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A two-sided operational calculus

by

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Abstract. A two-sided operational calculus on the real line is constructed using the algebraic method introduced by Mikusiński. A field of two-sided operators is obtained which contains a large subspace of distributions, including the Laplace transformable distributions of Schwartz. The operator field is shown to be isomorphic with a field of meromorphic functions. The isomorphism is an extension of the classical (and distributional) Fourier transform, and is expressed by an integral of the classical form which is defined relative to a type I-convergence notion. A similar expression for the inverse Fourier transform is obtained, and the two are also expressed by sequential limits relative to a type II-convergence notion. The representation of distributions by these operators is discussed.

1. Introduction. As is well known, Mikusiński obtained ([7]) certain generalized functions by considering an algebraic field of fractions for the convolution ring of continuous functions on the half-line $[0, \infty)$. The result is a one-sided operational calculus which possesses all the advantages of rigor supplied by the Laplace transform method and none of the limitations imposed by the underlying analysis.

Recently, Boehme and Wygant introduced ([1]) a two-sided operational calculus on the unit circle which is equivalent to a periodic operational calculus on the real line $\boldsymbol{R}$. They constructed from the ring $\mathscr{C}$ of continuous 2π-periodic functions on $\boldsymbol{R}$, under convolution and addition, the ring $\mathscr{M}$ of fractions f/g, where g has all of its Fourier coefficients nonzero. The latter are the nondivisors of zero in $\mathscr{C}$. These fractions are called *operators* (following Mikusiński's example), and the ring $\mathscr{M}$ of operators is found to contain (isomorphically) the ring $\mathscr{P}'$ of 2π-periodic distributions, under convolution. They showed that this ring of operators is isomorphic with the (convolution) ring of formal trigonometric series (equivalently, the ring of doubly infinite series of complex numbers under coordinate addition and multiplication), and that every operator can be expressed as a Fourier series. For the latter, a convergence notion is introduced into $\mathscr{M}$ which is analogous to that given by Mikusiński, and is called *type* I.

In this paper we introduce still another example of a two-sided operational calculus on $\boldsymbol{R}$ which results in a *field* $\mathscr{M}_{\mathrm{Exp}}$ of operators similar to the original Mikusiński field. $\mathscr{M}_{\mathrm{Exp}}$ contains (isomorphically) a large subspace $\mathscr{D}'_{\mathrm{Exp}}$ of two-sided distributions, including all of the Laplace transformable distributions of Schwartz [9] (the distributions with Laplace transforms which are analytic and are bounded by polynomials in vertical strips of the complex plane). It is shown that this field of two-sided operators is isomorphic with a field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ of meromorphic functions in neighborhoods of the real axis in $\boldsymbol{C}$. The isomorphism is an extension to $\mathscr{M}_{\mathrm{Exp}}$ of the classical Fourier transform and agrees in the subspace $\mathscr{D}'_{\mathrm{Exp}}$ with the distributional Fourier transform. It is found that these latter have Fourier transforms which are actually analytic functions in neighborhoods of the real axis. The Fourier transform on $\mathscr{M}_{\mathrm{Exp}}$ is expressed by an integral of the classical form which is defined relative to a type I-convergence notion. A similar expression for the inverse Fourier transform is obtained. Also the Fourier and inverse Fourier transforms are expressed by sequential limits (analogous to those in [2], [8], [11]) relative to a type II-convergence notion.

For our construction (§ 2) we use a convolution ring Exp of C^∞ functions $\varphi(t)$ on $\boldsymbol{R}$ which (along with all derivates) decay exponentially as $|t|\to\infty$. The ring has no divisors of zero, and so its field of fractions exists and becomes (§ 3) our field $\mathscr{M}_{\mathrm{Exp}}$ of two-sided operators. Each function $\varphi(t)\in$ Exp has a classical Fourier transform $\tilde{\varphi}(z)$ which is analytic in a neighborhood of the real axis $\operatorname{Im} z = 0$ and rapidly decays there as $\operatorname{Re} z\to\pm\infty$. The ring Z_{Exp} of such analytic functions is isomorphic with Exp, and its field of fractions becomes our field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ of meromorphic functions. Types I and II-convergences are then introduced (§ 4) in the field of operators $\mathscr{M}_{\mathrm{Exp}}$, in analogy with those in the Mikusiński field. These are employed to obtain expressions for the Fourier and inverse Fourier transforms between $\mathscr{M}_{\mathrm{Exp}}$ and $\tilde{\mathscr{M}}_{\mathrm{Exp}}$. Finally in § 5 we discuss the representation of general distributions by the two-sided operators of the field $\mathscr{M}_{\mathrm{Exp}}$. The theory of such representations was introduced earlier in [12] and further developed in [3]. Operational solutions in $\mathscr{M}_{\mathrm{Exp}}$ of differential equations (ordinary and partial) are found to represent all distributional solutions of the same differential equations, and distributions (in general) which are represented by operators in $\mathscr{M}_{\mathrm{Exp}}$ are found to have Fourier transforms represented by certain special meromorphic functions in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$.

2. Some commutative rings without zero divisors. Let Exp be the space of C^∞ functions φ on the real line $\boldsymbol{R}$ which satisfy families of inequalities of the form

$$|\varphi^{(k)}(t)| \leqslant c_{\varphi,k} e^{-b_\varphi |t|} \quad (t\in \boldsymbol{R},\ k = 1, 2, \ldots), \tag{1}$$

where $c_{\varphi,k}$ and b_φ are positive constants (depending upon φ, k and φ, respectively) and where $\varphi^{(k)}$ is the ordinary kth derivative of φ. We shall say that these functions *decay exponentially*. Because they do, their Fourier transforms

$$\tilde{\varphi}(z) = \int_{-\infty}^{\infty} e^{-izt}\varphi(t)\,dt \tag{2}$$

are analytic in strip-type neighborhoods $N_{b_\varphi} = \{z\colon\ |\operatorname{Im} z| < b_\varphi\}$ of the real axis $\mathscr{R} = \{z\colon \operatorname{Im} z = 0\}$ in $\boldsymbol{C}$, and decay there more rapidly than any power of $1/|z|$ as $\operatorname{Re} z\to\pm\infty$. The latter property is called *rapid descent* and is obtained upon integration by parts in (2), using (1).

Conversely, if $\tilde{\varphi}(z)$ is analytic in a neighborhood N_b for some $b > 0$, and if $\tilde{\varphi}(z)$ is of rapid descent there, then its inverse Fourier transform

$$\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{\varphi}(\omega)\,d\omega \qquad (\omega\in\mathscr{R}) \tag{3}$$

is C^∞ on $\boldsymbol{R}$ and satisfies a family of equalities of the form (1) for any $b_\varphi < b$. One sees this by shifting the path of integration in (3) to a line above and a line below the real axis $\mathscr{R}$, within N_b. Let us denote this space of analytic functions by Z_{Exp}. Then the Fourier transform becomes a one-to-one mapping from Exp onto Z_{Exp}. Moreover, if we denote the convolution of two elements φ, ψ of Exp by $\varphi*\psi$, where

$$\varphi*\psi(t) = \int_{-\infty}^{\infty}\varphi(t-u)\psi(u)\,du = \int_{-\infty}^{\infty}\varphi(u)\psi(t-u)\,du,$$

then one has

$$\widetilde{\varphi*\psi}(z) = \tilde{\varphi}(z)\tilde{\psi}(z) \quad \text{for} \quad |\operatorname{Im} z| < \min\{b_\varphi, b_\psi\}. \tag{4}$$

Thus with convolution as the product and with the usual addition of functions, Exp becomes a commutative ring (algebra) without zero divisors (since the right-hand member in (4) cannot vanish identically unless one of the factors does) and is isomorphic (via the Fourier transform) with the ring Z_{Exp} of analytic functions under the usual multiplication and addition of functions.

Let $\mathscr{D}$ denote (as usual) the test function space of all C^∞ functions on $\boldsymbol{R}$ with compact supports, and let Z denote the space of Fourier transforms of elements of $\mathscr{D}$. We recall ([15]) that the latter are entire functions $\tilde{\varphi}(z)$ which satisfy families of inequalities of the form

$$|z^k\tilde{\varphi}(z)| \leqslant c_{\tilde{\varphi},k} e^{a_{\tilde{\varphi}}|\operatorname{Im} z|} \quad (k = 1, 2, \ldots). \tag{5}$$

We consider $\mathscr{D}$ and Z also as subrings of Exp and Z_{Exp}, respectively, and endow them with their usual topologies ([15]) for which their topological duals are $\mathscr{D}' = \mathscr{D}'(\boldsymbol{R})$ (the space of all distributions on $\boldsymbol{R}$) and $Z' = Z'(\mathscr{R})$

(the space of all ultradistributions on $\mathscr{R}$). Recall also that the Fourier transform can be extended to a topological isomorphism from $\mathscr{D}'$ onto Z', relative to the weak topologies of these spaces, by the Parseval equation $\langle\tilde{f},\tilde{\varphi}\rangle = 2\pi\langle f,\check{\varphi}\rangle$, where $\check{\varphi}(t) = \varphi(-t)$.

We shall need topologies (convergences) for Exp and Z_{Exp} and these are defined in the usual way: A sequence $\{\varphi_n\}$, $n = 1, 2, \ldots$, of elements of Exp will be said to *converge in* Exp if the sequence, and all of its derived sequences, converge uniformly on compact subsets of $\boldsymbol{R}$, and if there exist positive constants c_k and b such that $|\varphi_n^{(k)}(t)| \leqslant c_k e^{-b|t|}$ holds for all $t \in \boldsymbol{R}$ and $k, n = 1, 2, \ldots$ A sequence $\{\tilde{\varphi}_n\}$ of elements of Z_{Exp} will be said to *converge in* Z_{Exp} if there exist positive constants c_k and b such that each $\tilde{\varphi}_n$ is analytic in N_b and satisfies

$$|z^k\tilde{\varphi}_n(z)| \leqslant c_k \quad \text{for} \quad |z| < b,\ k = 0, 1, 2, \ldots$$

and such that the sequence converges uniformly on compact subsets of N_b. With respect to these convergences, the Fourier transform becomes a topological isomorphism from Exp onto Z_{Exp}. Also the convergences in $\mathscr{D}$ and in Z become stronger than those in Exp and Z_{Exp}.

EXAMPLE. Let

$$\varphi(t) = \frac{2}{e^t+e^{-t}} = \operatorname{sech} t \quad \text{and} \quad \varphi_n(t,\lambda) = \sum_{j=0}^{n}\frac{\varphi^{(j)}(t)}{j!}\lambda^j.$$

Then $\varphi, \varphi_n \in \mathrm{Exp}$, and by Taylor's theorem, $\varphi(t+\lambda) = \varphi_n(t,\lambda) + \dfrac{\varphi^{(n+1)}(t+\lambda\theta)}{(n+1)!}\lambda^{n+1}$, with $0 \leqslant \theta < 1$. Now φ is analytic along the real axis $\boldsymbol{R}$, and at each $t \in \boldsymbol{R}$ its power series expansion has a radius of convergence greater than 1. Thus it follows that for $|\lambda| < 1$, and all t, $\varphi_n(t,\lambda) \to \varphi(t+\lambda)$ as $n \to \infty$. Moreover, since $\varphi(t+\lambda) - \varphi_n(t,\lambda) = \dfrac{\varphi^{(n+1)}(t+\lambda\theta)}{(n+1)!}\lambda^{n+1}$, we conclude from the Cauchy inequalities that the convergence is in Exp for $|\lambda| < \frac{1}{2}$, i.e. $\varphi(t+\lambda) = \sum_{j=0}^{\infty}\dfrac{\varphi^j(t)}{j!}\lambda^j$ for $|\lambda| < \frac{1}{2}$ and this series converges in Exp. Observe that λ need not be real here.

3. Some fields of operators and meromorphic functions. Associated with each of the rings (integral domains) mentioned in the previous section is its ordinary field of fractions. We recall that these are formal fractions of elements of the rings which are identified, added and multiplied, as ordinary numerical fractions are, but with operations corresponding to those in the rings. The fields of fractions for the convolution rings $\mathscr{D}$ and Exp are denoted by $\mathscr{M}_0$ and $\mathscr{M}_{\mathrm{Exp}}$, respectively, and are called *operator fields*. $\mathscr{M}_0$ can be considered as a subfield of $\mathscr{M}_{\mathrm{Exp}}$, and its elements are called *compact operators*. The elements $x = \dfrac{\psi}{\varphi}$ $(0 \neq \varphi, \psi \in \mathrm{Exp})$ of $\mathscr{M}_{\mathrm{Exp}}$ we shall call *exponential operators*. As is customary, we shall consider each of these convolution rings as subrings of their respective operator fields.

It is easy to imbed the space $\mathscr{E}'$ of distributions with compact supports as a subring, under convolution and addition, in $\mathscr{M}_0$. For if $f \in \mathscr{E}'$, then the fraction $\dfrac{f*\varphi}{\varphi}$, for any nonzero $\varphi \in \mathscr{D}$, identifies f uniquely in $\mathscr{M}_0$. Moreover, if f is any distribution such that $f*\varphi \in \mathrm{Exp}$ for *every* $\varphi \in \mathscr{D}$, then the fraction $\dfrac{f*\varphi}{\varphi}$, for any nonzero $\varphi \in \mathscr{D}$, identifies f uniquely in $\mathscr{M}_{\mathrm{Exp}}$. Let us denote by $\mathscr{D}'_{\mathrm{Exp}}$ the subspace of those distributions which possess this property, i.e. $f \in \mathscr{D}'_{\mathrm{Exp}}$ iff, $f*\varphi \in \mathrm{Exp}$ for every $\varphi \in \mathscr{D}$.

The fields of fractions for the rings Z and Z_{Exp}, of analytic functions, are denoted by $\tilde{\mathscr{M}}_0$ and $\tilde{\mathscr{M}}_{\mathrm{Exp}}$, respectively, and are fields of certain meromorphic functions, which consist of ordinary fractions with denominators and numerators coming from these rings. Thus $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ is the field of functions $H(z)$ which are meromorphic in (various) neighborhoods N_{b_H} of $\mathscr{R}$ and are expressible in the form $H(z) = \dfrac{\tilde{\psi}(z)}{\tilde{\varphi}(z)}$, with $\tilde{\psi}, \tilde{\varphi} \in Z_{\mathrm{Exp}}$. $\tilde{\mathscr{M}}_0$ is the subfield of such meromorphic functions which are expressible in this form with $\tilde{\psi}, \tilde{\varphi} \in Z$. Again we shall consider each of these rings of analytic functions as subrings of their respective fields of meromorphic functions.

Now the field $\mathscr{M}_{\mathrm{Exp}}$ of exponential operators is clearly isomorphic to the field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ of meromorphic functions under the mapping which sends an exponential operator $x = \dfrac{\psi}{\varphi}$ to the meromorphic function $H = \dfrac{\tilde{\psi}}{\tilde{\varphi}}$. This mapping will be called the *Fourier transform*, and the meromorphic function $H = \dfrac{\tilde{\psi}}{\tilde{\varphi}}$ will be called the Fourier transform of the operator x, and will be denoted by $\tilde{x}$.

We observe that if $f \in \mathscr{D}'_{\mathrm{Exp}}$, then the Fourier transform $\tilde{f}$ of f as an ultradistribution can be identified with its Fourier transform as an exponential operator, since in either sense, $\widetilde{f*\varphi} = \tilde{f}\tilde{\varphi}$, and thus (by (3) and the Parseval equation),

$$2\pi f*\varphi(t) = \int_{-\infty}^{\infty} e^{i\omega t}\tilde{f}(\omega)\tilde{\varphi}(\omega)\,d\omega = \langle\tilde{f}(\omega), e^{i\omega t}\tilde{\varphi}(\omega)\rangle$$

holds for all $\tilde{\varphi} \in Z$. Moreover, since $\tilde{f}\tilde{\varphi} \in Z_{\mathrm{Exp}}$ for all $\tilde{\varphi} \in Z$, it follows that $\tilde{f}$ (which in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ is a meromorphic function in some N_b) is, in fact, analytic in some N_b. In particular, if $\tilde{f}$ is analytic in some N_b and is bounded by a polynomial there, then it is the Fourier transform of a distribution $f \in \mathscr{D}'_{\mathrm{Exp}}$. These latter distributions are precisely those which possess Laplace transforms in strip-type neighborhoods of the imaginary axis [16]. Thus the field $\mathscr{M}_{\mathrm{Exp}}$ of exponential operators contains a rather large subspace $\mathscr{D}'_{\mathrm{Exp}}$ of distributions, and by allowing for the shifting of distributions via the exponential shifts $e^{\lambda t}$, can be considered to contain *all* Laplace transformable distributions.

Many of the more familiar operators are compact; for example the *numerical operators* $a = a\varphi/\varphi$, the differentiation operator $s = \varphi^{(1)}/\varphi$, the *integration operator* $h = 1/s = \varphi/\varphi^{(1)}$ and the (real) *translation operators* $e^{\lambda s} = \varphi(t+\lambda)/\varphi(t)$ are all compact, as one may choose any nonzero φ in $\mathscr{D}$. In the last instance, if λ is not real, then φ must be chosen to be an analytic function in Exp, such as that of the example in the previous section. Thus for nonreal λ, the translation $e^{\lambda s}$ is an exponential operator, but it is not a compact operator. It is not a Mikusiński operator either, since $\varphi(t)$ cannot be simultaneously, nonzero, analytic and right-sided. The Fourier transforms of these various operators are the meromorphic functions $\tilde{a} = a$, $\tilde{s} = iz$, $\tilde{h} = 1/iz$ and $\widetilde{(e^{\lambda s})} = e^{i\lambda z}$.

The *algebraic derivative* D is the transformation $D\colon \varphi(t) \mapsto -t\varphi(t)$ for all $\varphi \in \mathrm{Exp}$, and can be extended (as a derivation) to the field $\mathscr{M}_{\mathrm{Exp}}$ by the equation $Dx = D(\psi/\varphi) = (\varphi * D\psi - \psi * D\varphi)/\varphi * \varphi$. Since the Fourier transform of $D\varphi(t) = -t\varphi(t)$ is just the iz-derivative of the Fourier transform of $\varphi(t)$, it follows that the Fourier transform of Dx is the iz-derivative of the Fourier transform of x, i.e. $\widetilde{Dx}(z) = -id\tilde{x}(z)/dz$.

Examples. In [13] we have speculated that no Mikusiński operator satisfies the algebraic derivative equation $Dx = sx$. This equation transforms to the ordinary differential equation $d\tilde{x}/dz = -z\tilde{x}$, whose general solution is $\tilde{x} = ce^{-z^2/2}$. It is easy to check that each of these entire functions belongs to the ring Z_{Exp} and that its Fourier inverse is given by $x(t) = ce^{s^2/2} = ce^{-t^2/2}$ which, of course, belongs to Exp. However, the (reciprocal) entire function $e^{z^2/2}$ only belongs to the field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$, while its Fourier inverse $e^{-s^2/2}$ in $\mathscr{M}_{\mathrm{Exp}}$ satisfies the equation $Dx = -sx$. More generally, $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ contains any function $\tilde{f}$ which is analytic in some N_b and satisfies $\liminf_{\mathrm{Re}\,z\to\pm\infty} |\tilde{f}(z)| > 0$, there. For then we can select a nonzero $\tilde{\varphi} \in Z_{\mathrm{Exp}}$ with the same zeros and orders of zeros as $\tilde{f}$, so that $\tilde{\varphi}/\tilde{f} \in Z_{\mathrm{Exp}}$. Then, $\tilde{f} \in \tilde{\mathscr{M}}_{\mathrm{Exp}}$. We can show that the entire function $\tilde{x} = e^{iz^2}$ belongs to $\tilde{\mathscr{M}}_{\mathrm{Exp}}$. In fact, it satisfies $|\tilde{x}(z)| = e^{-(\mathrm{Im}\,z)(\mathrm{Re}\,z)2}$, and so we can select (an analytic) $\varphi \in \mathrm{Exp}$ (as that of the example in § 2) so that $\varphi_{\pm}(t) = \varphi(t \pm ib) \in \mathrm{Exp}$ for some $b > 0$. Then $\tilde{\psi}(z) = \tilde{x}(z)\tilde{\varphi}(z) \in Z_{\mathrm{Exp}}$, since $\tilde{\varphi}(z) = e^{\mp bz}\tilde{\varphi}_{\pm}(z)$ (and $\tilde{\psi}(z)$ is of rapid descent in $N_{b/2}$). Thus $\tilde{x}(z) = \tilde{\psi}(z)/\tilde{\varphi}(z) \in \tilde{\mathscr{M}}_{\mathrm{Exp}}$, and its Fourier inverse $x = e^{-is^2} \in \mathscr{M}_{\mathrm{Exp}}$. More generally, $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ contains any function f which is analytic in some N_b and of exponential growth there, i.e. $|\tilde{f}(z)| \leqslant ce^{a|\mathrm{Re}\,z|}$ for $z \in N_b$.

4. Convergences in the fields. The following convergence notion is the analogue for exponential operators of that defined in [7] for Mikusiński operators.

Definition. A sequence $\{x_n\}$, $n = 1, 2, \ldots$, of operators is said to be *type* I-*convergent in* $\mathscr{M}_{\mathrm{Exp}}$ if there exists a nonzero $\varphi \in \mathrm{Exp}$ such that $x_n\varphi \in \mathrm{Exp}$ for all n and $x_n\varphi \to \psi$ in Exp, as $n \to \infty$. The *type* I-*limit* of the sequence $\{x_n\}$ is the operator $x = \psi/\varphi$. The function φ is called a *convergence factor* for the sequence $\{x_n\}$.

It is clear what this definition means for the type I-convergence of the corresponding sequence $\{\tilde{x}_n\}$ of Fourier transforms: the sequence $\{\tilde{x}_n\}$ must possess a common denominator $\tilde{\varphi}$ (equal to the Fourier transform of a convergence factor) and the sequence of products $\{\tilde{x}_n\tilde{\varphi}\}$ of analytic functions must converge in Z_{Exp}.

Type I-convergence is much weaker than distributional weak convergence. For if $\{f_n\}$ is a sequence of distributions in $\mathscr{D}'_{\mathrm{Exp}}$, it converges weakly in the distributional sense iff, the sequence of functions $\{f_n * \varphi\}$ converges pointwise on $\boldsymbol{R}$ for every $\varphi \in \mathscr{D}$. It can be shown that pointwise convergence of $\{f_n * \varphi\}$, in this case, is equivalent to convergence of the sequence in Exp. This is well known except for the exponential decay property (1). But as operators, we have $f_n\varphi = f_n * \varphi$ whenever $\varphi \in \mathscr{D}$, and so the sequence $\{f_n\}$ of distributional operators converges in $\mathscr{D}'_{\mathrm{Exp}}$ iff the sequence of functions $\{f_n\varphi\}$ converges in Exp for *every* $\varphi \in \mathscr{D}$.

We extend type I-convergence to other analytical procedures in an analogous fashion. For example, we can formally express the Fourier transform by the type I-integral in $\mathscr{M}_{\mathrm{Exp}}$,

$$\tilde{x}(z) = \int_{-\infty}^{\infty} e^{-izt} x(t)\,dt. \tag{6}$$

Equality in (6) means that for some nonzero $\varphi \in \mathrm{Exp}$, we have

$$\tilde{x}(z)\tilde{\varphi}(z) = \int_{-\infty}^{\infty} e^{-izt} x\varphi(t)\,dt,$$

and this last is certainly the case if $x = \psi/\varphi$, so that $\tilde{\psi}(z) = \tilde{x}(z)\tilde{\varphi}(z)$ for all z with $|\mathrm{Im}\,z|$ sufficiently small, and $\psi(t) = x\varphi(t)$. Here, for each fixed value of z, $e^{-izt}x\varphi(t)$ is an operator in $\mathscr{M}_{\mathrm{Exp}}$ and the integral of it is interpreted as a numerical operator, with the numerical value $\tilde{x}(z)\tilde{\varphi}(z)$. Similarly, we can formally express the inverse Fourier transform by the type I-integral in $\tilde{\mathcal{M}}_{\mathrm{Exp}}$,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{x}(\omega)\,d\omega, \tag{7}$$

which means that for some nonzero $\tilde{\varphi}\in Z_{\mathrm{Exp}}$, we have

$$x\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{x}(\omega)\tilde{\varphi}(\omega)\,d\omega.$$

Again for each fixed value of t, $e^{izt}\tilde{x}(z)\tilde{\varphi}(z)$ is a meromorphic (analytic) function in $\tilde{\mathcal{M}}_{\mathrm{Exp}}$ and the integral of it is interpreted as a constant meromorphic function, with numerical value $x\varphi(t)$.

Mikusiński introduced another, even weaker convergence notion in his field of operators. It is usually called type II-convergence, and the following is its analogue in $\mathcal{M}_{\mathrm{Exp}}$.

Definition. A sequence $\{x_n\}$ of operators is said to be *type* II-*convergent in* $\mathcal{M}_{\mathrm{Exp}}$ if there exists a sequence $\{\varphi_n\}$ of nonzero elements of Exp such that $x_n\varphi_n\in \mathrm{Exp}$ for all n and $\varphi_n\to\varphi\neq 0$, $x_n\varphi_n\to\psi$ in Exp, as $n\to\infty$. The *type* II-*limit* of the sequence $\{x_n\}$ is the operator $\frac{\psi}{\varphi}$.

It is clear what this definition means for the convergence of the corresponding sequence $\{\tilde{x}_n\}$ of Fourier transforms: the members of the sequence $\{\tilde{x}_n\}$ must be expressible in the form $\tilde{x}_n = \tilde{\psi}_n/\tilde{\varphi}_n$ with $\tilde{\varphi}_n\to\tilde{\varphi}\neq 0$, $\tilde{\psi}_n\to\tilde{\psi}$ in Z_{Exp}.

We can formally express the Fourier transform as a type II-sequential limit in $\mathcal{M}_{\mathrm{Exp}}$,

$$\operatorname*{II-lim}_{n\to\infty} U_nT^{-iz}x(\cdot) = \tilde{x}(z), \tag{8}$$

for all z with $|\mathrm{Im}\,z|$ sufficiently small. First we need to explain the notation. For fixed $n = 1, 2, \ldots$ and complex p, we define two transformations in Exp by the equations

$$U_n\varphi(t) = n\varphi(nt) \text{ for all } \varphi, \qquad T^p\varphi(t) = e^{pt}\varphi(t) \text{ provided } |\mathrm{Re}\,p| < b_\varphi.$$

Then these are extended to the field $\mathcal{M}_{\mathrm{Exp}}$ by the equations

$$U_nx = \frac{U_n\psi}{U_n\varphi} \text{ for all } x, \qquad T^px = \frac{T^p\psi}{T^p\varphi} \text{ provided } |\mathrm{Re}\,p| < \min\{b_\varphi, b_\psi\}.$$

It is easy to check that these are well-defined in $\mathcal{M}_{\mathrm{Exp}}$, and U_nT^{iz} in (8) merely denotes the composition of these two transformations. To verify the limit in (8) we let $x = \frac{\psi}{\varphi}$, and consider $U_nT^{-iz}x(t) = \frac{U_nT^{-iz}\psi(t)}{U_nT^{-iz}\varphi(t)}$

and in particular, $U_nT^{-iz}\psi(t) = ne^{-iznt}\psi(nt)$. For any nonzero $\sigma\in\mathcal{D}$, we have

$$U_nT^{-iz}\psi*\sigma(t) = \int_{-\infty}^{\infty} ne^{-izn\tau}\psi(n\tau)\sigma(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-izu}\psi(u)\sigma\left(t-\frac{u}{n}\right)du, \tag{9}$$

and it is easily seen that the sequence in (9) converges in Exp to $\tilde{\psi}(z)\sigma(t)$ as $n\to\infty$, provided $|\mathrm{Im}\,z| < b_\psi$. Similarly, $U_nT^{-iz}\varphi*\sigma(t)$ converges to $\tilde{\varphi}(z)\sigma(t)$ in Exp, provided $|\mathrm{Im}\,z| < b_\varphi$. But

$$U_nT^{-iz}x(t) = \frac{U_nT^{-iz}\psi*\sigma}{U_nT^{-iz}\varphi*\sigma} \quad \text{for all } n,$$

and thus the limit in (8) exists in the sense of type II-convergence in $\mathcal{M}_{\mathrm{Exp}}$, provided $|\mathrm{Im}\,z| < \min\{b_\psi, b_\varphi\}$. For each such fixed value of z, the limit is the numerical operator $\frac{\tilde{\psi}(z)\sigma(t)}{\tilde{\varphi}(z)\sigma(t)} = \frac{\tilde{\psi}(z)}{\tilde{\varphi}(z)} = \tilde{x}(z)$, provided, of course, z is not a pole of $\tilde{x}(z)$. This is what is meant by (8).

An analogous result can be obtained for the inverse Fourier transform. To obtain the result we need to define certain linear transformations by employing convolution, rather than multiplication in Z_{Exp}. For fixed $n = 1, 2, \ldots$, $t\in\boldsymbol{R}$, and nonzero $\tilde{\sigma}\in Z$, we define a transformation on Z_{Exp} by the equation

$$\begin{aligned} U_nT^{it}_{\tilde{\sigma}}\tilde{\psi}(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} ne^{itn\omega}\tilde{\psi}(n\omega)\tilde{\sigma}(z-\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\xi}\tilde{\psi}(\xi)\tilde{\sigma}\left(z-\frac{\xi}{n}\right)d\xi = \psi*\sigma_{n,z}(t), \end{aligned} \tag{10}$$

analogous to (9). Here $\sigma_{n,z}(t) = ne^{inzt}\sigma(-nt)$, so that $\widetilde{\psi*\sigma_{n,z}}(\xi) = \tilde{\psi}(\xi)\tilde{\sigma}(z-\xi/n)$. It is not too difficult to see that $U_nT^{it}_{\tilde{\sigma}}\tilde{\psi}\in Z_{\mathrm{Exp}}$, whenever $\tilde{\psi}\in Z_{\mathrm{Exp}}$, and that the sequence in (10) converges in Z_{Exp} to $\psi(t)\tilde{\sigma}(z)$ as $n\to\infty$. Now if $\tilde{x} = \frac{\tilde{\psi}}{\tilde{\varphi}}\in\tilde{\mathcal{M}}_{\mathrm{Exp}}$, we can consider the meromorphic function in $\tilde{\mathcal{M}}_{\mathrm{Exp}}$,

$$U_nT^{it}_{\tilde{\sigma}}\frac{\tilde{\psi}(z)}{\tilde{\varphi}(z)} = \frac{U_nT^{it}_{\tilde{\sigma}}\tilde{\psi}(z)}{U_nT^{it}_{\tilde{\sigma}}\tilde{\varphi}(z)}, \tag{11}$$

which we would like to label $U_nT^{it}_{\tilde{\sigma}}\tilde{x}(z)$, but (in general) cannot, since it depends upon this particular representation of $\tilde{x}$ as a fraction. However, for any choice of representation, the type II-limit in $\tilde{\mathcal{M}}_{\mathrm{Exp}}$ of the sequence given in (11) exists and does *not* depend upon the particular representation of $\tilde{x}$. Moreover, this limit does not depend upon the particular choice of $\tilde{\sigma}$ either! Indeed, $U_n T_{\tilde{\sigma}}^{it}\tilde{\psi}(z) \to \psi(t)\tilde{\sigma}(z)$ and $U_n T_{\tilde{\sigma}}^{it}\tilde{\varphi}(z) \to \varphi(t)\tilde{\sigma}(z)$ in Z_{Exp}, as $n\to\infty$. Hence for any fixed value of t, the limit of the sequence given in (11) is the constant meromorphic function $\dfrac{\psi(t)\tilde{\sigma}(z)}{\varphi(t)\tilde{\sigma}(z)} = \dfrac{\psi(t)}{\varphi(t)}$, provided, of course, $\varphi(t) \neq 0$. Here we obtain the ordinary numerical ratio of $\psi(t)$ and $\varphi(t)$ for each fixed value of t, which is not very satisfactory. If we wish to identify the operator x, which is the *convolution quotient* ψ/φ, we must retain the quotient form $\dfrac{\psi(t)\tilde{\sigma}(z)}{\varphi(t)\tilde{\sigma}(z)}$ for the limit in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$, so as to identify ψ and φ individually. With appropriate interpretations then (see (10) and (11)), we can claim that

$$\text{(12)} \qquad \mathrm{II}\text{-}\lim_{n\to\infty} U_n T^{it}\tilde{x}(z) = x(t),$$

in analogy with (8).

There is another interesting interpretation which may be made of the II-limit in (12). We can interpret U_n and T^{it} as transformations of the ring Z_{Exp} in the variable z, much as we did for the ring Exp in the variable t: $U_n\tilde{\psi}(z) = n\tilde{\psi}(nz)$, $T^{it}\tilde{\psi}(z) = e^{itz}\tilde{\psi}(z)$. Then these may be extended to the field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ by the equations

$$U_n\tilde{x} = \frac{U_n\tilde{\psi}}{U_n\tilde{\varphi}}, \qquad T^{it}\tilde{x} = \frac{T^{it}\tilde{\psi}}{T^{it}\tilde{\varphi}} = \frac{\tilde{\psi}}{\tilde{\varphi}} = \tilde{x}, \qquad \text{for all } \tilde{x},$$

so that $U_n T^{it}$ becomes a well-defined transformation on $\tilde{\mathscr{M}}_{\mathrm{Exp}}$. Actually, it is independent of t, but any representation is made to depend upon t in the form

$$\text{(13)} \qquad U_n T^{it}\tilde{x} = \frac{U_n T^{it}\tilde{\psi}}{U_n T^{it}\tilde{\varphi}}.$$

Now $U_n T^{it}\tilde{\psi}(z) = ne^{itnz}\tilde{\psi}(nz)$ can be considered as a (regular) ultradistribution and converges *as an ultradistribution* to $2\pi\psi(t)\delta(z)$, where $\langle\delta(z), \tilde{\sigma}(z)\rangle = \tilde{\sigma}(0)$ for all $\tilde{\sigma}\in Z$, as $n\to\infty$. This is because

$$\langle U_n T^{it}\tilde{\psi}(z), \tilde{\sigma}(z)\rangle = \int_{-\infty}^{\infty} ne^{itnz}\tilde{\psi}(nz)\tilde{\sigma}(z)\,dz$$
$$= \int_{-\infty}^{\infty} e^{it\xi}\tilde{\psi}(\xi)\tilde{\sigma}\left(\frac{\xi}{n}\right)d\xi \to 2\pi\psi(t)\tilde{\sigma}(0) \qquad \text{as } n\to\infty.$$

Thus if the type II-limit in (12) is interpreted to mean that the limits in the numerator and denominator in (13) each exist in the sense of convergence in Z', then the limit is the formal ratio $\psi(t)\delta(z)/\varphi(t)\delta(z)$ of ultradistributions. For each fixed value of t with $\varphi(t) \neq 0$, this ratio may be considered to be a constant meromorphic function with numerical value $\psi(t)/\varphi(t)$. Again, we need to obtain the limits in the numerator and denominator separately in order to identify the convolution quotient $\psi/\varphi = x$.

Following [10], we can define an interesting topology in the field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$. For each fixed $b > 0$, let M_b denote the subcollection of functions in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ which are meromorphic in the neighborhood N_b. We endow M_b with the topology of uniform convergence on compact subsets of N_b with respect to the chordal metric. Thus a sequence $\{\tilde{x}_n\}$ converges to $\tilde{x}$ in M_b iff,

$$\lim_{n\to\infty}\left[\sup_{z\in K}\frac{|\tilde{x}_n(z)-\tilde{x}(z)|}{\sqrt{1+|\tilde{x}_n(z)|^2}\sqrt{1+|\tilde{x}(z)|^2}}\right] = 0$$

for all compact subsets K of N_b. We endow $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ with the finest topology $\tilde{\mathscr{T}}$ for which all the injections $M_b \nearrow \tilde{\mathscr{M}}_{\mathrm{Exp}}$ are continuous. Then a sequence $\{\tilde{x}_n\}$ is convergent to $\tilde{x}$ in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ if it is eventually in some M_b, containing $\tilde{x}$, and converges to $\tilde{x}$ there. We remark that the converse is not necessarily true since $\tilde{\mathscr{T}}$ is not a *strict* limit topology. Indeed, the injections $M_{b_1} \nearrow M_{b_2}$ $(b_1 > b_2)$ are continuous mappings, but are not homeomorphisms. It is easy to see that a type I-convergent sequence in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ is convergent in this topology if, in its convergent form, it has a common denominator which does not vanish in some neighborhood N_b. For its numerator sequence then converges in Z_{Exp}, and so the sequence itself converges uniformly on compact subsets of N_b. Similarly, any sequence of analytic functions in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ which converges uniformly on compact subsets of some N_b to an element of $\tilde{\mathscr{M}}_{\mathrm{Exp}}$, is convergent in the topology $\tilde{\mathscr{T}}$. This applies, in particular, to what is customarily considered a convergent sequence of Laplace transforms [16]. The inverse mapping $\tilde{x}\mapsto 1/\tilde{x}$ on $\tilde{\mathscr{M}}_{\mathrm{Exp}} - \{0\}$ is continuous, since the chordal distance between nonzero $\tilde{x}$ and $\tilde{y}$ is the same as the chordal distance between their inverses.

Since the inverse Fourier transform is a bijection, we can transform the topology $\tilde{\mathscr{T}}$ to an equivalent topology $\mathscr{T}$ for the operator field $\mathscr{M}_{\mathrm{Exp}}$. Then, of course, a sequence $\{x_n\}$ of exponential operators converges to the operator x in the topology $\mathscr{T}$ iff its sequence $\{\tilde{x}_n\}$ of Fourier transforms converges to $\tilde{x}$ in the topology $\tilde{\mathscr{T}}$. Again, type I-convergent sequences in $\mathscr{M}_{\mathrm{Exp}}$ are $\mathscr{T}$-convergent if they have convergence factors whose Fourier transforms do not vanish in some N_b, and the inverse mapping in $\mathscr{M}_{\mathrm{Exp}}$ is $\mathscr{T}$-continuous. In particular, the series $\sum s^j\lambda^j/j!$ is $\mathscr{T}$-convergent to $e^{\lambda s}$ since its Fourier transform is $\tilde{\mathscr{T}}$-convergent to $e^{i\lambda z}$. Actually, this series of operators is type I-convergent in $\mathscr{M}_{\mathrm{Exp}}$, since a convergence factor, like the example in § 2, can readily be constructed. Similarly, the examples in § 3 show that the series $\sum(-i)^j s^{2j}/j!$ and $\sum s^{2j}/j!2^j$ are type I-convergent to e^{-is^2} and $e^{s^2/2}$ in $\mathscr{M}_{\mathrm{Exp}}$. On the other hand, the series $\sum(-1)^j s^{2j}/j!$ is only type II-convergent to e^{-s^2} in $\mathscr{M}_{\mathrm{Exp}}$.

We can show that these topologies are finer than the metric topology obtained from the chordal metric restricted to the real axis $\mathscr{R}$. This is defined by the family of pseudo-metrics (ϱ_k for $\mathscr{M}_{\mathrm{Exp}}$, $\tilde{\varrho}_k$ for $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ with $k = 1, 2, \ldots$)

$$\varrho_k(x, y) = \tilde{\varrho}_k(\tilde{x}, \tilde{y}) = \sup_{-k\leqslant\omega\leqslant k} \frac{|\tilde{x}(\omega)-\tilde{y}(\omega)|}{\sqrt{1+|\tilde{x}(\omega)|^2}\sqrt{1+|\tilde{y}(\omega)|^2}}.$$

For suppose that $\tilde{x}\in\tilde{\mathscr{M}}_{\mathrm{Exp}}$, $\varepsilon>0$ and $S(\tilde{x}, \varepsilon) = \{\tilde{y}\in\tilde{\mathscr{M}}_{\mathrm{Exp}}\colon \tilde{\varrho}_k(\tilde{x}, \tilde{y}) < \varepsilon\}$, for some fixed k. We need to show that for any $b>0$, the intersection $M_b\cap S(\tilde{x}, \varepsilon)$ is open in M_b. If $\tilde{w}$ belongs to this intersection, then $\tilde{\varrho}_k(\tilde{x}, \tilde{w}) < \varepsilon$. Let

$$W = \left\{\tilde{y}\in M_b\colon \sup_{\substack{|\mathrm{Im}\,z|\leqslant b/2\\ |\mathrm{Re}\,z|\leqslant k}} \frac{|\tilde{y}(z)-\tilde{w}(z)|}{\sqrt{1+|\tilde{y}(z)|^2}\sqrt{1+|\tilde{w}(z)|^2}} < \varepsilon-\tilde{\varrho}_k(\tilde{x}, \tilde{w})\right\}.$$

Then W is a neighborhood of $\tilde{w}$ in M_b, and if $\tilde{y}\in W$, it follows that $\tilde{\varrho}_k(\tilde{w}, \tilde{y}) < \varepsilon-\tilde{\varrho}_k(\tilde{x}, \tilde{w})$. But then $\tilde{\varrho}_k(\tilde{x}, \tilde{y}) \leqslant \tilde{\varrho}_k(\tilde{x}, \tilde{w}) + \tilde{\varrho}_k(\tilde{w}, \tilde{y}) < \varepsilon$, and so $W \subseteq S(\tilde{x}, \varepsilon)$. Thus, $M_b\cap S(\tilde{x}, \varepsilon)$ is open in M_b for every $b>0$, and the metric topology defined by the $\tilde{\varrho}_k$ is such that all the injections $M_b \hookrightarrow \tilde{\mathscr{M}}_{\mathrm{Exp}}$ are continuous. But $\tilde{\mathscr{T}}$ is the finest topology for $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ which has this property, and so is finer than this metric topology.

Now suppose that the sequence $\{x_n\}$ is type II-convergent to x in $\mathscr{M}_{\mathrm{Exp}}$. Then if $\tilde{x}_n = \tilde{\psi}_n/\tilde{\varphi}_n$ and $\tilde{x} = \tilde{\psi}/\tilde{\varphi}$, with $\tilde{\psi}_n\to\tilde{\psi}$, $\tilde{\varphi}_n\to\tilde{\varphi}$, we have

$$\varrho_k(x_n, x) = \sup_{-k\leqslant\omega\leqslant k} \frac{|\tilde{\varphi}(\omega)\tilde{\psi}_n(\omega)-\tilde{\varphi}_n(\omega)\tilde{\psi}(\omega)|}{\sqrt{|\tilde{\varphi}_n(\omega)|^2+|\tilde{\psi}_n(\omega)|^2}\sqrt{|\tilde{\varphi}(\omega)|^2+|\tilde{\psi}(\omega)|^2}}$$
$$\leqslant \sup_{-k\leqslant\omega\leqslant k} \frac{|\tilde{\varphi}(\omega)-\tilde{\varphi}_n(\omega)|+|\tilde{\psi}(\omega)-\tilde{\psi}_n(\omega)|}{\sqrt{|\tilde{\varphi}(\omega)|^2+|\tilde{\psi}(\omega)|^2}}.$$

If the fraction $\tilde{\psi}(\omega)/\tilde{\varphi}(\omega)$ is such that $\tilde{\psi}(\omega)$ and $\tilde{\varphi}(\omega)$ do not vanish simultaneously, it follows that $\varrho_k(x_n, x)\to 0$ as $n\to\infty$. Thus a type II-convergent sequence is convergent in the metric topology if its limit, in its convergence form ψ/φ, satisfies $\sqrt{|\tilde{\varphi}(\omega)|^2+|\tilde{\psi}(\omega)|^2} \neq 0$ for all $\omega\in\mathscr{R}$. A similar argument shows that a type II-convergent sequence is $\mathscr{T}$ (or $\tilde{\mathscr{T}}$)-convergent if its limit, in its convergence form ψ/φ, satisfies $\sqrt{|\tilde{\varphi}(z)|^2+|\tilde{\psi}(z)|^2} \neq 0$ for all z in some N_b. In particular, the type II-limit in (8) can be replaced by either the metric or the $\mathscr{T}$ limit, since $\tilde{\sigma}(z)$ can be chosen to satisfy this condition. The same situation holds for (12).

5. Representation of distributions by exponential operators. A theory of representation of general distributions (called *operator homomorphisms*) by various Mikusiński-type operators was introduced in [12] and was further expanded in [3]. Exponential operators furnish us with additional interesting examples of such representations.

We recall that an operator $x\in\mathscr{M}_{\mathrm{Exp}}$ is said to *represent* a distribution f whenever $f*\varphi = x\varphi\in\mathrm{Exp}$ holds for some nonzero $\varphi\in\mathscr{D}$. Of course, this requires that $x = \psi/\varphi$ with $\psi\in\mathrm{Exp}$, and such operators, which can be expressed as fractions with denominators from $\mathscr{D}$, are said to be *semi-compact*.

In [3], it is shown that operators which represent distributions have certain continuity properties, just as distributions have, and they are said to be *neocontinuous*. Probably the most interesting examples of neocontinuous operators are the (Mikusiński) operational solutions of inhomogeneous differential (specially partial differential) equations which have been shown to represent all distributional solutions of the same differential equations. These are significant examples of neocontinuous Mikusiński operators which have been discussed previously in [12] and [3]. However, the very same considerations (applied to the higher dimensional analogues of $\mathscr{M}_{\mathrm{Exp}}$), show that exponential operational solutions of these differential equations also represent all distributional solutions of the same differential equations.

Suppose an exponential operator x represents a distribution f, and φ is a nonzero member of $\mathscr{D}$ which satisfies $x\varphi = f*\varphi\in\mathrm{Exp}$. This need not mean that $f*\varphi\in\mathrm{Exp}$ holds for *every* $\varphi\in\mathscr{D}$, and so "representation" of a distribution in $\mathscr{M}_{\mathrm{Exp}}$ is a weaker link between $\mathscr{D}'$ and $\mathscr{M}_{\mathrm{Exp}}$ than is "identification" of a distribution in $\mathscr{M}_{\mathrm{Exp}}$. It is only a distribution f in $\mathscr{D}'_{\mathrm{Exp}}$ which is *identified* with an exponential operator, and for which $f*\varphi\in\mathrm{Exp}$ holds for every $\varphi\in\mathscr{D}$. Operators can represent many different distributions, but a distribution can be represented by at most one exponential operator.

Now if $x\in\mathscr{M}_{\mathrm{Exp}}$ represents $f\in\mathscr{D}'$, then there exists a nonzero $\varphi\in\mathscr{D}$ such that $x\varphi = f*\varphi\in\mathrm{Exp}$. This means that the Fourier transform $\widetilde{x\varphi} = \widetilde{f*\varphi} = \tilde{f}\tilde{\varphi}$ ($\tilde{f}\in Z'$, $\tilde{\varphi}\in Z$) necessarily belongs to Z_{Exp}. Thus the Fourier transform $\tilde{f}$ of a distribution f, which is represented by an exponential operator, can itself be represented by a meromorphic function $H(z) = \tilde{\psi}(z)/\tilde{\varphi}(z)$ in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$ with $\tilde{\psi}\in Z_{\mathrm{Exp}}$, $\tilde{\varphi}\in Z$. This latter then constitutes the definition of *representation of an ultradistribution $\tilde{f}$ by a meromorphic function in $\tilde{\mathscr{M}}_{\mathrm{Exp}}$*, i.e. for some nonzero $\tilde{\varphi}\in Z$, one has $\tilde{f}\tilde{\varphi}\in Z_{\mathrm{Exp}}$. On the other hand, suppose that $0 \neq \tilde{\varphi} \in Z$, $\tilde{\psi} \in Z_{\mathrm{Exp}}$ and that the meromorphic function $H(z) = \tilde{\psi}(z)/\tilde{\varphi}(z)$ is such that for some fixed real $\varrho(|\varrho| < b_H)$, the function $H(\omega+i\varrho)$ of the real variable ω is bounded by a polynomial in ω. Then we can define an ultradistribution $\tilde{f}$ by the equation

$$(14) \qquad \langle \tilde{f}(\omega), \tilde{\sigma}(\omega)\rangle = \int_{-\infty}^{\infty} H(\omega+i\varrho)\tilde{\sigma}(\omega+i\varrho)\,d\omega \quad (\text{for all } \tilde{\sigma} \in Z),$$

and this ultradistribution satisfies

$$\langle \tilde{f}(\omega)\tilde{\varphi}(\omega), \tilde{\sigma}(\omega)\rangle = \langle \tilde{f}(\omega), \tilde{\varphi}(\omega)\tilde{\sigma}(\omega)\rangle = \int_{-\infty}^{\infty} H(\omega+i\varrho)\tilde{\varphi}(\omega+i\varrho)\tilde{\sigma}(\omega+i\varrho)\,d\omega$$

$$= \int_{-\infty}^{\infty} \tilde{\psi}(\omega+i\varrho)\tilde{\sigma}(\omega+i\varrho)\,d\omega = \int_{-\infty}^{\infty} \tilde{\psi}(\omega)\tilde{\sigma}(\omega)\,d\omega,$$

or all $\tilde{\sigma} \in Z$. Hence $\tilde{f}\tilde{\varphi} = \tilde{\psi}$, which means that the meromorphic function $H = \tilde{\psi}/\tilde{\varphi}$ represents the ultradistribution $\tilde{f}$ defined by (14). Thus a semi-compact, exponential operator $x = \dfrac{\psi}{\varphi}$ $(\varphi \in \mathscr{D}, \psi \in \mathrm{Exp})$ represents a distribution if the restriction of its Fourier transform $H(z) = \tilde{\psi}(z)/\tilde{\varphi}(z)$ to some horizontal line $\operatorname{Im} z = \varrho$ is bounded by a polynomial in $\operatorname{Re} z = \omega$. We remark that various choices of ϱ in (14) will, in general, result in various ultradistributions $\tilde{f}$. Moreover, *whenever* an ultradistribution $\tilde{f}$ is defined by an integral like (14), using some path of integration γ which results in $\int_\gamma \tilde{\psi}(z)\tilde{\sigma}(z)\,dz = \int_{-\infty}^{\infty}\tilde{\psi}(\omega)\tilde{\sigma}(\omega)\,d\omega$ holding for all $\tilde{\sigma} \in Z$, then this ultradistribution $\tilde{f}$ is represented by $H(z)$. This suggests the conjecture: a meromorphic function of the form $H(z) = \tilde{\psi}(z)/\tilde{\varphi}(z)$, with $\tilde{\psi} \in Z_{\mathrm{Exp}}$ and $0 \neq \tilde{\varphi} \in Z$, represents an ultradistribution iff, there exists some path γ, lying in N_{b_H}, such that the equation $\langle \tilde{f}(\omega), \tilde{\sigma}(\omega)\rangle = \int_\gamma H(z)\tilde{\sigma}(z)\,dz$ defines an ultradistribution $\tilde{f}$ and $\int_\gamma \tilde{\psi}(z)\tilde{\sigma}(z)\,dz = \int_{-\infty}^{\infty}\tilde{\psi}(\omega)\tilde{\sigma}(\omega)\,d\omega$ holds for all $\tilde{\sigma} \in Z$.

There are known classes of semi-compact, exponential operators (actually compact operators) which do not represent any distributions. These are the field inverses (reciprocals) of the nonzero elements of $\mathscr{D}$ [3]. The examples e^{-is^2} and $e^{-s^2/2}$ in § 3 also appear to be nonneocontinuous. Indeed it seems highly unlikely that $e^{iz^2}\tilde{\varphi}(z) \in Z_{\mathrm{Exp}}$ or $e^{z^2/2}\tilde{\varphi}(z) \in Z_{\mathrm{Exp}}$ for any nonzero $\tilde{\varphi} \in Z$.

Example. Suppose $H(z) = Q(z)/P(z)$ is a rational function with P and Q polynomials. Then it is certainly possible to choose ϱ so that no poles of $H(z)$ lie on the line $\operatorname{Im} z = \varrho$. Moreover, $H(\omega+i\varrho)$ is bounded by a polynomial in ω, (certainly by $|Q(\omega+i\varrho)|$). Hence $H(z)$ represents an ultradistribution $\tilde{f}$, defined as in (14). It is easy to see (using Residue Theory) that if the path of integration is modified slightly so as to "cross over" a pole of $H(z)$ of order j at $z = \lambda$, then the ultradistribution is modified by a term proportional to $\delta^{(j)}(\lambda - z)$, where $\langle \delta^{(j)}(\lambda - z), \tilde{\sigma}(z)\rangle = \tilde{\sigma}^{(j)}(\lambda)$ for all $\tilde{\sigma} \in Z$, which is not in the field $\tilde{\mathscr{M}}_{\mathrm{Exp}}$. None-the-less the modified ultradistribution is still represented by $H(z)$. The inverse Fourier transform is similarly modified by a term proportional to $t^j e^{i\lambda t}$, which is not in the field $\mathscr{M}_{\mathrm{Exp}}$. Of course, $H(z) = Q(z)/P(z)$ is the Fourier transform of the unique operator x satisfying the equation $P[s]x = Q[s]$, where $P[s]$ is the polynomial $P(z)$ with z replaced by s. The term $t^j e^{i\lambda t}$ is just one of the (classical) solutions of the corresponding homogeneous differential equation, which can be added to or subtracted from the solutions of the inhomogeneous differential equation without invalidating them.

The operator $x = Q[s]/P[s]$, on the other hand, represents all distributional solutions f of the differential equation $P[d/dt]f = Q[\delta(t)]$. The Fourier transforms $\tilde{f}$ of the various solutions f of this inhomogeneous differential equation are the various ultradistributions represented by the meromorphic function $H(z) = Q(z)/P(z)$. In particular, $\dfrac{1}{P(z)}$ represents all distributional fundamental solutions belonging to the differential operator $P[d/dt]$.

6. Final remarks. Working independently, Száz [14] and Krabbe [4]–[6] have also developed an operational calculus they call two-sided. It is isomorphic to a direct sum of convolution rings on each of the two half-lines $(-\infty, 0]$ and $[0, \infty)$, and so reflects more the algebraic structure of a one-sided calculus. Indeed, their ring operations are equivalent to those defined pairwise for ordered pairs of functions or distributions with supports in these two half-lines. In particular, their multiplication for the direct sum is *not* convolution on the entire real line $\boldsymbol{R} = (-\infty, \infty)$. Their type of operational calculus is well suited for treating initial-value problems, but is not comparable with the two-sided calculus developed in this paper or with the classical (i.e. Fourier and Laplace) transform procedures on $\boldsymbol{R}$.

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A compact convex set with no extreme points

by

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Abstract. The purpose of the paper is to show the existence of a Fréchet space X containing a compact convex set K such that K contains no extreme points.

1. Introduction. The Krein–Milman theorem states that if K is a compact convex subset of a Hausdorff locally convex topological vector space X, then K is the closed convex hull of its extreme points ([2], p.70). In this paper we shall produce a Hausdorff topological vector space X containing a compact convex set K such that K has no extreme points. The question of the existence of such a compact convex set is mentioned in [1], p. 124, and [2], p. 70. The first step in the construction of the space X will be to construct some fairly pathological paranorms on finite-dimensional spaces. This will be done in Section 2. In Section 3 we shall inductively piece together the finite-dimensional spaces to obtain a linear metric space V. The space X will be obtained by taking the completion of V.

Lastly, the author would like to thank the referee for his very helpful suggestions.

2. Paranorms on finite-dimensional spaces. This section will deal almost exclusively with paranorms on finite-dimensional vector spaces. Throughout this paper all vector spaces will be over the reals and θ will always denote the zero element of the vector space. If V is a vector space, then a nonnegative real valued function N on V is called a *paranorm* if for every $x, y \in V$,

(1) $N(\theta) = 0$,

(2) $N(x) = N(-x)$,

(3) $N(x+y) \leqslant N(x)+N(y)$,

(4) $\lim_{\alpha\to 0} N(\alpha x) = 0$.

A paranorm N is *total* if $N(x) \neq 0$ for every $x \in V$ such that $x \neq \theta$. N is *monotone* if, for every $x \in V$ and $\alpha \in [0,1]$, $N(\alpha x) \leqslant N(x)$ (equivalently, if $|\beta| \leqslant |\gamma|$, then $N(\beta x) \leqslant N(\gamma x)$). If N is a total paranorm on a vector