On a class of multi-valued vector fields in Banach spaces

by

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Abstract. An upper-semicontinuous map $\Phi\colon X\to Y$ is called an *n-admissible map*, $n\geqslant 1$, provided there exist two maps $p\colon Z\to X$ and $q\colon Z\to Y$ such that:

(i) p is a Vietoris n-map or, in particular, a Vietoris map,

(ii) $q(p^{-1}(x))\subset\Phi(x)$ for all $x\in X$.

In this paper we consider n-admissible compact fields in Banach spaces, i.e., maps of the type $\varphi = I-\Phi$, where Φ is a compact n-admissible map from a subset of a Banach space E into E. The topological degree theory for n-admissible compact fields $\varphi\colon A\to E$ is given in the following two cases:

(i) A is a closed ball in E,

(ii) A is the boundary of a closed ball in E.

This theory is applied to the fixed point theory of n-admissible fields and to the proofs of the theorem on antipodes and the theorem on the invariance of domain for some n-admissible fields.

Note that the class of n-admissible fields is essentially larger than the class of acyclic fields and, in particular the class of convex-valued fields.

In this paper we continue the study of n-admissible maps which was undertaken in [3]. We note that the class of n-admissible maps contains n-acyclic maps (see [7], [1]) and hence acyclic maps.

The object of this paper is to extend the classical Brouwer–Leray–Schauder degree theory from n-admissible maps in finite-dimensional vector spaces to n-admissible compact vector fields in Banach spaces.

From this theory we obtain some fixed-point theorems and a theorem on antipodes for n-admissible compact fields. Moreover, we obtain a theorem on antipodes for admissible compact vector fields and a theorem on the invariance of domain for strongly admissible compact vector fields.

We would like to recall that this theory was considered with by:

(i) A. Granas [9], [10], A. Cellina and A. Lasota [4], Ju. G. Borisovic, B. D. Gelman, E. Muhamadiev and V. V. Obuhovskii [2], T. W. Ma [12] for convex-valued compact vector fields,

(ii) S. A. Williams [15], M. Furi and M. Martelli [6] for acyclic compact vector fields,

(iii) D. G. Bourgin [1] for n-acyclic compact vector fields.

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1. Preliminaries. Let H denote the Čech cohomology functor with integer coefficients Z from the category of metric spaces and continuous maps to the category of graded abelian groups and homomorphisms of degree zero. Thus, for a space X,

$$H(X) = \{H^k(X)\}_{k\geqslant 0}$$

is a graded abelian group, and for a continuous map $f\colon X\to Y$, $H(f)$ is the induced homomorphism

$$H(f) = f^* = \{f^{*k}\}\colon H(Y)\to H(X),$$

where $f^{*k}\colon H^k(Y)\to H^k(X)$.

A non-empty space X is called 0-*acyclic* if $H^0(X) = Z$; let us call X k-*acyclic*, $k\geqslant 1$, provided $H^k(X) = 0$; call X *acyclic* if X is k-acyclic for each $k\geqslant 0$.

Let A be a subset of a space X. Denote by $\mathrm{rd}_X A$ the relative dimension of A in X. From the definition given in [13] we have:

$$\mathrm{rd}_X A = \sup_{C\subset A}\dim C,$$

where C is a closed subset of X and by $\dim C$ we denote the topological dimension of C. We assume that $\mathrm{rd}_X A<0$ iff the set A is empty and in this case we put $\mathrm{rd}_X A = -\infty$. We observe that:

(1.1) *Let X_0 be a closed subset of X. Assume further that A is a subset of X_0, B is a subset of X and $A\subset B$. Then we have*

$$\mathrm{rd}_{X_0} A\leqslant \mathrm{rd}_X B.$$

A continuous map $f\colon Y\to X$ is *proper* if for each compact subset $A\subset X$ the counter-image $f^{-1}(A)$ of A under f is compact; f is closed provided for each closed subset $B\subset Y$ the image of B under f is closed in X.

The following fact is evident:

(1.2) *If $f\colon Y\to X$ is a proper map, then f is closed.*

The Vietoris–Begle Theorem (see [13]) and (1.2) gives

(1.3) Theorem. *Let $f\colon Y\to X$ be a proper and surjective map, let M_f^i be the set of all $x\in X$ such that $f^{-1}(x)$ fails to be i-acyclic. Let $n = 1+\max\limits_{i\geqslant 0}(\mathrm{rd}_X M_f^i+i)$. Then for each $k>n$ the induced homomorphism $f^{*k}\colon H^k(X)\to H^k(Y)$ is an isomorphism.*

Remark. We observe that if M_f^i is the empty set for each $i\geqslant 0$, then $\mathrm{rd}_X M_f^i = -\infty$ and hence Theorem (1.3) implies that

$$f^*\colon H(X)\to H(Y)\ \text{is an isomorphism}.$$

(1.4) Definition. A map $p\colon Y\to X$ is called a *Vietoris n-map* provided the following two conditions are satisfied:

(i) p is a proper and surjective map,
(ii) $\mathrm{rd}_X M_p^i\leqslant n-2-i$ for each $i\geqslant 0$.

Definition (1.4) implies that if p is a Vietoris 1-map, then $\mathrm{rd}_X M_p^i<0$ for each $i\geqslant 0$, and from the above remark we deduce that $p^*\colon H(X)\to H(Y)$ is an isomorphism. In what follows a Vietoris 1-map is called simply a *Vietoris map*.

Finally, we note that if p is a Vietoris n-map, then we have

$$1+\max_{i\geqslant 0}(\mathrm{rd}_X M_p^i+i)\leqslant 1+\max_{i\geqslant 0}[(n-2-i)+i] = n-1$$

and (1.3) implies that $p^{*k}\colon H^k(X)\to H^k(Y)$ is an isomorphism for each $k\geqslant n$.

From (1.1) we deduce

(1.5) *Let $p\colon Y\to X$ is a Vietoris n-map and $A\subset X$ is closed, then the map $\tilde p\colon p^{-1}(A)\to A$ is a Vietoris n-map, where $\tilde p$ is given by $\tilde p(y) = p(y)$ for all $y\in p^{-1}(A)$.*

(1.6) *If $p\colon Y\to X$ is a Vietoris map, then for each $A\subset X$ the map $\tilde p\colon p^{-1}(A)\to A$ is a Vietoris map, where $\tilde p$ is given by $\tilde p(y) = p(y)$ for all $y\in p^{-1}(A)$.*

From (1.6) we deduce

(1.7) *If $p_1\colon Y\to X$ and $p_2\colon X\to Z$ are Vietoris maps, then the composition $p_2\circ p_1\colon Y\to Z$ of p_1 and p_2 is also a Vietoris map.*

(1.8) Definition. Let X be a space and let A, B be two subsets of X. Denote by $i\colon (A, A\cap B)\to(X, B)$ and $j\colon (B, A\cap B)\to (X, A)$ the inclusion maps. A triple (X, A, B) is called a k-*triad*, $k\geqslant 0$, provided: (i) $X = A\cup B$ and (ii) $i^{*l}\colon H^l(X, B)\to H^l(A, A\cap B)$ and $j^{*l}\colon H^l(X, A)\to H(B, A\cap B)$ are isomorphisms for each $l\geqslant k+1$.

If (X, A, B) is a 0-triad, then (X, A, B) is called simply a *triad*.

(1.9) Mayer–Vietoris Theorem. *Let (X, A, B) be a k-triad. Then the sequence*

$$H^k(A\cap B)\xrightarrow{\Delta} H^{k+1}(X)\xrightarrow{\alpha} H^{k+1}(A)\oplus H^{k+1}(B)\xrightarrow{\beta} H^{k+1}(A\cap B)\to\dots,$$

in which Δ, α, β are Mayer–Vietoris homomorphisms, is exact.

The proof of (1.9) is analogous to the proof of the Mayer–Vietoris Theorem for triads (comp. [5], pp. 57–67).

Similary to [5], if $f\colon (X, A, B)\to (Y, C, D)$ is a map of k-triads, i.e., $f\colon X\to Y$ is a continuous map and $f(A)\subset C$, $f(B)\subset D$, then f induces a homomorphism of the respective Mayer–Vietoris sequences.

Let E be a Banach space and let $E^{k+1}\subset E^{k+2}$ be two subspaces of E with dimensions $k+1$ and $k+2$ respectively. Denote by E_+^{k+2} and E_-^{k+2} the two closed half-spaces of E^{k+2} such that $E^{k+1} = E_+^{k+2}\cap E_-^{k+2}$ and $S_+^{k+1} = E_+^{k+2}\cap S$ and $S_-^{k+1} = E_-^{k+2}\cap S$, where S is the unit sphere in E. Clearly, $S^k = S\cap E^{k+1} = S_+^{k+1}\cap S_-^{k+1}$.

We note that $(S^{k+1}, S_+^{k+1}, S_-^{k+1})$ is a triad and the Mayer–Vietoris homomorphism $\Delta\colon H^k(S^k)\to H^{k+1}(S^{k+1})$ is an isomorphism.

The following lemma is of importance:

(1.10) LEMMA. *Let $p, q\colon Y \to S^{k+1}$ be two continuous map such that*: (i) *p is a Vietoris n-map, $n \leqslant k$,* (ii) *$qp^{-1}(S_+^{k+1}) \subset S_-^{k+1}$ and* (iii) *$qp^{-1}(S_-^{k+1}) \subset S_+^{k+1}$. Then the following diagram commutes*:

$$\begin{array}{ccc} H^k(S^k) & \xrightarrow{\Delta} & H^{k+1}(S^{k+1}) \\ {\scriptstyle \bar{q}^{*k}(\bar{p}^{*k})^{-1}}\downarrow & & \downarrow{\scriptstyle q^{*k+1}(p^{*k+1})^{-1}} \\ H^k(S^k) & \xrightarrow{\Delta} & H^{k+1}(S^{k+1}) \end{array}$$

where $\bar{q}, \bar{p}\colon p^{-1}(S^k) \to S^k$ are contractions of q and p, respectively.

Proof. Let $Y_+ = p^{-1}(S_+^{k+1})$ and $Y_- = p^{-1}(S_-^{k+1})$. Then from (1.5) and (1.3) we deduce that the triple (Y, Y_+, Y_-) is a k-triad. Observe that $Y_+ \cap Y_- = p^{-1}(S^k)$. By assumptions (i) and (ii) we infer that p, q are maps from the k-triad (Y, Y_+, Y_-) to the k-triad $(S^{k+1}, S_+^{k+1}, S_-^{k+1})$. Finally, from the naturality of the Mayer–Vietoris Theorem for k-triads we deduce (1.10).

The following fact is well known (comp. [9] pp. 24–25).

(1.11) APPROXIMATION THEOREM. *Let E be a Banach space and let $q\colon Y \to E$ be a compact map. Then for every $\varepsilon > 0$ there exists a continuous finite dimensional map $q_\varepsilon\colon Y \to E^n \subset E$ such that*

$$\|q_\varepsilon(y) - q(y)\| < \varepsilon \quad \textit{for every } y \in Y.$$

In this case the map q_ε is called an ε-approximation of q.

2. Multi-valued maps. Let X and Y be two spaces and assume that for every point $x \in X$ a non-empty subset $\varphi(x)$ of Y is given; in this case, we say that φ is a multi-valued map from X to Y and we write $\varphi\colon X \to Y$. In what follows the symbols φ, ψ, $\varkappa$ will be reserved for multi-valued maps; the single-valued maps will be denoted by f, g, h, p, q.

Let $\varphi\colon X \to Y$ be a multi-valued map. We associate with φ the following diagram of continuous maps:

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y,$$

in which $\Gamma_\varphi = \{(x, y) \in X \times Y;\ y \in \varphi(x)\}$ is the *graph* of φ and *natural projections p_φ and q_φ* are given by $p_\varphi(x, y) = x$ and $q_\varphi(x, y) = y$.

The image of a subset $A \subset X$ under φ is $\varphi(A) = \bigcup_{a \in A} \varphi(a)$. The counter-image of a subset $B \subset Y$ under φ is $\varphi^{-1}(B) = \{x \in X;\ \varphi(x) \subset B\}$.

Let $\varphi\colon X \to Y$ be a multi-valued map, A is a subset of X and B a subset of Y. If $\varphi(A) \subset B$, then the *contraction* of φ to the pair (A, B) is a multi-valued map $\varphi'\colon A \to B$ defined by $\varphi'(a) = \varphi(a)$ for each $a \in A$. The contraction of φ to the pair (A, Y) is simply the *restriction* $\varphi|_A$ of φ to A.

Let $\psi, \varphi\colon X \to Y$ be two multi-valued maps such that $\varphi(x) \subset \psi(x)$ for every $x \in X$; in this case we say that φ is a *selector* of ψ and indicate this by $\varphi \subset \psi$.

Let $\varphi\colon X \to X$ be a multi-valued map and let A be a subset of X. A point $x \in X$ is called a fixed point for φ whenever $x \in \varphi(x)$; if moreover $x \in A$, we say that φ has a fixed point in A.

A multi-valued map $\varphi\colon X \to Y$ is called *upper semi-continuous* (u.s.c.) provided: (i) $\varphi(x)$ is compact for each $x \in X$ and (ii) for each open set $V \subset Y$ the counter-image $\varphi^{-1}(V)$ is an open subset of X.

(2.1) PROPOSITION. *If $\varphi\colon X \to Y$ is a* u.s.c. *multi-valued map, then the graph Γ_φ of φ is a closed subset of $X \times Y$.*

A u.s.c. multi-valued map $\varphi\colon X \to Y$ is called *compact provided the image $\varphi(X)$* of X under φ is contained in a compact subset of Y.

A u.s.c. multi-valued map $\varphi\colon X \to Y$ is said to be *acyclic* provided the set $\varphi(x)$ is acyclic for every point $x \in X$.

(2.2) DEFINITION. A map $\varphi\colon X \to Z$ is called *n-admissible*, $n \geqslant 1$, provided there exists a metric space Y and a pair of single-valued (continuous) maps of the form $X \xleftarrow{p} Y \xrightarrow{q} Z$ such that the following two conditions are satisfied:

(i) p is a Vietoris n-map,

(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for each $x \in X$.

In this case the pair of maps (p, q) is called a *selected pair* of φ, written $(p, q) \subset \varphi$.

The class of all n-admissible maps from X to Z will be denoted by $C^n(X, Z)$. A map $\varphi \in C^1(X, Z)$ is called simply *admissible*.

(2.3) DEFINITION. An admissible map $\varphi\colon X \to Z$ is called *strongly admissible* (s-admissible) provided there exists a selected pair $(p, q) \subset \varphi$ such that: $q(p^{-1}(x)) = \varphi(x)$ for each $x \in X$.

If a selected pair (p, q) of φ satisfies the above condition, then we write $(p, q) = \varphi$.

The class of all s-admissible maps from X to Z will be denoted by $C_s(X, Z)$.

REMARKS 1. We observe that if $\varphi\colon X \to Z$ is a compact acyclic map, then, for example, the pair (p_φ, q_φ) is a selected pair of φ and hence $(p_\varphi, q_\varphi) = \varphi$.

2. Denote by $C(X, Z)$ the class of all compact acyclic maps from X to Z; then we have the following inclusions:

$$C(X, Z) \subset C_s(X, Z) \subset C^1(X, Z) \subset C^2(X, Z) \ldots$$

(2.4) PROPOSITION. *If $\varphi_1 \in C_s(X_1, X_2)$ and $\varphi_2 \in C_s(X_2, X_3)$, then $\varphi_2 \circ \varphi_1 \in C_s(X_1, X_3)$ and for every $(p_1, q_1) = \varphi_1$, $(p_2, q_2) = \varphi_2$ there exists a pair $(p, q) = \varphi_2 \circ \varphi_1$ such that*

$$(p^*)^{-1}q^* = (p_1^*)^{-1}q_1^*(p_2^*)^{-1}q_2^*.$$

(2.5) PROPOSITION. *If $\varphi \in C(X, Z)$, then for every selected pair $(p, q) \subset \varphi$ we have $(p^*)^{-1}q^* = (p_\varphi^*)^{-1}q_\varphi^*$.*

For the proof of (2.4) and (2.5) see (3.4) and (3.5) in [3].

It is well known (see example (3.6) in [3]) that $C(X,Z) \neq C_s(X,Z)$.

(2.6) DEFINITION. Two maps $\varphi, \psi \in C^n(X,Z)$, $n \geqslant 1$, $[\varphi, \psi \in C_s(X,Z)]$ are called *homotopic* $(\varphi \sim \psi)$ if there exists a map

$$\chi \in C^n(X \times [0,1], Z)\,[\chi \in C_s(X \times [0,1], Z)]\,,$$

where $[0,1]$ is the unit interval, such that $\chi(x,0) \subset \varphi(x)$ and $\chi(x,1) \subset \psi(x)$ $[\chi(x,0) = \varphi(x)$ and $\chi(x,1) = \psi(x)$ for each $x \in X$.

(2.7) PROPOSITION. *If two maps $\varphi, \psi \in C^n(X,Z)\,[\varphi, \psi \in C_s(X,Z)]$ are homotopic, then there are selected pairs $(p_0, q_0) \subset \varphi$ and $(p_1, q_1) \subset \psi$ such that for each $k \geqslant n$*

$$(p_0^{*k})^{-1} q_0^{*k} = (p_1^{*k})^{-1} q_1^{*k}$$

(*for each* $k \geqslant 0$, $(p_0^{*k})^{-1} q_0^{*k} = (p_1^{*k})^{-1} q_1^{*k}$).

For the proof of (2.7) see [3] (Proposition (3.8)). Consider two continuous maps of the form $X \xleftarrow{p} Y \xrightarrow{q} Z$, where p is a Vietoris n-map. We define a multi-valued map $\varphi_{p,q}\colon X \to Z$ by putting $\varphi_{p,q}(x) = q(p^{-1}(x))$ for each $x \in X$. Evidently, $\varphi_{p,q}$ is an n-admissible and u.s.c. map. Moreover, if q is a compact map, then $\varphi_{p,q}$ is also compact.

3. Admissible maps of subsets of finite dimensional vector spaces. Let S^n be the unit sphere in the $(n+1)$-dimensional vector space E^{n+1} and K^{n+1} be the unit closed ball in E^{n+1}; by P^{n+1} we denote the space E^{n+1} without the point 0.

In this section we define the degree of an n-admissible map $\varphi\colon S_1^n \to S_2^n$ where S_i^n $(i = 1, 2)$ are two spaces which have the cohomology of an n-sphere S^n. We orient S_i^n by choosing generators $\beta_i = H^n(S_i^n)$.

Let $\varphi\colon S_1^n \to S_2^n$ be an n-admissible map and let (p, q) be a pair of the form $S_1^n \leftarrow Y \rightarrow S_2^n$ such that $(p, q) \subset \varphi$.

(3.1) DEFINITION. The degree $\deg(p, q)$ of the pair (p, q) is the unique integer which satisfies

$$(p^{*n})^{-1} q^{*n}(\beta_2) = \deg(p, q)'\beta_1\,.$$

(3.2) DEFINITION. Let $\varphi\colon S_1^n \to S_2^n$ be an n-admissible map. By the degree $\operatorname{Deg}(\varphi)$ of φ we be understand the following set of integers:

$$\operatorname{Deg}\varphi = \{\deg(p, q);\ (p, q) \subset \varphi\}\,.$$

(3.3) *Let* $\varphi, \psi \in C^n(S_1^n, S_2^n)$. *Then* (see (4.2) in [3]):

(i) $\varphi \sim \psi$ *implies that* $\operatorname{Deg}\varphi \cap \operatorname{Deg}\psi \neq \emptyset$,

(ii) $\varphi \subset \psi$ *implies that* $\operatorname{Deg}\varphi \subset \operatorname{Deg}\psi$.

From Remark 1 in Section 2 we obtain:

(3.4) *If φ is acyclic, then the set $\operatorname{Deg}\varphi$ is a singleton, in particular, $\varphi \sim \psi$ implies that* $\operatorname{Deg}\varphi = \operatorname{Deg}\psi$.

Let $\varphi\colon K^{n+1} \to E^{n+1}$ be an n-admissible map and assume that $\varphi(S^n) \subset E^{n+1} \setminus \{x_0\}$. By $\varphi|_{S^n}\colon S^n \to E^{n+1} \setminus \{x_0\}$ we denote the contraction of φ to the pair $(S^n, E^{n+1} \setminus \{x_0\})$. From (1.1) we infer that $\varphi|_{S^n} \in C^n(S^n, E^{n+1} \setminus \{x_0\})$. In this case with every selected pair $(p, q) \subset \varphi$ we associate a pair $(p_1, q_1) \subset \varphi|_{S^n}$ as follows: let $p\colon Y \to K^{n+1}$, $q\colon Y \to E^{n+1}$ be two maps such that $(p, q) \subset \varphi$; then $p_1\colon p^{-1}(S^n) \to S^n$, $q_1\colon p^{-1}(S^n) \to E^{n+1} \setminus \{x_0\}$ are given as contractions of p and q, respectively. Evidently $(p_1, q_1) \subset \varphi|_{S^n}$. We define the degree $\operatorname{Deg}(\varphi, x_0)$ of φ by putting

$$\operatorname{Deg}(\varphi, x_0) = \{\deg(p_1, q_1);\ (p, q) \subset \varphi\}\,. \tag{3.5}$$

Clearly, $\operatorname{Deg}(\varphi, x_0) \subset \operatorname{Deg}(\varphi|_{S^n})$, but $\operatorname{Deg}(\varphi, x_0) \neq \operatorname{Deg}(\varphi|_{S^n})$ (see [3], Section 4).

(3.6) LEMMA ((4.4) in [3]). *Let $\varphi\colon K^{n+1} \to E^{n+1}$ be an n-admissible such that $\varphi(S^n) \subset E^{n+1} \setminus \{x_0\}$ for some point $x_0 \in E^{n+1}$. If $\operatorname{Deg}(\varphi, x_0) \neq \{0\}$, then there exists a point $x \in K^{n+1}$ such that $x_0 \in \varphi(x)$.*

(3.7) THEOREM ((4.5) in [3]). *Let $\varphi\colon K^{n+1} \to E^{n+1}$ be an n-admissible map such that $\varphi(S^n) \subset K^{n+1}$. Then φ has a fixed point.*

(3.8) *Let $\varphi\colon S^n \to S^n$ be an n-admissible map such that $\operatorname{Deg}\varphi \neq \{0\}$. Then $\varphi(S^n) = S^n$.*

Proof. The proof of (3.8) is analogous to the proof (6.4) in [3].

(3.9) THEOREM (Theorem on antipodes for n-admissible maps ([3], Theorem 5.1)). *Let $\varphi\colon S^n \to P^{n+1}$ be an n-admissible map. If for every $x \in S^n$ there exists an n-subspace of E^{n+1} strictly separating $\varphi(x)$ and $\varphi(-x)$, then $0 \notin \operatorname{Deg}\varphi$.*

(3.10) THEOREM (Theorem on antipodes for admissible maps). *Let $\varphi\colon S^n \to P^{n+1}$ be an admissible map such that following condition is satisfied: every radius with origin of the zero point of E^{n+1} has an empty intersection with the set $\varphi(x)$ or $\varphi(-x)$ for each $x \in S^n$. Then $0 \notin \operatorname{Deg}\varphi$.*

Theorem (3.10) clearly follows from Theorem (6.2) in [3].

An s-admissible map $\varphi\colon X \to Z$ is called an *ε-map* provided the condition $\varphi(x) \cap \varphi(x) \neq \emptyset$ implies $d(x, x') < \varepsilon$ for each $x, x' \in X$.

(3.11) LEMMA. *Let $\varphi\colon K_\varepsilon^{n+1} \to E^{n+1}$ be an ε-map. Then*:

(i) $\varphi(S^n) \subset E^{n+1} \setminus \{z_0\}$ *for each* $z_0 \in \varphi(0)$.

(ii) $0 \notin \operatorname{Deg}(\varphi, z_0)$.

Lemma (3.11) clearly follows from Lemma (7.3) in [3].

(3.12) THEOREM ((7.5) in [3]). *Let $\varepsilon > 0$ be a positive real number. If $\varphi\colon E^{n+1} \to E^{n+1}$ is an ε-map, then $\varphi(E^{n+1})$ is an open subset of E^{n+1}.*

(3.13) THEOREM ((7.6) in [3]). *Let U be an open subset of E^{n+1} and $\varphi\colon U \to E^{n+1}$ an s-admissible map. Assume further that for any points $x_1, x_2 \in U$ the condition $x_1 \neq x_2$ implies $\varphi(x_1) \cap \varphi(x_2) = \emptyset$. Then $\varphi(U)$ is an open subset of E^{n+1}.*

4. Admissible compact vector fields in Banach spaces. Let E be a Banach space, X a subset of E and $\Phi\colon X \to E$ a multi-valued map. We define a multi-valued map $\varphi\colon X \to E$ by putting $\varphi = I - \Phi$, where

$$(I - \Phi)(x) = \{x - y;\ y \in \Phi(x)\} \quad \text{for every } x \in X\,.$$

(4.1) Definition. A map $\varphi\colon X\to Y\subset E$ is called an *n-admissible* (*s-admissible*) *compact vector field* if and only if there exists an n-admissible (s-admissible) compact map $\Phi\colon X\to E$ such that $\varphi = I-\Phi$.

If Φ is an admissible compact map, then $\varphi = I-\Phi$ is called an *admissible compact vector field.*

A point $x_0\in X$ is called a *singular point of the vector field* $\varphi\colon X\to Y$ *if the image* $\varphi(x_0)$ contains the origin 0 of E. If there are no singular points, we say that φ is *singularity free* (written $\varphi\colon X\to P$).

(4.2) Definition. Two n-admissible (s-admissible) compact vector fields $\varphi_1 = I-\Phi_1$, $\varphi_2 = I-\Phi_2$ ($\varphi_1,\varphi_2\colon X\to Y\subset E$) are said to be *homotopic*, written $\varphi_1\sim\varphi_2$, provided there exists a map $\chi\colon X\times J\to Y$, where J is a unit interval, which can be represented in the form $\chi(x,t) = x-X(x,t)$, where $X\colon X\times J\to E$ is an n-admissible (s-admissible) compact homotopy between φ_1 and φ_2.

The following evident remark is of importance:

(4.3) *Let A be a closed subset of E and let $\varphi\colon A\to E$ be an n-admissible compact vector field. Then the image $\varphi(A)$ is a closed subset of E.*

(4.4) *Consider two maps of the form $X\xleftarrow{p} Y\xrightarrow{q} E$ such that X is a subset of E and p is a Vietoris n-map. Define a map $\tilde q\colon Y\to E$ by putting $\tilde q(y) = p(y)-q(y)$. Then $I-\varphi_{p,q} = \varphi_{p,\tilde q}$,* (for the definition of $\varphi_{p,q}$ see Section 2) *and hence*

(i) *if q is a compact map, then $I-\varphi_{p,q}$ is an n-admissible compact vector field,*

(ii) *every n-admissible compact field is an n-admissible map.*

5. Degree of n-admissible compact vector fields in Banach spaces. Let E be a Banach space and let $\varphi\colon S\to P$ be an n-admissible compact vector field from the unit sphere S to $P = E\setminus\{0\}$.

Consider an arbitrary but fixed selected pair $(p,q)\subset\Phi$ of the form $S\xleftarrow{p} Y\xrightarrow{q} E$. First, for such a pair $(p,q)\subset\Phi$ we define an integer $\deg(p,q)$ which is called the *degree* of (p,q).

From Lemma (4.3) we obtain a positive number δ such that $d(0,\varphi(S)) = \delta$. We observe that $d(0,(I-\varphi_{p,q})(S))\geqslant\delta$. Let ε be a positive number such that $\varepsilon<\delta$. Since Φ is a compact map we infer that q is also compact. Applying the Approximation Theorem to the map q and the number ε, we obtain a map $q_\varepsilon\colon Y\to E^{k+1}$ such that $\|q(y)-q_\varepsilon(y)\|<\varepsilon$ for every $y\in Y$. We may assume without loss of generality that $k+1\geqslant 2$ and $k+1\geqslant n$.

Let $Y_k = p^{-1}(S^k)$, where $S^k = S\cap E^{k+1}$. Consider the diagram

$$S^k\xleftarrow{p_k} Y_k\xrightarrow{q_k} E^{k+1},$$

in which p_k and q_k are contractions of p and q, respectively. Applying (1.5) and (4.4) to the pair (p_k, q_k), we obtain the diagram

$$S^k\xleftarrow{p_k} Y_k\xrightarrow{\tilde q_k} P^{k+1},$$

in which $\tilde q_k(y) = p_k(y)-q_k(y)$. We observe that $\tilde q_k(y)\neq 0$ for every $y\in Y_k$. Indeed, since $\|x-q(y)\|\geqslant\delta$ for every $x\in S$ and $y\in p^{-1}(x)$, we have

$$\begin{aligned}\|q_k(y)\| &= \|p_k(y)-q_k(y)\| = \|x-q(y)\| = \|(x-q(y))-(q_\varepsilon(y)-q(y))\|\\ &\geqslant \|x-q(y)\|-\|q(y)-q_\varepsilon(y)\|\geqslant\delta-\varepsilon>0 .\end{aligned}$$

We define $\deg(p,q)$ of the pair $(p,q)\subset\Phi$ by putting

$$\deg(p,q) = \deg(p_k,\tilde q_k)$$

where $\deg(p_k,\tilde q_k)$ is given in (3.1).

(5.1) Lemma. *Let $\varphi = I-\Phi\colon S\to P$ be an n-admissible compact vector field and let (p,q) be a selected pair of Φ of the form $S\xleftarrow{p} Y\xrightarrow{q} E$. Assume further that $q_\varepsilon, q'_\varepsilon\colon Y\to E^{k+1}$ are two ε-approximations of q. Then $\deg(p_k,\tilde q_k) = \deg(p_k,\tilde q'_k)$.*

Proof. Define the map $h\colon Y_k\times[0,1]\to E^{k+1}$ by putting

$$h(y,t) = t\tilde q_k(y)+(1-t)\tilde q'_k(y) .$$

Then h is a homotopy between $\tilde q_k$ and $\tilde q'_k$. We prove that $h(y,t)\neq 0$ for each $y\in Y_k$ and $t\in[0,1]$:

$$\begin{aligned}\|t\tilde q_k(y)+(1-t)\tilde q'_k(y)\| &= \|t(p_k(y)-q_k(y))+(1-t)(p_k(y)-q'_k(y))\|\\ &= \|p_k(y)-tq_\varepsilon(y)+(1-t)q'_\varepsilon(y)\|\\ &= \|p(y)-q(y)-[t(q_\varepsilon(y)-q(y))+(1-t)(q'_\varepsilon(y)-q(y))]\|\\ &\geqslant\delta-\varepsilon>0 .\end{aligned}$$

Therefore we have $\tilde q'^*_k = \tilde q^*_k$ and the proof is completed.

(5.2) Lemma. *Let $\varphi = I-\Phi$ and $(p,q)\subset\Phi$ be as in (5.1). Assume further that E^{k+1}, E^{k+2} are two subspaces of E such that $E^{k+1}\subset E^{k+2}$. If $q_\varepsilon\colon Y\to E^{k+1}$ is an ε-approximation of q and $q'_\varepsilon\colon Y\to E^{k+2}$ is the map given by $q_\varepsilon(y) = q'_\varepsilon(y)$ for every $y\in Y$, then $\deg(p_k,\tilde q_k) = \deg(p_{k+1},\tilde q'_{k+1})$.*

Proof. Define a map $r\colon P^{k+2}\to S^{k+1}$ by putting $r(z) = z/\|z\|$. We orient S^{k+1} and P^{k+2} so that $\deg(p_{k+1},\tilde q'_{k+1}) = \deg(p_{k+1}, r\tilde q'_{k+1})$. Applying Lemma (1.10) to the pair (p_{k+1}, rq_{k+1}), we obtain (5.2).

Finally, from (5.1) and (5.2) we deduce that $\deg(p,q)$ of the pair (p,q) is well defined.

Now, we define $\mathrm{Deg}(I-\Phi)$ of an n-admissible compact vector field $\varphi = I-\Phi\colon S\to P$ by putting

$$\mathrm{Deg}(I-\Phi) = \{\deg(p,q);\ (p,q)\subset\Phi\} . \tag{5.3}$$

(5.4) Proposition. *Let $\varphi,\psi\colon S\to P$ be two n-admissible compact vector fields. Then*

(i) *$\varphi\sim\psi$ implies $\mathrm{Deg}(\varphi)\cap\mathrm{Deg}(\psi)\neq\emptyset$,*

(ii) *$\varphi\subset\psi$ implies $\mathrm{Deg}(\varphi)\subset\mathrm{Deg}(\psi)$.*

Proof. Let $\chi = I - X$ be a homotopy between φ and ψ and let (p, q) be a selected pair of χ. The set $\chi(S \times [0, 1])$ is closed and does not contain the origin. Then from the above construction of the degree for the selected pairs (2.7) and (3.3) we obtain (i).

The proof of (ii) is evident.

(5.5) PROPOSITION. *Let $\varphi = I - \Phi$: $S \to P$ be an admissible compact vector field. If Φ is an acyclic map, then the set* $\mathrm{Deg}(\varphi)$ *is a singleton.*

Proof. Let $(p, q) \subset \Phi$ be a selected pair of Φ of the form $S \xleftarrow{p} Y \xrightarrow{q} E$. Consider the commutative diagram

$$\begin{tikzcd} S & \Gamma_\Phi \arrow[l, "p_\Phi"'] \arrow[r, "q_\Phi"] & E \\ & Y \arrow[ul, "p"] \arrow[u, "f"] \arrow[ur, "q"'] & \end{tikzcd}$$

in which $f(y) = (p(y), q(y))$ for each $y \in Y$. Let $(q_\varphi)_\varepsilon$: $\Gamma_\Phi \to E^{k+1}$ be an ε-approximation of q_Φ. For the proof we take an ε-approximation of q such that $q_\varepsilon(y) = (q_\Phi)_\varepsilon f(y)$ for each $y \in Y$. Denote by Γ_k the graph of $\Phi|_{S^k}$ ($S^k = S \cap E^{k+1}$). Let

$$(p_\varphi)_k\colon \Gamma_k \to S^k \quad \text{and} \quad (q_\Phi)_k\colon \Gamma_k \to E^{k+1}$$

be contractions of p_φ and $(q_\varphi)_\varepsilon$, respectively. Finally, we obtain (comp. the definition of $\deg(p, q)$ in this section) the commutative diagram

$$\begin{tikzcd} S^k & \Gamma_k \arrow[l, "(p_\Phi)_k"'] \arrow[r, "(\tilde{q}_\Phi)_k"] & P^{k+1} \\ & p^{-1}(S^k) \arrow[ul, "p_k"] \arrow[u, "\bar{f}"] \arrow[ur, "\tilde{q}_k"'] & \end{tikzcd}$$

in which p_k and $\bar{f}$ are contractions of p and f, respectively, and the map $(\tilde{q}_\Phi)_k$: $\Gamma_k \to P^{k+1}$ is given by $(\tilde{q}_\Phi)_k(x, y) = (p_\Phi)_k(x, y) - (q_\Phi)_k(x, y)$ for each $(x, y) \in \Gamma_k$. The map $\tilde{q}_k$: $p^{-1}(S^k) \to P^{k+1}$ is given by $\tilde{q}_k(y) = p_k(y) - q_\varepsilon(y)$ for each $y \in p^{-1}(S^k)$. Now, the proof of (5.5) is strictly analogous to the proof of (3.5) in [3].

(5.6) EXAMPLE. Let E be a Banach space and $y_0 \in E$ a point such that $\|y_0\| > 1$. Consider the map Φ: $S \to E$ given by $\Phi(x) = \{0, y_0\}$ for each $x \in S$. Clearly, Φ is an admissible and compact map. We have the following selected pairs of Φ:

1) $(\mathrm{id}_S, f) \subset \Phi$, where f: $S \to E$ is given by $f(x) = 0$ for $y \in S$,

2) (Id_S, g), where g: $S \to E$ is given by $g(x) = y_0$ for each x.

Moreover, we infer that $\deg(\mathrm{Id}_S, f) \neq 0$ and $\deg(\mathrm{Id}_S, g) = 0$ and hence $\mathrm{Deg}(I - \Phi)$ is not singleton.

The following theorem is a generalization of the following classical result: If f: $S^n \to S^n$ is a continuous single-valued map on the n-dimensional unit sphere with a non-vanishing degree, then f must be surjective.

(5.7) THEOREM. *Let φ: $S \to P$ be an n-admissible compact field such that* $\mathrm{Deg}\,\varphi \neq \{0\}$. *Then for each $x \in S$ there is a positive real number λ such that $\lambda x \in \varphi(S)$.*

Proof. Suppose that there exists an $x_0 \in S$ such that

$$L_{x_0} = \{\lambda x_0;\ \lambda \geqslant 0\} \cap \varphi(S) = \emptyset.$$

Let $\varepsilon = \frac{1}{2}\min\big(d(\varphi(S), L_{x_0}), d(0, \varphi(S))\big)$. We observe that $d(\varphi(S), L_{x_0}) > 0$, and by assumption we have $\varepsilon > 0$.

Let $(p, q) \subset \Phi$ be a selected pair of the form $S \xleftarrow{p} Y \xrightarrow{q} E$ such that $\deg(p, q) \neq 0$. We take an ε-approximation q_ε: $Y \to E^{k+1}$ such that $x_0 \in E^{k+1}$.

Consider the diagram

$$S^k \xleftarrow{p_k} Y_k \xrightarrow{\tilde{q}_k} P^{k+1}$$

(comp. the definition of $\deg(p, q)$ in this section). Since q_ε is an ε-approximation of q, we obtain

$$\tilde{q}_k(p_k^{-1}(S^k)) \subset O_\varepsilon(\varphi(S)),$$

where $O_\varepsilon(\varphi(S)) = \{y \in E$; there is a point $z \in \varphi(S)$ such that $\|z - y\| < \varepsilon\}$ and hence

(1) $$L_{x_0} \cap \tilde{q}_k(p_k^{-1}(S^k)) = \emptyset.$$

Consider the map ψ_k: $S^k \to P^{k+1}$ given by $\psi_k(x) = \tilde{q}_k(p_k^{-1}(x))$ for each $x \in S^k$. Then $(p_k, \tilde{q}_k) \subset \psi_k$ is a selected pair of ψ_k and hence ψ_k is an n-admissible map. Moreover, $\deg(p_k, \tilde{q}_k) = \deg(p, q) \neq 0$.

Let r: $P^{k+1} \to S^k$ be retraction ($r(x) = x/\|x\|$). Then $\mathrm{Deg}(r\psi_k) \neq \{0\}$ and from (1) we have $x_0 \notin (r\psi_k)(S^k)$, but this contradicts (3.8). The proof of (5.7) is completed.

Let $\varphi = I - \Phi$: $K \to E$ be an n-admissible compact field such that $\varphi(S) \subset P$, where S is the boundary of the closed ball K. By $\varphi|_S$: $S \to P$ we denote the contraction of φ to the pair (S, P). From (1.6) we infer that $\varphi|_S$ is an n-admissible compact vector field on S. In this case with every selected pair $(p, q) \subset \Phi$ we associate a pair $(p_1, q_1) \subset \Phi|_S$ as follows: let p: $Y \to K$, q: $Y \to E$ be two maps such that $(p, q) \subset \Phi$; then $p_1: p^{-1}(S) \to S$, $q_1: p^{-1}(S) \to E$ are given as contractions of p and q, respectively. Evidently $(p_1, q_1) \subset \Phi|_S$. We define degree $\mathrm{Deg}(\varphi, 0)$ of φ by putting

(5.8) $$\mathrm{Deg}(\varphi, 0) = \{\deg(p_1, q_1);\ (p, q) \subset \Phi\}.$$

Clearly, $\mathrm{Deg}(\varphi, 0) \subset \mathrm{Deg}(\varphi|_S)$.

Let φ: $K \to E$ be an n-admissible compact field such $\varphi(S) \subset E \setminus \{z_0\}$. By $(\varphi - z_0)$: $K \to E$ we denote the n-admissible compact field given by

$$(\varphi - z_0)(x) = \{y - z_0,\ y \in \varphi(x)\}$$

for each $x \in K$. Observe that $(\varphi - z_0)(S) \subset P$. We define $\mathrm{Deg}(\varphi, z_0)$ by putting

(5.9) $$\mathrm{Deg}(\varphi, z_0) = \mathrm{Deg}(\varphi - z_0, 0).$$

The following lemma is of importance:

(5.10) LEMMA. *Let φ: $K \to E$ be an n-admissible compact vector field such that $\varphi(S) \subset P$. If $\mathrm{Deg}(\varphi, 0) \neq \{0\}$, then there exists a point $x_0 \in K$ such that $0 \in \varphi(x_0)$.*

Proof. Let $\varphi = I - \Phi$, where Φ: $K \to E$ is an n-admissible compact map. Assume that $0 \notin \varphi(x)$ for all $x \in K$. From Lemma (4.3) we obtain a positive number δ such that $d(0, \varphi(K)) = \delta$. Let ε be a positive number such that $\varepsilon < \delta$. Let (p, q) be a selected pair of Φ of the form $K \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} E$. Let q_ε: $Y \to E^{k+1}$ be an ε-approximation of q. Then, as in the definition of degree $\deg(p, q)$, we obtain the following diagram:

$$K^{k+1} \overset{p_k}{\leftarrow} Y_k \overset{\tilde{q}_k}{\rightarrow} P^{k+1}.$$

Consider the map ψ: $K^{k+1} \to P^{k+1}$ given by $\psi(x) = q_k(p_k^{-1}(x))$ for each $x \in K^{k+1}$. Applying Lemma (3.6) to the map ψ we have $\mathrm{Deg}(\psi, 0) = \{0\}$. Consequently, $\deg(p_1, q_1) = 0$, where (p_1, q_1) is the pair associated with (p, q). Since (p, q) is an arbitrary selected pair of Φ, we obtain $\mathrm{Deg}(\varphi, 0) = \{0\}$ and the proof is completed.

The following theorem is an extension to the case of n-admissible maps of a well known theorem of Rothe (cf. [9,6]).

(5.11) THEOREM. *If Φ: $K \to E$ is an n-admissible compact map such that $\Phi(S) \subset K$, then Φ has a fixed point.*

Proof. Let φ: $K \to E$ be an n-admissible compact field given by $\varphi = I - \Phi$. We may assume without loss of generality that $\varphi(S) \subset P$ and by Lemma (5.10) it suffices to prove that $\mathrm{Deg}(\varphi, 0) \neq \{0\}$. For this purpose let

$$\psi(x, t) = x - t\Phi(x) \quad \text{for an arbitrary } x \in S,\ 0 \leqslant t \leqslant 1.$$

It follows from our assumption that for an arbitrary $z \in \psi(x, t)$ we have

$$\|z\| = \|x - ty\| \geqslant \|x\| - t\|y\| > 0 \quad \text{for} \quad 0 \leqslant t < 1$$

and thus ψ: $S \times [0, 1] \to P$.

It is evident that $\psi(S \times [0, 1])$ is a closed subset of E and hence $d(0, \psi(S \times [0, 1])) = \delta > 0$.

Let $(p, q) \subset \Phi$ be a selected pair of the form $K \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} E$ and let $S \overset{p_1}{\leftarrow} p^{-1}(S) \overset{q_1}{\rightarrow} E$ be the pair associated with (p, q) (comp. the definition of $\mathrm{Deg}(\varphi, 0)$). Let q_ε: $p^{-1}(S) \to E^{k+1}$ be an ε-approximation of q_1, where $0 < \varepsilon < \delta$. We put $S^k = S \cap E^{k+1}$ and $Y_k = p^{-1}(S^k)$.

We have the diagram

$$S^k \overset{p_k}{\leftarrow} Y_k \overset{q_k}{\rightarrow} E^{k+1},$$

in which p_k, q_k are contractions of p_1 and q_ε, respectively.

Define the map $\tilde{q}_k$: $Y_k \to P^{k+1}$ by putting $\tilde{q}_k(y) = p_k(y) - q_k(y)$ for each $y \in Y_k$. We claim that $\deg(p_1, q_1) = \deg(p_k, \tilde{q}_k) \neq 0$. In this order, consider the map f: $Y_k \to P^{k+1}$ given by $f(y) = p_k(y)$ and a homotopy h: $Y_k \times [0, 1] \to P^{k+1}$ given by $h(y, t) = p_k(y) - tq_k(y)$. Since $\varphi(S) \subset P$ and q_ε is an ε-approximation of q_1, $0 < \varepsilon < \delta$,

we deduce that $h(Y_k \times [0, 1]) \subset P^{k+1}$. Then the maps f and $\tilde{q}_k$ are homotopic and hence $f^* = \tilde{q}_k^*$. Finally, we obtain

$$\deg(p_1, q_1) = \deg(p_k, \tilde{q}_k) = \deg(p_k, f) \neq 0,$$

and the proof is completed.

In fact, from the above proof we infer that $\mathrm{Deg}(\varphi, 0)$ is a singleton and $0 \neq \mathrm{Deg}(\varphi, 0)$.

6. Theorems on antipodes. Let A be a subset of Banach space E and let η be a positive real number. Define the set $O_\eta(A)$ by putting $O_\eta(A) = \{z \in E;$ there is an $x \in A$ such that $\|z - x\| < \eta\}$.

The following theorem is a certain extension of the theorem on antipodes ([9], [12]) to n-admissible fields.

(6.1) THEOREM. *Let $\varphi = I - \Phi$: $S \to P$ be an n-admissible compact field. Suppose that there exists a real positive number η such that for each $x \in S$ there exists a subspace E_x of E, of codimension equal to 1, strictly separating $O_\eta(\varphi(x))$ and $O_\eta(\varphi(-x))$. Then $0 \notin \mathrm{Deg}\,\varphi$.*

Proof. Consider a selected pair $(p, q) \subset \Phi$ of the form $S \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} E$. Let $\varepsilon_0 = \min(\eta, d(0, \varphi(S)))$. By assumption $\varepsilon_0 > 0$. We take an ε-approximation q_ε: $Y \to E^{k+1}$ of q, with $0 < \varepsilon < \varepsilon_0$ $(k \geqslant n)$. Consider the diagram

$$S^k \overset{p_k}{\leftarrow} p^{-1}(S^k) \overset{\tilde{q}_k}{\rightarrow} P^{k+1},$$

where p_k is the contraction map of p to the pair $(S^k, p^{-1}(S^k))$ and $\tilde{q}_k(y) = p_k(y) - q_k(y)$ for each $y \in p^{-1}(S^k)$. Then we have an n-admissible map ψ: $S^k \to P^{k+1}$ given as the composition $\psi = \tilde{q}_k p_k^{-1}$.

Let $E_x^k = E_x^k \cap E^{k+1}$. Observe that $\dim E_x^k = k$. Since q_ε is an ε-approximation of q, $\varepsilon < \eta$, by assumption we have $\psi(x) \subset O_\eta(\varphi(x))$ for each $x \in S^k$. This implies that E_x^k strictly separate $\psi(x)$ and $\psi(-x)$ for each $x \in S^k$. Applying Theorem (3.8) to ψ, we obtain $0 \notin \mathrm{Deg}(\psi)$ and hence $0 \neq \deg(p, q)$. Since (p, q) is an arbitrary selected pair of Φ, we have $0 \notin \mathrm{Deg}(\varphi)$ and the proof is completed.

From (5.10) and (6.1) we obtain

(6.2) COROLLARY. *Let φ: $K \to E$ be an n-admissible compact field such that $\varphi|_S$ satisfies all the assumptions of (6.1). Then there is a point $x_0 \in K \setminus S$ such that $O \in \varphi(x_0)$.*

Now, for admissible compact fields we prove a stronger version of Theorem (6.1).

(6.3) THEOREM. *Let φ: $S \to P$ be an admissible compact field. Suppose that there exists a positive real number $\eta > 0$ such that the following condition is satisfied:*

every half-ray $L_y = \{z \in E;\ z = ty$ for some $t \geqslant 0\}$ has an empty intersection with the set $O_\eta(\varphi(x))$ or $O_\eta(\varphi(-x))$ for each $x \in S$. Then $0 \notin \mathrm{Deg}(\varphi)$.

Outline of the proof. Consider the admissible map ψ given in the same way as in the proof of (6.1). Applying Theorem (3.9) to the map ψ, we deduce (6.3).

From (5.10) and (6.3) we infer

(6.4) COROLLARY. *Let φ: $K \to E$ be an admissible compact field such that $\varphi|_S$ satisfies all the assumptions of* (6.3). *Then there is a point $x_0 \in K \setminus S$ such that $0 \in \varphi(x_0)$.*

7. Theorem on the invariance of domain. In this section we denote by K_ε a closed ball in a Banach space E with the centre 0 and radius ε, and by S_ε the boundary of K_ε in E. Let A be a subset of E.

A compact s-admissible field φ: $A \to E$ is called an *ε-field* provided the condition

$$\varphi(x_1) \cap \varphi(x_2) \neq \emptyset \Rightarrow \|x_1 - x_2\| < \varepsilon$$

is satisfied for any $x_1, x_2 \in A$.

A compact s-admissible field φ: $A \to E$ is called an *ε-field in the narrow sense* if for some constant $\eta > 0$ the condition

$$O_\eta(\varphi(x_1)) \cap O_\eta(\varphi(x_2)) \neq \emptyset \Rightarrow \|x_1 - x_2\| < \varepsilon$$

is satisfied for every $x_1, x_2 \in A$.

The proof of the theorem on the invariance of domain for ε-fields in the narrow sense is based on the following lemmas.

(7.1) LEMMA. *Let φ: $K \to E$ be an ε-field in the narrow sense. Then:*

(i) $\varphi(S_\varepsilon) \subset E \setminus \{y_0\}$ *for each* $y_0 \in \varphi(0)$

and

(ii) $0 \notin \mathrm{Deg}(\varphi, y_0)$.

Proof. For the proof of (i) we observe that if $\varphi(0) \cap \varphi(x) \neq \emptyset$ for some $x \in K_\varepsilon$, then $O_\eta(\varphi(0)) \cap O_\eta(\varphi(x)) \neq \emptyset$ and this implies that $\|x\| < \varepsilon$; hence $\varphi(S) \subset E \setminus \{y_0\}$ for each $y_0 \in \varphi(0)$.

Let Φ: $K \to E$ be a compact part of φ, i.e., $\varphi = I - \Phi$. Consider a selected pair $(p, q) \subset \Phi$ of the form $K \xleftarrow{p} Y \xrightarrow{q} E$ and a point $y_0 \in E$ such that $y_0 \in \varphi(0)$.

Let $\delta = \min(\eta, d(y_0, \varphi(S_\varepsilon)))$, where φ is the ε-field in the narrow sense with the constant η. It is evident that δ is a positive real number. We take an δ-approximation q_δ: $Y \to E^{k+1}$ of the compact map q such that $y_0 \in E^{k+1}$.

Let $K_\varepsilon^{k+1} = K_\varepsilon \cap E^{k+1}$ and $Y_k = p^{-1}(K^{k+1})$. We have the diagram

$$K_\varepsilon^{k+1} \xleftarrow{p_k} Y_k \xrightarrow{\tilde{q}_k} E^{k+1},$$

in which p_k is the contraction of p to the pair (Y_k, K^{k+1}) and $\tilde{q}_k$ is given by $\tilde{q}_k(y) = p_k(y) - q_\delta(y)$ for each $y \in Y_k$.

Let φ_k: $K^{k+1} \to E^{k+1}$ be a multi-valued map given by $\varphi_k(x) = \tilde{q}_k(p_k^{-1}(x))$ for each $x \in K^{k+1}$. From (4.4) we deduce that φ_k is an s-admissible map. We assert that φ_k is an ε-map. Indeed, because $0 < \delta \leqslant \eta$ we have $\varphi_k(x) \subset O_\delta(\varphi(x)) \subset O_\eta(\varphi(x))$ for each $x \in K_\varepsilon^{k+1}$ and hence the condition $\varphi_k(x_1) \cap \varphi_k(x_2) \neq \emptyset$ implies $O_\eta(\varphi(x_1)) \cap \cap O_\eta(\varphi(x_2)) \neq \emptyset$. Then, by assumption, we obtain $\|x_1 - x_2\| < \varepsilon$ and φ_k is an ε-map.

Applying Lemma (3.10) to the map φ_k, we obtain $0 \notin \mathrm{Deg}(\varphi_k, y_0)$ and hence $\deg(p, q) \neq 0$. Since (p, q) is an arbitrary selected pair of Φ, we obtain (ii) and the proof is completed.

(7.2) LEMMA. *If φ: $K \to E$ is an admissible compact field and $y_0 \notin \varphi(S)$, then for every $y_1 \in E$ such that $\|y_0 - y_1\| < d(y_0, \varphi(S))$ we have* $\mathrm{Deg}(\varphi, y_0) \cap \mathrm{Deg}(\varphi, y_1) \neq \emptyset$.

Proof. Consider the map χ: $K \times [0, 1] \to E$ given by $\chi(x, t) = x - X(x, t)$, where $X(x, t) = \Phi(x) + (ty_1 + (1-t)y_0)$. It is evident that $\chi(S \times [0, 1]) \subset E \setminus \{0\}$ and $\chi(S \times [0, 1])$ is a closed subset of E. Therefore χ is a homotopy between $\varphi - y_0$ and $\varphi - y_1$ and by similar arguments to those used in the proof of (5.4) we deduce (7.2).

Remark. It is possible to prove that $\mathrm{Deg}(\varphi, y_0) = \mathrm{Deg}(\varphi, y_1)$ for φ as in (7.2) but we only need (7.2).

We now prove the main result of this section.

(7.3) THEOREM. *If φ: $E \to E$ is an ε-field in the narrow sense, then $\varphi(E)$ is an open subset of E.*

Proof. Let $y_0 \in \varphi(x_0)$ be a point in $\varphi(E)$. Consider the closed ball $K_\varepsilon = K(y_0, \varepsilon)$ and let $\psi = \varphi|_{K_\varepsilon}$ be the restriction of φ to K_ε. Then ψ: $K_\varepsilon \to E$ is an ε-field in the narrow sense. Applying Lemma (7.1) to ψ, we obtain $0 \notin \mathrm{Deg}(\psi, y_0)$. Let $y_1 \in E$ be a point such that $\|y_0 - y_1\| < d(y_0, \psi(S_\varepsilon))$. Then from (7.2) and (7.1) we infer $\mathrm{Deg}(\psi, y_1) \neq \{0\}$ and, in view of (5.10), we have $y_1 \in \psi(K_\varepsilon)$. This implies that $B(y_0, \delta) \subset \psi(K_\varepsilon) \subset \varphi(E)$, where $B(y_0, \delta)$ is the open ball in E with centre y_0 and radius $\delta = d(y_0, \psi(S_\varepsilon))$. The proof of (7.3) is completed.

Because E is a connected space, from Theorems (7.3) and (4.3) we deduce

(7.4) COROLLARY. *If φ: $E \to E$ is an ε-field in the narrow sense, then $\varphi(E) = E$.*

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Symmetric words in nilpotent groups of class $\leqslant 3$

by

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Abstract. Let G be a group. A word $w = w(x_1, \ldots, x_n)$ is said to be *symmetric* if $w(a_1, \ldots, a_n) = w(a_{\pi 1}, \ldots, a_{\pi n})$ for all $a_1, \ldots, a_n \in G$ and all permutations π from the symmetric group S_n on n-letters. In this note we describe symmetric words in nilpotent groups of class $\leqslant 3$.

1. Introduction and notation. Let G be a group, and let $F_G(x_1, \ldots, x_r)$ be the group freely generated by $x_1, \ldots, x_r$ in the smallest variety $\mathrm{var}(G)$ of groups containg G. Let A be the group of automorphisms of $F_G(x_1, \ldots, x_r)$ induced by the mappings

$$x_i \to x_{\mu i}, \quad 1 \leqslant i \leqslant r,$$

μ belonging to the symmetric group S_r on r letters. Let $S^{(r)}(G)$ be the set of all fixed points of A, i.e.,

$$S^{(r)}(G) = \{w\colon\ \xi w = w \text{ for all } \xi \in A\}.$$

The elements of $S^{(r)}(G)$ are called *symmetric words* (of r variables) in G.

Clearly, $S^{(r)}(G)$ is a group. The aim of this note is to describe symmetric words in nilpotent groups of class $\leqslant 3$. We prove that in this case $S^{(r)}(G)$ is Abelian.

2. Symmetric words. In an Abelian group every word w of r variables is of the form

$$w = \prod_{1 \leqslant i \leqslant r} x_i^{a(i)}.$$

We thus have

THEOREM 1. *If G is an Abelian group, then $w \in S^{(r)}(G)$ if and only if*

$$w = \prod_{1 \leqslant i \leqslant r} x_i^a.$$

In [3] all elements of $S^{(r)}(G)$ for a nilpotent G of class 2 are described. Namely

THEOREM 2. *If G is a nilpotent group of class 2, then $w \in S^{(r)}(C)$ if and only if*

$$w = \prod_{1 \leqslant i \leqslant r} x_i^a \prod_{1 \leqslant j < i \leqslant r} [x_i, x_j]^b,$$