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A rigid topological vector space

by

L. WAELBROECK (Bruxelles)

Abstract. A topological vector space is *rigid* if its only continuous endomorphisms are multiples of the identity mapping. We give here an example of such a rigid space, it is metrisable, separable, and p-normable for all $p < 1$.

The space we describe is not complete. It is a dense subspace of a Fréchet space that we describe first. This Fréchet space E has few continuous endomorphisms. The algebra $\mathscr{L}(E)$ of all these endomorphisms is isomorphic to $L_\infty(I)$.

Our construction will use a result of P. Turpin (cf. [3], or [4], paragraph 3.4, Theorem 3.4.8, p. 95). Assume that ϱ_1 and ϱ_2 are two increasing, concave mappings $\boldsymbol{R}_+ \to \boldsymbol{R}_+$, continuous and vanishing at the origin. Let L_{ϱ_i} be the space of measurable functions on a diffuse probability space, defined modulo null sets, and such that

$$\nu_{\varrho_i}(u) = \int \varrho_i(|u|)\, dm < \infty$$

with the topology defined by the $\mathfrak{F}$-norm ν_{ϱ_i}. Turpin then shows that a non-zero continuous linear mapping $L_{\varrho_1} \to L_{\varrho_2}$ can only exist if $L_{\varrho} \subseteq L_{\varrho}$, i.e. if

$$\limsup_{t\to\infty} \varrho_2(t)/\varrho_1(t) < \infty.$$

1. Γ will be a compact metrisable space with a diffuse probability measure, $\varrho(x, t)$ will be a continuous mapping $\Gamma \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ which is increasing and concave as a function of t for each constant x, and such that $\varrho(x, 0) = 0$. And $L_\varrho(\Gamma)$ will be the space of measurable functions u on Γ, defined modulo null sets, such that

$$\nu_\varrho(u) = \int \varrho\big(x, |u(x)|\big)\, dm < \infty \tag{1}$$

with the $\mathfrak{F}$-norm ν_ϱ. The situation is thus a little bit more general than the one described in the introduction, where ϱ was independent of x.

It is clear that $L_\varrho(\Gamma)$ is a separable $\mathfrak{F}$-space.

L_ϱ is a *p-normable* space if some t_0 exists such that

$$\varrho(x, t') \geqslant \varrho(x, t) \cdot \left(\frac{t'}{t}\right)^p \tag{2}$$

when $t' \geqslant t \geqslant t_0$. This is easy. We let $\varrho_1(x,t) = \varrho(x,t)$ when $t \geqslant t_0$, $\varrho_1(x,t) = \varrho(x,t_0)(t/t_0)^p$ when $t < t_0$. The function ϱ_1 satisfies relation (2) but is not concave in t for constant x. We next let ϱ_2 be the smallest concave function of t which is larger than ϱ_1. Then ϱ_2 is concave, clearly satisfiee relation (2), for $t' \geqslant t > 0$, and is less than

$$\varrho(x,t) + \varrho(x,t_0)\left(\frac{t}{t_0}\right)^p,$$

when $t > t_0$, which shows that ν_ϱ and ν_{ϱ_2} are equivalent $\mathfrak{F}$-norms.

Now, if $\varepsilon > 0$, if

$$B_\varepsilon = \{f \mid \nu_{\varrho_2}(f) < \varepsilon\},$$

we see that

$$B_\varepsilon + B_\varepsilon \subseteq B_{2\varepsilon}$$

because ϱ_2 is concave, also that

$$B_{2\varepsilon} \subseteq 2^{1/p} B_\varepsilon$$

because ϱ_2 satisfies relation (2) for $t' \geqslant t \geqslant 0$, therefore $L_{\varrho_2} = L_\varrho$ is p-normable by the Rolewicz theorem ([2]; or [1], p. 165).

2. $\Gamma = \{0,1\}^N$ will be the Cantor set, with its usual probability measure, i.e. the countable direct product of the measure we obtain when we put on $\{0,1\}$ that measure for which $\{0\}$ and $\{1\}$ each have measure $1/2$.

To fix the notations, if $q \in N$, if $k = 0, 1, \ldots, 2^q - 1$, we let

$$\gamma_{qk} = \Big\{(a_n)_{n \in N} \in \Gamma \Big| \sum_0^{q-1} a_n 2^n = k\Big\}.$$

The sets γ_{qk} (q constant) are the 2^q elements of the obvious partition of Γ "of order q". The measure of each γ_{qk} is 2^{-q}.

We shall later prove the existence of a function $\varrho(x,t)$, and of functions $\varrho_{q+}(x,t)$, $\varrho_{q-}(x,t)$, defined for $x \in \Gamma$ and $t \in \boldsymbol{R}_+$, taking their values in $\boldsymbol{R}_+$, and having several properties:

a. The functions ϱ, ϱ_{q+}, ϱ_{q-} are continuous, $\varrho_{q-} \leqslant \varrho \leqslant \varrho_{q+}$;

b. For x constant, each of these functions is an increasing concave function of t which vanishes when $t = 0$;

c. $\varrho_{q+}(x,t) = \varrho_{q+}(x',t)$, $\varrho_{q-}(x,t) = \varrho_{q-}(x',t)$ when x and x' belong to the same element γ_{qk} of the partition of order q of Γ;

d. $\limsup_{t\to\infty} \varrho_{q-}(x',t)/\varrho_{q+}(x,t) = \infty$ when x and x' belong to different elements γ_{qk}, γ_{qk} ($k \neq k'$) of that partition of Γ.

The aim of Sections 3 and 4 will be the proof of

PROPOSITION 1. *If ϱ is the function whose existence is postulated above, the only continuous endomorphisms of $L_\varrho(\Gamma)$ are multiplications by the measurable functions t such that $tL_\varrho(\Gamma) \subseteq L_\varrho(\Gamma)$.*

3. Let $A \subseteq \Gamma$ be a measurable subset. We shall call $L_\varrho(A)$ the set of $u \in L_\varrho(\Gamma)$ which vanish on the complement of A.

LEMMA. *$L_\varrho(A)$ is an invariant subspace of $L_\varrho(\Gamma)$ under endomorphisms, this for all measurable subsets A of Γ.*

The proof of this lemma will be done in a few steps. Let $\mathscr{A}$ be the set of measurable subsets of Γ such that $L_\varrho(A)$ is invariant under continuous endomorphisms. We must prove that all measurable sets belong to $\mathscr{A}$.

a. *The sets γ_{qk} belong to $\mathscr{A}$.* A continuous endomorphism of $L_\varrho(\Gamma)$ has a restriction to $L_\varrho(\gamma_{qk})$ which is a continuous linear mapping

$$L_\varrho(\gamma_{qk}) \to L_\varrho(\Gamma) = \oplus_{k'} L_\varrho(\gamma_{qk'}).$$

We must show that a continuous linear T: $L_\varrho(\gamma_{qk}) \to L_\varrho(\gamma_{qk'})$ vanishes if $k \neq k'$.

We have

$$L_{\varrho_{q+}}(\gamma_{qk}) \subseteq L_\varrho(\gamma_{qk}) \xrightarrow{T} L_\varrho(\gamma_{qk'}) \subseteq L_{\varrho_{q-}}(\gamma_{qk'}).$$

The first imbedding is dense and continuous. The last imbedding is continuous, and of course, injective. The composition is a continuous linear mapping $L_{\varrho_{q+}}(\gamma_{qk}) \to L_{\varrho_{q-}}(\gamma_{qk'})$. And if T were different from zero, this composition would also be different from zero.

But Turpin's result, quoted in the introduction, implies that every continuous linear map $L_{\varrho_{q+}}(\gamma_{qk}) \to L_{\varrho_{q-}}(\gamma_{qk})$ vanishes. As a matter of fact, $\varrho_{q+}(x,t)$ and $\varrho_{q-}(x',t)$ are independent of x or x' when $x \in \gamma_{qk}$, $x' \in \gamma_{qk'}$, we can put $\varrho_{qk+}(t) = \varrho_{q+}(x,t)$, $\varrho_{qk'-}(x,t) = \varrho_{q-}(x',t)$ when $x \in \gamma_{qk}$, $x' \in \gamma_{qk'}$, $L_{\varrho_{q+}}(\gamma_{pk}) = L_{\varrho_{qk+}}(\gamma_{pk})$, $L_{\varrho_{q-}}(\gamma_{pk'}) = L_{\varrho_{pk'-}}(\gamma_{pk'})$ and $\limsup_{t\to\infty} \varrho_{qk'-}/\varrho_{qk+} = \infty$.

b. *$\mathscr{A}$ is stable under finite unions.* Let $A_1 \in \mathscr{A}$, $A_2 \in \mathscr{A}$, and $A = A_1 \cup A_2$. Then $L_\varrho(A) = L_\varrho(A_1) + L_\varrho(A_2)$. A continuous endomorphism of $L_\varrho(\Gamma)$ leaves both terms of the right-hand side invariant, and leaves therefore $L_\varrho(A)$ invariant.

c. *$\mathscr{A}$ is stable under countable unions.* Let $A = \bigcup_1^\infty A_n$, where each $A_n \in \mathscr{A}$. We may assume $A_n \subseteq A_{n+1}$. Then $\bigcup_1^\infty L_\varrho(A_n)$ is a vector space, is invariant, and is dense in $L_\varrho(A)$.

d. $\mathscr{A}$ *is stable under countable intersections.* Let $A = \bigcap_1^\infty A_n$, where each $A_n \in \mathscr{A}$. Then $L_\varrho(A) = \bigcap L_\varrho(A_n)$, and an intersection of invariant subspaces is invariant.

e. $A \in \mathscr{A}$ *if* $A = A' \Delta N$, *where* N *is a null set and* $A' \in \mathscr{A}$. (Here, Δ designates the symmetric difference.) Of course, since $L_\varrho(A) = L_\varrho(A')$.

The lemma, i.e. the fact that all measurable sets belong to $\mathscr{A}$, follows from the results a, b, c, d, e, above.

4. It is now easy to prove Proposition 1. Let 1 be the constant function equal to unity, let $t = T\cdot 1$, where $T\colon L_\varrho(\Gamma) \to L_\varrho(\Gamma)$ is a continuous linear mapping. Then $T(u) = t\cdot u$ for all $u \in L_\varrho(\Gamma)$ ($t\cdot u$ is the pointwise product of t and u).

We first observe that $T\cdot 1_A = t\cdot 1_A$ if 1_A is the characteristic function of the measurable set A. As a matter of fact, if A' is the complement of A, $1_A + 1_{A'} = 1$, $T\cdot 1_A + T\cdot 1_{A'} = t$, but $T\cdot 1_A$ vanishes on the complement of A (by Proposition 1) and $T\cdot 1_{A'}$ vanishes on A. This proves the result.

Let next $f = \sum s_n 1_{A_n}$ be a step function. Then $Tf = tf$ by linearity.

If f is a positive real-valued function, $f \in L_\varrho(\Gamma)$, we choose an increasing sequence of step functions f_n, tending pointwise to f. Then $f_n \to f$ in $L_\varrho(\Gamma)$. Also $tf_n = Tf_n \to Tf$, but tf_n cannot have any limit in $L_\varrho(\Gamma)$ other than tf, hence $tf \in L_\varrho(\Gamma)$ and $Tf = tf$.

Proposition 1 is proved by writing a general $f \in L_\varrho(\Gamma)$ as a linear combination of positive real-valued functions.

5. We have used, but have not constructed, the function $\varrho(x, t)$ whose existence was postulated in Section 2. The construction of ϱ will involve two functions σ_+, σ_-, $\boldsymbol{R}_+ \to \boldsymbol{R}_+$, both continuous, concave, increasing, vanishing at the origin, such that $\sigma_-(t) \leqslant \sigma_+(t)$ for all t, such that a sequence $t_n \to \infty$ exists with $\sigma_+(t_n) = \sigma_-(t_n)$ and

$$\lim_{n\to\infty} \sup_{t_n \leqslant t \leqslant t_{n+1}} \sigma_+(t)/\sigma_-(t) = \infty.$$

We shall prove the existence of such functions in Section 7, and construct σ_+, σ_- in such a way that

$$\sigma_+(t') \geqslant \left(\frac{t'}{t}\right)^{1-1/n} \sigma_+(t),$$

$$\sigma_-(t') \geqslant \left(\frac{t'}{t}\right)^{1-1/n} \sigma_-(t),$$

when $t' > t \geqslant t_n$.

Let σ_+ and σ_- be two functions with the properties described above. Let $\varepsilon \in \Gamma$, then $\varepsilon = (\varepsilon_n)_{n\in \boldsymbol{N}}$, where for each n, $\varepsilon_n = 0$ or $\varepsilon_n = 1$. We define $\sigma_\varepsilon(t)$ by the relations

$$\sigma_\varepsilon(t) = \sigma_+(t) \quad \text{if } t_n \leqslant t \leqslant t_{n+1},\ \varepsilon_n = +1,$$

$$\sigma_\varepsilon(t) = \sigma_-(t) \quad \text{if } t_n \leqslant t \leqslant t_{n+1},\ \varepsilon_n = 0.$$

Clearly, the mapping $(\varepsilon, t) \mapsto \sigma_\varepsilon(t)$ is a continuous mapping $\Gamma \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$, and for each constant ε, $t \mapsto \sigma_\varepsilon(t)$ is concave, increasing, and vanishes at the origin. For each $p < 1$, we can find t_p such that $\sigma_\varepsilon(t') \geqslant \sigma_\varepsilon(t)(t'/t)^p$. Also,

$$\limsup_{t\to\infty} \frac{\sigma_\varepsilon(t)}{\sigma_{\varepsilon'}(t)} = \infty$$

if an infinite number of indices n_k exist such that $\varepsilon_{n_k} = 1$, $\varepsilon'_{n_k} = 0$.

We define mappings $f, f_{q+}, f_{q-}\colon \Gamma \to \Gamma$ in the following way. Let first $\boldsymbol{N} = X_1 \cup \ldots \cup X_n \cup \ldots$ be an infinite partition of $\boldsymbol{N}$ into infinite sets. Let also $X_n = X'_n \cup X''_n$ be a partition of X_n into two infinite subsets. For each $x = (x_k)_{k\in\boldsymbol{N}}$, let $f(x) = (\varepsilon_k)_{k\in\boldsymbol{N}}$, where

$$\varepsilon_k = 0 \quad \text{if } k \in X'_n \text{ and } x_n = 0 \text{ or } k \in X''_n \text{ and } x_n = 1,$$

$$\varepsilon_k = 1 \quad \text{if } k \in X'_n \text{ and } x_n = 1 \text{ or } k \in X''_n \text{ and } x_n = 0.$$

We also let $f_{q+}(x) = \delta'$, $f_{q-}(x) = \delta''$, where $\delta'_k = \delta''_k = \varepsilon_k$, when $k \in \bigcup_{n\leqslant q-1} X_n$, while $\delta'_k = 1$ and $\delta''_k = 0$ when $k \in \bigcup_{n\geqslant q} X_n$. These mappings are such that $f_{q+}(x) \geqslant f(x) \geqslant f_{q-}(x)$ when we put on Γ its natural order. Also, f_{q+} and f_{q-} remain constant on each element of the partition $\Gamma = \bigcup \gamma_{qk}$ of order q of Γ, and, if x and x' are in two different elements of this partition, i.e. if there is an $n \leqslant q-1$ such that $x_n \neq x'_n$, an infinite set of indices m can be found such that $[f_{q-}(x)]_m = 1$, $[f_{q+}(x)]_m = 0$.

The functions

$$\varrho(x, t) = \sigma_{f(x)}(t),$$

$$\varrho_{q+}(x, t) = \sigma_{f_{q+}}(x, t),$$

$$\varrho_{q-}(x, t) = \sigma_{f_{q-}}(x, t)$$

have all the properties announced in Section 2.

6. Proposition 2. *The function ϱ being constructed as in Section 5, $L_\varrho(\Gamma)$ is a p-normable for all $p < 1$. The only endomorphisms of $L_\varrho(\Gamma)$ are multiplications by elements of $L_\infty(\Gamma)$.*

We know that $\varrho(x, t') \geqslant \varrho(x, t)(t'/t)^{1-1/n}$ when $t' \geqslant t \geqslant t_n$. This has been shown in Section 1 to be sufficient for p-normability when $p = 1-1/n$.

The endomorphisms of $L_\varrho(\Gamma)$ are multiplications by functions τ such that $\tau L_\varrho \subseteq L_\varrho$. The mapping $u \mapsto \tau u$ is then continuous. We shall show that τu does not depend continuously on u when τ is unbounded.

Let $E_n = \{x \mid \tau(x) \geqslant n\}$; choose $a_n(\varepsilon)$ in such a way that

$$\int_{E_n} \varphi\big(x, a_n(\varepsilon)\big)\, dm = \varepsilon.$$

We notice that $a_n(\varepsilon) \to \infty$ for ε constant when $n \to \infty$, hence $a_n(\varepsilon) > t_2$ when $n \geqslant N(\varepsilon)$ for some constant $N(\varepsilon)$. But, for $t \geqslant t_2$, we know that $\varphi(x, \tau t) \geqslant \tau^{1/2}\varphi(x, t)$. Also $\tau(x) \geqslant n$ for $x \in E_n$; hence

$$\int_{E_n} \varphi\big(x, \tau(x)\, a_n(\varepsilon)\big) \geqslant n^{1/2}\varepsilon.$$

We choose $\varepsilon_n = n^{-1/2}$, then $\varepsilon_n 1_{E_n} \to 0$ but $\tau\cdot\varepsilon_n 1_{E_n}$ does not tend to zero in L_ϱ.

7. We must still construct the functions σ_+ and σ_- that were used in Section 5. The sequence of real numbers t_n will be constructed by induction, along with the functions σ_+, σ_- on the interval $(0, t_n)$. To start the induction, we let $t_1 = 1$, $\sigma_+(t) = \sigma_-(t) = t$ for $0 \leqslant t \leqslant 1$.

We now assume that t_n has been constructed, along with σ_+ and σ_- on the interval $(0, t_n)$, in such a way that these functions are continuous, concave, increasing, vanishing at the origin, that $\sigma_+(t_n) = \sigma_-(t_n)$, and that $\sigma_+ \geqslant \sigma_-$ on the interval $(0, t_n)$. We shall extend σ_+ and σ_- to the right of t_n in such a way that σ_+ remains concave, that $\sigma_-(t) = \sigma_-(t_n)(t/t_n)^{1-1/n}$ for values of t larger than t_n but near to t_n, and that $\sigma_+(t) \geqslant \sigma_-(t)$. This is only possible if the left-hand derivative of σ_+ at t_n is larger than

$$\frac{d}{dt}\left[\sigma_-(t_n)\left(\frac{t}{t_n}\right)^{1-1/n}\right]_{t=t_n} = \left(1-\frac{1}{n}\right)\frac{\sigma_+(t_n)}{t_n}$$

so we assume that we have the required inequality.

On an interval (t_n, t_n'), we let thus

$$\sigma_+(t) = \sigma_+(t_n) + (t - t_n)\left(1-\frac{1}{n}\right)\frac{\sigma_+(t_n)}{t_n},$$

$$\sigma_-(t) = \sigma_-(t_n)\left(\frac{t}{t_n}\right)^{1-1/n},$$

and we choose t_n' in such a way that $\sigma_+(t_n')/\sigma_-(t_n') \geqslant n$. We next extend σ_+ and σ_- to an interval (t_n', t_n''), letting on this interval

$$\sigma_+(t) = \sigma_+(t_n')\left(\frac{t}{t_n'}\right)^{1-1/n},$$

$$\sigma_-(t) = \sigma_-(t_n') + (t - t_n')\left(1-\frac{1}{n}\right)\frac{\sigma_-(t_n')}{t_n'},$$

and choose t_n'' in such a way that $\sigma_+(t_n'') = \sigma_-(t_n'')$.

On the interval (t_n'', t_{n+1}) we let

$$\sigma_+(t) = \sigma_-(t) = \sigma_+(t_n'') + (t - t_n'')\left(1-\frac{1}{n}\right)\frac{\sigma_+(t_n'')}{t_n''}$$

and choose t_{n+1} in such a way that

$$\left(1-\frac{1}{n}\right)\frac{\sigma_+(t_n'')}{t_n''} \geqslant \left(1-\frac{1}{n+1}\right)\frac{\sigma_+(t_{n+1})}{t_{n+1}},$$

i.e. the left-hand derivative of σ_+ at the point t_{n+1} must be larger than the derivative of $\sigma_-(t)$ in the interval (t_{n+1}, t_{n+1}').

We must check that the successive choices of t_n', t_n'', t_{n+1} are possible. The proof is easy as far as t_n' and t_n'' are concerned. The expression of σ_+ in the interval (t_n, t_n') has a larger order of magnitude than the expression of σ_- on the interval, so the ratio must become larger than n somewhere. On (t_n', t_n''), the expression of σ_- has a larger order of magnitude than that of σ_+, but $\sigma_-(t_n') < \sigma_+(t_n')$, so some t_n'' must exists where these two expressions become equal.

On (t_n'', t_{n+1}), the common expression of σ_+ and σ_- has the form $a + bt$ with $a > 0$, $b > 0$ (it is the tangent to a strictly increasing concave function which vanishes at the origin). Of course,

$$b = \left(1-\frac{1}{n}\right)\frac{\sigma_+(t_n'')}{t_n''}.$$

Further $(a+bt)/t \to b$ as $t \to \infty$, for t_{n+1} large enough we have

$$b \geqslant \left(1-\frac{1}{n+1}\right)\frac{a + bt_{n+1}}{t_{n+1}}$$

as required.

8. Proposition 3. *The space $L_\varrho(\Gamma)$ has a dense rigid subspace.*

If E is a topological vector space and E_0 a dense subspace, a continuous endomorphism of E_0 extends to E. In other words, the algebra of endomorphisms of E_0 can be identified with the algebra of endomorphisms of E which leave E_0 invariant.

All we need, therefore, is to find a dense subspace E_0 of $L_\varrho(\Gamma)$ such that τ is constant if τ is a measurable function such that $\tau E_0 \subseteq E_0$.

We identify, as usual, Γ with the unit interval and the Lebesgue measure, mapping $(x_n)_{n\in N}$ onto $\sum x_n 2^{-n-1}$. This is injective on the complement of a countable set and is measure preserving. We obtain in this way an Orlicz space of functions on the interval whose only endomorphisms are multiplications by elements of $L_\infty(I)$.

The space E_0 of continuous piecewise linear functions on I is a vector lattice, is dense in $C(I)$ by the Stone–Weierstrass theorem, but $C(I)$ is dense in $L_\varrho(I)$, so E_0 is dense in $L_\varrho(I)$. And clearly, τ is constant if τ is a function and $\tau E_0 \subseteq E_0$.

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Multipliers of L^p_E, II*

by

DANIEL M. OBERLIN (Tallahassee, Fla.)

Abstract. Let X be an abelian group, the character group of a compact group G. For a subset E of X let L^p_E be the subspace of E-spectral functions in $L^p(G)$. We show that if X is infinite and $p > 2$ is an even integer, then E can be chosen so that not every multiplier of $\widehat{L^p_E}$ extends to a multiplier of $\widehat{L^p}(G)$.

1. Let G be a compact abelian group with character group X. For $1 \leqslant p \leqslant \infty$, let $L^p(G)$ be the usual Lebesgue space with respect to normalized Haar measure on G, and for $E \subseteq X$, let L^p_E be the translation invariant subspace of $L^p(G)$ consisting of those functions whose Fourier transforms vanish off of E. Let M^p_E denote the set of functions in $l^\infty(E)$ which are multipliers for the Fourier transform space of L^p_E. Thus $\varphi \in M^p_E$ if and only if for every $f \in L^p_E$ there exists $g \in L^p_E$ with $\hat{g}(x) = \varphi(x)\hat{f}(x)$ for each $x \in E$. For $1 \leqslant p < \infty$, M^p_E can be identified with the space of operators on L^p_E which commute with translation by elements of G. Let $M^p|_E$ denote the set of restrictions to E of functions in $M^p (= M^p_X)$. Then, clearly, $M^p|_E \subseteq M^p_E$. We are interested in the following questions:

(i) Does $M^p|_E = M^p_E$?

(ii) For $1 \leqslant p_1 < p_2 < 2$ or $2 < p_2 < p_1 \leqslant \infty$, is $M^{p_1}_E \subseteq M^{p_2}_E$? (If $M^{p_i}_E$ is replaced by $M^{p_i}|_E$, $i = 1, 2$, the answer is yes, by the Riesz–Thorin theorem.)

Question (i) is posed for the circle group T in [3], pp. 280–281, and has an affirmative answer for any G if $p = 2$ (trivially) or if $p = \infty$ (see Proposition 1.2 below). On the other hand, in [12] the following theorem is proved.

THEOREM 1.1. *If G is infinite and $1 \leqslant p < 2$, there exists $E \subseteq X$ for which $M^p|_E$ is a proper subset of M^p_E.*

The present paper is a sequel to [12], and our main result here, Theorem 4.1, will be an analogue of Theorem 1.1 for the case when $p > 2$ is an even integer. In Section 5 we will show that question (ii) sometimes has a negative answer.

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* The results contained in this paper, together with those of [12], were announced in [11].