $s\in S_\omega(\varkappa)$ every equivalence class of ϱ is disjoint with $h(s)$ or included in $h(s)$. Now let $i,j\in x$ and $x\in I/\varrho$. Then there is an $s\in S_\omega(\varkappa)$ such that $x\subseteq h(s)$, in particular $x\subseteq g(s)$. Consequently by the definition of $\varkappa$-goodness we have $(i,j)\in F(s)$, which finishes the proof that $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good*.

THEOREM 1.6. *$(\mathscr{F},\mathscr{G})$ is $\varkappa$-good if and only if $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good*.*

Proof. If $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good, then it is $\varkappa$-good* by Lemma 1.5. If $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good*, then puting $\varrho = h*F$ and $g(s) = u_h(s)$ we obtain an equivalence relation and a function which satisfy properties (g.1)-(g.3).

2. Now we are ready to state and prove the main results of the paper. The necessity of the assumptions of $\varkappa$-goodness in the theorem below will be proved in the next section.

THEOREM 2.1. *If $\mathscr{F}\subseteq S(I)$ is an ultrafilter and $\mathscr{G}\subseteq E(I)$ is a filter such that $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good, then for every relational structure $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}$ is $\varkappa^+$-saturated.*

Proof. Let $\mathfrak{A}$ be a relational structure with $|L(\mathfrak{A})|\leqslant\varkappa$. Let $\{a_\xi\colon \xi<\varkappa\}$ be a sequence of elements of $A^I|\mathscr{G}$ and finally let Σ denote a set (of power $\leqslant\varkappa$) of elements of $L(\varkappa)$ with one free variable v. We assume that Σ is finitely satisfiable in $\mathfrak{B} = (\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}, a_\xi/\mathscr{F})_{\xi<\varkappa}$. For $s\in S_\omega(\Sigma)$ and $i\in I$ we put $F(s) = \bigcap\{eq(a_\xi)\colon a_\xi/\mathscr{F}$ appears in $\bigwedge s\}$, $\mathfrak{A}_i = (\mathfrak{A}, a_\xi(i))_{\xi<\varkappa}$ and $f(s) = \{i\in I\colon \mathfrak{A}_i\models\exists v\bigwedge s\}$. It is obvious that F is an additive function and f is a monotonic function. Moreover, for every $s\in S_\omega(\Sigma)$ we have $\mathfrak{B}\models\exists v\bigwedge s$; consequently $f(s)\in\mathscr{F}$. Finally (g.0) follows from the fact that if $a_\xi/\mathscr{F}$ appears in $\bigwedge s$, then a_ξ is constant on every equivalence class of $F(s)$. Now, by Theorem 1.6 $(\mathscr{F},\mathscr{G})$ is $\varkappa$-good*, whence there is a function h which satisfies (*.1)-(*.3). Let $i\in h(s)$. Then, since $h(s)\subseteq u_h(s)\subseteq f(s)$, we have $i\in f(s)$. Hence by the definition of f we get $\mathfrak{A}_i\models\exists v\bigwedge s$. Let x be an equivalence class of $h*F$ such that $i\in x$. Then every function a_ξ is constant on x provided $a_\xi/\mathscr{F}$ appears in $\bigwedge s$. Consequently, there is an $a_x\in A$ such that for every $i\in x$, $\mathfrak{A}_i\models\bigwedge s[a_x]$ holds (if $x = h(0)$ then a_x is an arbitrary element of A). We put $a(i) = a_z$ where z is an element of $I/(h*F)$ such that $i\in z$. By definition a is constant on every equivalence class of $h*F$, whence $a\in A^I|\mathscr{G}$. We claim that $(\mathfrak{B}, a/\mathscr{F})\models\Sigma$. In fact, let $\sigma\in\Sigma$, and $E_\sigma = \{i\colon \mathfrak{A}_i\models\sigma[a(i)]\}$. Then

$$E_\sigma = \bigcup_{\sigma\in t\in S_\omega(\Sigma)}\{i\colon \mathfrak{A}_i\models\bigwedge t[a(i)] \text{ and } i\in h(t)\}.$$

But it follows from the definition of a that if $i\in h(t)$, then $\mathfrak{A}_i\models\bigwedge t[a(i)]$, whence

$$E_\sigma\supseteq\bigcup_{\sigma\in t\in S_\omega(\Sigma)} h(t) = u_h(t)\in\mathscr{F}.$$

THEOREM 2.2. *If $\mathscr{F}\subseteq S(I)$ is an ultrafilter and $\mathscr{G}\subseteq E(I)$ is a filter such that $(\mathscr{F},\mathscr{G})$ is $(\omega,\varkappa)$-regular, then for every structure $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}$ is $\varkappa^+$-universal.*

Proof. We proceed almost exactly as in the proof of Theorem 1.5 in [4]. The only difference is that for $s\in S_\omega(\varkappa)$ we ~~keep~~ defined functions constant on $h(s)$.

Theorem 2.2 was obtain independently by B. Węglorz as a corollary to his embedding theorem (see [12]).

Now we shall give an application of Theorem 2.2 to the problem of homogeneity of limit ultrapowers.

THEOREM 2.3. *If for every relational structure $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\omega$ the limit reduced power $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-homogeneous, then the pair $(\mathcal{F},\mathcal{G})$ is $(\omega,\varkappa)$-regular.*

Proof. Let $B_\varkappa = S_\omega(\varkappa+\varkappa)\cup S^\omega(\varkappa+\varkappa)$, $C_\varkappa = S(\varkappa+\varkappa)\times\{0\}$ and $A_\varkappa = B_\varkappa\cup C_\varkappa$ (here $+$ denotes the addition of ordinals). For $\alpha<\varkappa$ we put $a_\alpha = \{\alpha\}$, $b_\alpha = \varkappa+\varkappa-\{\varkappa+\alpha\}$, $c_\alpha = (\{\alpha\},0)$ and $d_\alpha = (\varkappa+\varkappa-\{\varkappa+\alpha\},0)$. Moreover, $c_\varkappa = (\varkappa,0)$. Now assume that for every relational structure $\mathfrak{A}$ the limit reduced power $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-homogeneous. Hence in particular $\mathfrak{A}^* = \mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-homogeneous, where $\mathfrak{A} = \langle A_\varkappa,\subseteq\rangle$. If $a\in A_\varkappa$, then by $\bar{a}$ we denote the element of $(A_\varkappa)^I|\mathcal{G}$ such that $\bar{a}(i) = a$ for $i\in I$. Since every infinite atomic Boolean algebras are elementary equivalent, we have

$$(\mathfrak{A}^*,\bar{a}_\alpha/\mathcal{F},\bar{b}_\alpha/\mathcal{F})_{\alpha<\varkappa}\equiv(\mathfrak{A}^*,\bar{c}_\alpha/\mathcal{F},\bar{d}_\alpha/\mathcal{F})_{\alpha<\varkappa}\,.$$

Now since $\mathfrak{A}^*$ is $\varkappa^+$-homogeneous, there is an $a\in(A_\varkappa)^I|\mathcal{G}$ such that

$$(1)\qquad (\mathfrak{A}^*,\bar{a}_\alpha/\mathcal{F},\bar{b}_\alpha/\mathcal{F},a/\mathcal{F})\equiv(\mathfrak{A}^*,\bar{c}_\alpha/\mathcal{F},\bar{d}_\alpha/F,\bar{c}_\varkappa/\mathcal{F})\,.$$

Let

$$h_1(s) = \{i\in I\colon a(i)\supseteq a_\alpha\leftrightarrow\alpha\in s\}\quad\text{and}\quad h_2(s) = \{i\in I\colon a(i)\subseteq b_\alpha\leftrightarrow\alpha\in s\}\,.$$

Finally $h(s) = h_1(s)\cup h_2(s)$. Of course h is a partition function and $u_h(s) = \{i\in I\colon a_\alpha\subseteq a(i)\subseteq b_\alpha\}$, whence by (1) $u_h(s)\in\mathcal{F}$. Moreover, since $a\in(A_\varkappa)^I_{\mathcal{F}}|\mathcal{G}$ and all a_α's and b_α's are constant, the equivalence relation ϱ defined by $I/\varrho = \{h(s)\colon s\in S_\omega(\varkappa)\}$ is an element of $\mathcal{G}$.

As a corollary to Theorems 2.2 and 2.3 we get the following theorem of J. Wierzejewski [13].

THEOREM 2.4. *If for any relational structure $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\omega$ the limit ultrapower $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-homogeneous, then for every $\mathfrak{A}$ such that $|L(\mathfrak{A})|\leqslant\varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-saturated.*

Proof. By Theorem 2.3 $(\mathcal{F},\mathcal{G})$ is $(\omega,\varkappa)$-regular, whence by Theorem 2.1 $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-universal; whence $\varkappa^+$-saturated.

THEOREM 2.5. *If for every $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\omega$, $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is ω-homogeneous, then $(\mathcal{F},\mathcal{G})$ is (ω,ω)-regular.*

Proof. Let η denote the set of rationals. We consider the structure $\mathfrak{A} = \langle\eta,<,1/n\rangle_{n<\omega}$. Let $a(i) = 0$ and $b(i) = -1$ for $i\in I$. Then $(\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G},\bar{a}/\mathcal{F})\equiv(\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G},\bar{b}/\mathcal{F})$ and since $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is ω-homogeneous, there is a $c\in A^I|\mathcal{G}$ such that

$$(\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G},a/\mathcal{F},c/\mathcal{F})\equiv(\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G},b/\mathcal{F},a/\mathcal{F})\,.$$

We put $h(s) = \{i\in I\colon c_i\leqslant 1/n\leftrightarrow n\in s\}$ and then proceed as in the proof of Theorem 2.4.

3. Now we shall prove that the assumption in Theorems 2.1 and 2.2 are necessary.

THEOREM 3.1. *If for every $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-universal, then the pair $(\mathcal{F},\mathcal{G})$ is $(\omega,\varkappa)$-regular.*

Proof. Let $\mathfrak{A} = \langle S_\omega(\varkappa),\subseteq,\{\alpha\}\rangle_{\alpha<\varkappa}$ and let $\Sigma = \{\{\alpha\}\subseteq v\}_{\alpha<\varkappa}$. Since Σ is finitely satisfiable in $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ and since $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-universal, there is an element $a\in A^I|\mathcal{G}$ such that $a/\mathcal{F}$ satisfies Σ in $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$. We put $h(s) = \{i\in I\colon a(i) = s\}$.

The proof above is a slight modification of a proof given by Keisler in [4]. Also the proof of the theorem below is based on an idea of Keisler's (see [3]).

THEOREM 3.2. *If for every $\mathfrak{A}$ such that $|L(\mathfrak{A})|\leqslant\varkappa$ the limit reduced power $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-saturated, then $(\mathcal{F},\mathcal{G})$ is $\varkappa$-good.*

Proof. The proof we are going to present is divided into several steps. From now on we assume that for every $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\varkappa$ the limit reduced power $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-saturated. Let f and F be arbitrary but fixed functions which satisfy the hypotheses of the definition of $\varkappa$-goodness.

3.2.1. *There is a $\varrho_1\in\mathcal{G}$ and a function $d\colon\varkappa\to\mathcal{F}$ such that*

$$(1.\text{a})\qquad d(\alpha)\in 2^I|\mathcal{G}$$

$$(1.\text{b})\qquad \textit{if } y\in I/\varrho_1 \textit{ and } i,j\in d(\alpha)\cap y \textit{ then } (i,j)\in F(\{\alpha\}).$$

Proof. Let $\mathfrak{B} = \langle S(I),\subseteq,\neq\rangle$. For $\alpha<\varkappa$ we define

$$b_\alpha(i) = \{j\in I\colon (i,j)\in F(\{\alpha\})\}$$

and we put $\Sigma = \{v\subseteq b_\alpha/\mathcal{F}\}_{\alpha<\varkappa}\cup\{v\neq 0\}$. Since $F(s)\in\mathcal{G}$ for every $s\in S_\omega(\varkappa)$ and since F is additive, Σ is finitely satisfiable in $\mathfrak{B}^I_{\mathcal{F}}|\mathcal{G}$. By the hypotheses $\mathfrak{B}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-saturated, whence there is an element b of $B^I|\mathcal{G}$ such that $b/\mathcal{F}$ satisfies Σ in $\mathfrak{B}^I_{\mathcal{F}}|\mathcal{G}$. We put $\varrho_1 = eq(b)$ and $d(\alpha) = \{i\in I\colon 0\neq b(i)\subseteq b_\alpha(i)\}$. Of course $d(\alpha)\in\mathcal{F}$ and $\varrho_1\in\mathcal{G}$. Now let $\varrho(\alpha) = \varrho_1\cap F(\{\alpha\})$. Since b is constant on every equivalence class of ϱ_1 and b_α is constant on every equivalence class of $F(\{\alpha\})$, the logical value of $0\neq b(i)\subseteq b_\alpha(i)$ is constant on every equivalence class of $\varrho(\alpha)$. But $\varrho(\alpha)\in\mathcal{G}$, whence (1.a) holds. Now assume that $y\in I/\varrho_1$ and $i,j\in y\cap d(\alpha)$. Then, since $i,j\in y$, we have $b(i) = b(j)$; moreover, since $i,j\in d(\alpha)$, we have $0\neq b(i)$, $b(i)\subset b_\alpha(i)$ and $b(j)\subseteq b_\alpha(j)$. This proves that $b_\alpha(i)$ and $b_\alpha(j)$ are equivalence classes of $F(\{\alpha\})$ and have non-empty intersection. Hence $b_\alpha(i) = b_\alpha(j)$ and consequently (1.b) holds.

3.2.2. *Assume that $f_0\colon S_\omega(\varkappa)\to\mathcal{F}$ is a monotonic function such that $f_0(s)\in 2^I|\mathcal{G}$ for every $s\in S_\omega(\varkappa)$. Then there is a $\varrho_2\in\mathcal{G}$ and $f_1\colon S_\omega(\varkappa)\to S(I)$ such that, for every $s\in S_\omega(\varkappa)$,*

$$(2.\text{a})\qquad f_1(s)\in 2^I|\varrho_2\,,\quad f_1(s)\subseteq f_0(s)\,,\quad f_1(s)\in\mathcal{F}$$

and moreover

$$(2.\text{b})\qquad |\{s\in S_\omega(\varkappa)\colon i\in f_1(s)\}|<\omega\quad \textit{for every } i\in I\,.$$

Proof. Let $\mathfrak{S} = \langle S_\omega(S_\omega(\varkappa)) \cup \{S_\omega(\varkappa)\}, \subseteq, \neq \rangle$. For $s \in S_\omega(\varkappa)$ and $i \in I$ we put

$$c_s(i) = \begin{cases} \{s\} & \text{if} \quad i \in f_0(s), \\ S_\omega(\varkappa) & \text{if} \quad i \notin f_0(s) \end{cases}$$

and

$$c_\varkappa(i) = S_\omega(\varkappa).$$

Let $\Sigma = \{v \supseteq c_s/\mathcal{F}\} \cup \{v \neq c_\varkappa/\mathcal{F}\}$. We claim that Σ is finitely satisfiable in $\mathfrak{S}^I_{\mathcal{F}}|\mathcal{G}$. In fact, if Σ_0 is a finite subset of Σ and $t = \bigcup \{s\colon c_s \text{ appears in } \Sigma_0\}$, then putting $c'(i) = S(t)$ we obtain an element of $\mathfrak{S}^I_{\mathcal{F}}|\mathcal{G}$ which satisfies Σ_0.

Now, since $\mathfrak{S}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-saturated, there is an element c of $\mathfrak{S}^I|\mathcal{G}$ such that $c/\mathcal{F}$ satisfies Σ. Let $\varrho_2 = eq(c)$. Of course $\varrho_2 \in \mathcal{G}$. Since $c/\mathcal{F}$ satisfies Σ, we have

(2.c) $$E_s = \{i \in I\colon c_s(i) \subseteq c(i) \neq S_\omega(\varkappa)\} \in \mathcal{F}.$$

But

(2.d) $$E_s = \bigcup \{\{i \in I\colon c(i) = t\}, \{s\} \subseteq t \in S_\omega(S_\omega(\varkappa))\}.$$

Moreover, if $i \in E_s$, then in particular $c_s(i) \neq S_\omega(\varkappa)$, whence $i \in f_0(s)$, i.e.,

(2.e) $$E_s \subseteq f_0(s).$$

Let $f_1(s) = E_s$, then by (2.c), (2.d) and (2.e) f_1 satisfies (2.a) and (2.b).

3.2.3. *There exists a monotonic function $f_2\colon S_\omega(\varkappa) \to \mathcal{F}$ and equivalence relations $\varrho_1, \varrho_2 \in \mathcal{G}$ such that*

(3.a) $$f_2 \leqslant f,$$

(3.b) $$f_2(s) \in \mathbf{2}^I|\varrho_2 \quad \textit{for} \quad s \in S_\omega(\varkappa),$$

(3.c) *if $y \in I/\varrho_1$ and $i, j \in y \cap f_2(s)$ then $(i, j) \in F(s)$.*

Proof. Recall that $f\colon S_\omega(\varkappa) \to \mathcal{F}$ is monotonic and $F\colon S_\omega(\varkappa) \to \mathcal{G}$ is additive. Moreover $f(s) \in \mathbf{2}^I|F(s)$. By 3.2.1 there is a $\varrho_1 \in \mathcal{G}$ and $d\colon \varkappa \to \mathcal{F}$ such that (1.a) and (1.b) hold. For $s \in S_\omega(\varkappa)$ we put $f_0(s) = \bigcap_{\alpha \in s} d(\alpha) \cap f(s)$. Of course f_0 is a monotonic function and $f_0^* S_\omega(\varkappa) \subseteq \mathcal{F}$. Let $s \in S_\omega(\varkappa)$; then $f(s) \in \mathbf{2}^I|\mathcal{G}$ and $d(\alpha) \in \mathbf{2}^I|\mathcal{G}$ for every $\alpha \in s$, whence $f_0(s) \in \mathbf{2}^I|\mathcal{G}$. We proved that f_0 satisfies the hypotheses of 3.2.2. Consequently, there is a $\varrho_2 \in \mathcal{G}$ and $f_1\colon S_\omega(\varkappa) \to \mathcal{F}$ such that (2.a) and (2.b) hold. We put $f_2(s) = \bigcap_{t \subseteq s} f_1(t)$. Clearly $f_2 \leqslant f_1$ and $f_2(s) \in \mathcal{F}$ for every $s \in S_\omega(\varkappa)$, but by the definition of f_1 we also have $f_1 \leqslant f$, whence $f_2 \leqslant f$. Of course f_2 is monotonic. Since $f_1(s) \in \mathbf{2}^I|\varrho_2$, (3.b) follows from the definition of f_2. To prove (3.c) let us notice that $f_2 \leqslant f_1 \leqslant f_0$. But $f_0(s) \subseteq \bigcap_{\alpha \in s} d(\alpha)$, whence for every $s \in S_\omega(\varkappa)$, $f_2(s) \subseteq \bigcap_{\alpha \in s} d(\alpha)$. Consequently, if $y \in I/\varrho_1$ and $i, j \in y \cap f_2(s)$, then $i, j \in y \cap d(\alpha)$ for every $\alpha \in s$. This by (1.b) implies that $(i, j) \in F(\{\alpha\})$ for every $\alpha \in s$. But F is an additive function, whence $(i, j) \in F(s)$.

3.2.4. *There is an additive function $g\colon S_\omega(\varkappa) \to \mathcal{F}$ and $\varrho_3 \in \mathcal{G}$ such that $g \leqslant f_2$ and $g(s) \in \mathbf{2}^I|\varrho_3$ for every $s \in S_\omega(\varkappa)$.*

Proof. Let $\mathfrak{A} = \langle S_\omega(S_\omega(\varkappa)), \subseteq, \neq \rangle$ and let, for $\alpha < \varkappa$ and $i \in I$,

$$a_\alpha(i) = \{t \in S_\omega(\varkappa)\colon i \in f_2(t), \alpha \in t\}.$$

Since $f_2 \leqslant f_0$, it follows from (2.b) that for every $i \in I$ and every $\alpha < \varkappa$, $a_\alpha(i) \in S_\omega(S_\omega(\varkappa))$. Moreover, by (3.b) a_α is constant on every equivalence class of ϱ_2, whence $a_\alpha \in A^I|\mathcal{G}$. Let $s \in S_\omega(\varkappa)$; then it is a matter of simple calculation to check that

(4.a) $$\{i \in I\colon \bigcap_{\alpha \in s} a_\alpha(i) \neq 0\} = f_2(s).$$

Now, let $\Sigma = \{v \subseteq a_\alpha/\mathcal{F}\}_{\alpha < \varkappa} \cup \{v \neq 0\}$. By (4.a) Σ is finitely satisfiable in $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$. By $\varkappa^+$-saturatedness of $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ there exists an $a \in A^I|\mathcal{G}$ such that $a/\mathcal{F}$ satisfies Σ. We put

(4.b) $$g(s) = \bigcap_{\alpha \in s} \{i \in I\colon 0 \neq a(i) \subseteq a_\alpha(i)\}.$$

Clearly $g(s)$ is an additive function; moreover, $a/\mathcal{F}$ satisfies Σ, whence, for every $s \in S_\omega(\varkappa)$, $g(s) \in \mathcal{F}$. From (4.b) we get $g(s) \subseteq \{i \in I\colon \bigcap_{\alpha \in s} a_\alpha(i) \neq 0\}$. Consequently, by (4.a) $g \leqslant f_2$. Now let $\varrho_3 = \varrho_2 \cap eq(a)$. Of course $\varrho_3 \in \mathcal{G}$. Since a is constant on every equivalence class of $eq(a)$ and a_α's are constant on every equivalence class of ϱ_2, we have $g(s) \in \mathbf{2}^I|\varrho_3$ for every $s \in S_\omega(\varkappa)$.

Now to complete the proof of Theorem 3.2 we put $\varrho_3 \cap \varrho_1 = \varrho$. Of course $g(s) \in \mathbf{2}^I|\varrho$. Assume that $x \in I/\varrho$ and $i, j \in x \cap g(s)$; then there is a $y \in I/\varrho_1$ such that $x \subseteq y$. But $g \leqslant f_2$, whence by (3.c) we have $(i, j) \in F(s)$. Finally notice that it follows from (2.b) that if $H = \{g(s)\colon s \in S_\omega(\varkappa)\}$ then $|H| = \varkappa$ and for every infinite $H_0 \subseteq H$ the intersection $\bigcap H_0$ is empty. This proves that $\mathcal{F}$ is $(\omega, \varkappa)$-regular and completes the proof.

From Theorems 2.1 and 3.2 we get the following corollary:

THEOREM 3.3. *Let $\mathcal{F}$ be an ultrafilter on I and $\mathcal{G}$ a filter in $E(I)$. Then $(\mathcal{F}, \mathcal{G})$ is $\varkappa$-good if and only if for every relational structure $\mathfrak{A}$ with $|L(\mathfrak{A})| \leqslant \varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathcal{F}}|\mathcal{G}$ is $\varkappa^+$-saturated.*

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Note on decompositions of metrizable spaces I

by

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Abstract. In this note we investigate, in the class of metrizable spaces, the property of being σ-locally of weight $<\mathfrak{t}$, introduced by A. H. Stone in his theory of non-separable absolutely Borel spaces [8], and we prove some facts related to the questions raised in [8].

In this note we investigate, in the class of metrizable spaces, the property of being σ-locally of weight $<\mathfrak{t}$, introduced by A. H. Stone in his theory of non-separable absolutely Borel spaces [8], and we prove some facts related to the questions raised in [8].

Our topological terminology and notation is from [2] and [5]; our set—theoretical terminology will follow [4]. *All of our spaces are assumed to be metrizable.* For a given space X we say that ϱ is a metric on X if ϱ is any metric compatible with the topology of X. For a metric ϱ, a set $A\subset X$ and $\varepsilon>0$, we write $B(A,\varepsilon)=\{x\in X\colon \varrho(x,A)<\varepsilon\}$. The symbol $w(X)$ denotes the weight of a space X and $|S|$ the cardinality of a set S. The set of all ordinals less than a given ordinal λ is denoted by $W(\lambda)$. For an initial ordinal λ of a regular cardinality $\mathfrak{t}$ we call a set $\Gamma\subset W(\lambda)$ *stationary* if and only if for every function $\Phi\colon \Gamma\to W(\lambda)$ with $\Phi(\xi)<\xi$, there exists $\alpha<\lambda$ such that $|\Phi^{-1}(\alpha)|=\mathfrak{t}$. The successor of a cardinal number $\mathfrak{t}$ is denoted by $\mathfrak{t}^+$.

We say that a space X is $\mathfrak{h}$-*locally of weight* $<\mathfrak{t}$ (in symbols, $X\in\mathfrak{h}-\mathrm{Lw}(<\mathfrak{t})$; see [8], 2.1) provided $X=\bigcup\{X_a\colon a\in A\}$, where $|A|\leqslant\mathfrak{h}$ and each X_a is locally of weight $<\mathfrak{t}$. It is easy to verify (cf. [8], 2.1) that for a metric ϱ on X this is equivalent to the following condition: there are families $\mathscr{F}_s$ of subsets of X of weight $<\mathfrak{t}$ and $\varepsilon_s>0$ for $s\in S$, where $|S|\leqslant\mathfrak{h}$, such that

$$(1)\qquad X=\bigcup\{\bigcup\mathscr{F}_s\colon s\in S\}\quad\text{and}\quad \varrho(F',F'')\geqslant\varepsilon_s\quad\text{for different } F',F''\in\mathscr{F}_s.$$

For $\mathfrak{h}=\aleph_0$ we write $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$; if $X\in\mathfrak{h}-\mathrm{Lw}(<\aleph_0)$ we say that X is $\mathfrak{h}$-*discrete*.

PROPOSITION (cf. [8], Theorem 3). *Suppose that* $\mathfrak{t}$ *is a regular or sequential cardinal and* $\mathfrak{h}<\mathfrak{t}$. *If* $X\in\mathfrak{h}-\mathrm{Lw}(<\mathfrak{t})$, *then* $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$.