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Universal spaces and universal bases in metric linear spaces

by

N. J. KALTON (Swansea, U. K.)

Abstract. Our main result is the existence of a separable F-space (complete metric linear space) universal for all separable F-spaces. We also show that there are complementably universal members of the families of all bases in Banach spaces and all bases in Fréchet spaces (locally convex metric linear spaces). In the other direction, we show that there is no separable F-space with a separating dual which is universal for all F-spaces with separating duals.

1. Introduction. A topological vector space X is [*complementably*] *universal* for a family $\mathscr{A}$ of topological vector spaces if for each $Y \in \mathscr{A}$, there exists a [complemented] subspace Z of X isomorphic to Y. X is *co-universal* for $\mathscr{A}$, if for each $Y \in \mathscr{A}$ there is a closed subspace N of X such that $Y \cong X/N$.

The set of all separable Banach spaces contains a universal member $C[0,1]$ (Banach–Mazur [1], p. 185), a co-universal member (l_1, see [10], p. 280) but no complementably universal member (Johnson–Szankowski [5]). On the other hand, the set of all Banach spaces with bases has a unique complementably universal member (Pełczyński [13]) which is also the unique complementably universal member for the set of all Banach spaces with the Bounded Approximation Property (Kadec [6], Pełczyński [14]). There are a number of other existence and non-existence results known on classes of separable Banach spaces (see for example [21], [24] and [27]).

It is the purpose of this paper to study universal spaces for general F-spaces (complete metric linear spaces) where much less is known. Our main result (Theorem 4.3) which answers a question of Rolewicz (see Problem II. 4.1 of [16]; also see [12]) is that there exists a separable F-space universal for all separable F-spaces. The proof depends on a method of packing expounded in § 2, which is a slight modification of that of Pełczyński [13] (see also Gurarii [3]). The modification is significant, in that we are able to apply the techniques of § 2 to Banach spaces and obtain some new results. We construct a universal linearly independent sequence and then answer a question of Pełczyński [13], p. 266, Problem 1, by showing that there is a universal basis for the family of bases in Banach spaces

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(without any normalization conditions). By modifying the technique we can also produce a universal basis for all bases in Fréchet spaces. We do not however know whether there is a universal basis for all bases in F-spaces (see Theorem 4.4).

In § 5 we answer a question of Rolewicz ([16], p. 47) by giving a co-universal separable F-space; we also show that there is no complementably universal separable F-space. In § 6 we answer a question of Pełczyński by showing that there is no nearly convex (i.e. with separating dual) separable F-space universal for all nearly convex separable F-spaces. Finally, in § 7 we show nearly convex F-spaces can always be embedded in F-spaces with the approximation property. In § 8 we list some open problems.

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The following notation will be used in the paper. An F^*-space is a metric linear space and an *F-semi-norm* on a real (or complex) vector space X is a map $q\colon X\to[0,\infty)$ satisfying (a) $q(0)=0$, (b) $q(x+y)\leqslant q(x)+q(y)$, $x,y\in X$, (c) $q(tx)\leqslant q(x)$ for $|t|\leqslant 1$, and (d) $\lim_{t\to 0} q(tx)=0$ for $x\in X$. If in addition $q(x)=0$ implies $x=0$, then q is an *F-norm*. A *p-norm* is an F-norm q satisfying $q(tx)=|t|^p q(x)$, $x\in X$. A set A is *absolutely p-convex* if $|\alpha|+|\beta|^p\leqslant 1$, $x,y\in A$ imply $\alpha x+\beta y\in A$. An F^*-space with a base of absolutely p-convex neighbourhoods of 0 is called *locally p-convex*. An F^*-space is *locally bounded* if it has a bounded neighbourhood of 0. Every locally bounded F^*-space is *locally p-convex* for some p, $0<p<1$ (see [16], p. 61) and its topology may be given by a p-norm. A locally p-convex locally bounded F-space is called a *p-Banach space*.

2. A Packing technique. Suppose $(E,\leqslant)$ is a partially ordered set. A subset A of E is (i) a *chain* if it is totally ordered and (ii) a *section* if $b\leqslant a\in A$ implies $b\in A$. For each $c\in E$, let $E[c]=\{a\colon a\leqslant c\}$. If for each $c\in E$, $E[c]$ is a finite chain we shall say that E is *treelike*.

LEMMA 2.1. *If E is countable and treelike, then there is a non-decreasing bijection $\sigma\colon N\to E$.*

Proof. Let $\{a_n\}$ be a sequence of elements of E such that each element occurs infinitely often. Define an increasing sequence $\{m_n\}$ by induction: if $\{m_j\colon j<k\}$ have been chosen (where if $k=1$, this set is empty), choose m_k to be the least $n>m_{k-1}$ such that $\{a_{m_1},\ldots,a_{m_{k-1}},a_n\}$ is a section of E and $a_n\notin\{a_{m_1},\ldots,a_{m_{k-1}}\}$. Then let $\sigma(k)=a_{m_k}$. It can be proved by induction on the number of elements of $E[c]$ that each $c\in E$ is in the range of σ.

For each subset A of E we let $\mathscr{F}(A)$ be the space of real finitely supported functions on A. Let P_A denote the natural projection $P_A\colon\mathscr{F}(E)\to\mathscr{F}(A)$. By a *consistent family* of F-norms on E we mean a collection $\{\pi_c\colon c\in E\}$ such that

(i) π_c is an F-norm on $\mathscr{F}(E[c])$.

(ii) If $a\leqslant c$, then for $x\in\mathscr{F}(E[a])$, $\pi_a(x)=\pi_c(x)$.

If $\{\pi_c\colon c\in E\}$ is a consistent family, we define the *limit F-norm* of $\{\pi_c\colon c\in E\}$ on $\mathscr{F}(E)$ by

$$\pi(x)=\inf\Big\{\sum_{i=1}^{n}\pi_{c_i}(u_i)\colon \sum_{i=1}^{n}u_i=x;\ u_i\in\mathscr{F}(E[c_i]);\ n\in N\Big\}. \tag{1}$$

It is clear that π is an F-semi-norm; the fact that it is an F-norm is proved below

PROPOSITION 2.2. *Suppose E is treelike and $\{\pi_c\colon c\in E\}$ is a consistent family of F-norms on E, with limit π. Then*

(i) *If $A=\operatorname{supp}x$, then*

$$\pi(x)=\inf\Big\{\sum_{a\in A}\pi_a(u_a)\colon u_a\in\mathscr{F}(E[a]),\ \sum u_a=x\Big\}.$$

(ii) *If $x\in\mathscr{F}(E[c])$, $\pi(x)=\pi_c(x)$.*

(iii) *π is an F-norm on $\mathscr{F}(E)$.*

Proof. (i) Given $\varepsilon>0$, choose a minimal collection $u_i\in\mathscr{F}(E[c_i])$ $(1\leqslant i\leqslant n)$ such that

$$\sum_{i=1}^{n}u_i=x,$$

$$\sum_{i=1}^{n}\pi_{c_i}(u_i)\leqslant\pi(x)+\varepsilon.$$

We may suppose that $c_i=\max[\operatorname{supp}u_i]$ by the consistency of $\{\pi_c\colon c\in E\}$. It is enough to show that for each i there exists $a_i\in A$ with $c_i\leqslant a_i$. Suppose not; then there exists a maximal c such that $c_j=c$ for some $j\leqslant n$, and c non $\leqslant a$ for any $a\in A$. As $\sum u_i(c)=0$, we have $c_k\geqslant c$ for some $k\neq j$, and hence $c_k=c$. Then

$$\sum_{i\neq j,k}u_i+(u_j+u_k)=x$$

and

$$\sum_{i\neq j,k}\pi_{c_i}(u_i)+\pi_c(u_j+u_k)\leqslant\pi(x)+\varepsilon,$$

contradicting the minimality of $[u_1,\ldots,u_n]$.

(ii) This follows immediately from (i); if $u_a \in \mathscr{F}(E[a])$ and $\sum u_a = x$, then $\sum \pi_a(u_a) = \sum \pi_c(u_a) \geqslant \pi_c(\sum u_a) = \pi_c(x)$.

(iii) If $\pi(x) = 0$, then for each $n \in N$ there exist $u_a^n \in \mathscr{F}(E[a])$, $a \in A = \operatorname{supp} x$, such that

$$\sum_{a \in A} \pi_a(u_a^n) \leqslant 1/n,$$

$$\sum u_a^n = x.$$

Then $\lim_{n\to\infty} \pi_a(u_a^n) = 0$ for each $a \in A$. As each $\mathscr{F}(E[a])$ is finite-dimensional $\lim_{n\to\infty} u_a^n(a) = 0$ for $a \in A$ (the functional $u \to u(a)$ is continuous on $(\mathscr{F}(E[a]), \pi_a)$). Hence $x(a) = 0$ for all $a \in A$, i.e. $x = 0$.

An F-norm π on $\mathscr{F}(E)$ is called *monotone* if $\pi(P_A x) \leqslant \pi(x)$ for each $x \in \mathscr{F}(E)$ and for each section $A \subset E$.

PROPOSITION 2.3. *Suppose E is treelike and $\{\pi_c\colon c \in E\}$ is a consistent family of F-norms satisfying $\pi_c(P_{E[a]}x) \leqslant \pi_c(x)$ whenever $a \leqslant c$. Then the limit F-norm π is monotone.*

Proof. Suppose A is any section of E and $x \in \mathscr{F}(E)$. For $\varepsilon > 0$ suppose $x = \sum_{i=1}^n u_i$, where $u_i \in \mathscr{F}(E[c_i])$ and $\sum \pi_{c_i}(u_i) \leqslant \pi(x) + \varepsilon$. Then $A \cap E[c_i] = E[a_i]$ for some a_i and hence

$$\pi(P_A x) \leqslant \sum \pi_{c_i}(P_A u_i) \leqslant \sum \pi_{c_i}(u_i) \leqslant \pi(x) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary this concludes the proof.

Consider now the space $\mathscr{F}(N)$ of all finitely non-zero sequences. For each $n \in N$ we define $e_n \in \mathscr{F}(N)$ by $e_n(k) = \delta_{nk}$. We shall also denote $\mathscr{F}(N[n])$ by the more natural R_n; of course $R_n = \operatorname{lin}(e_1, \ldots, e_n)$. Let Φ be the family of all F-norms on $\mathscr{F}(N)$ and Φ_n be the family of all F-norms on R_n. The distance $d(p, q)$ between two F-norms p and q on $X = \mathscr{F}(N)$ or R_n is defined by

$$d(p, q) = \sup_{\substack{x \in X \\ x \neq 0}} \big|\log\big(p(x)/q(x)\big)\big|.$$

Strictly d is not a metric; however we can define a metric d^* on X by

$$d^*(p, q) = \arctan d(p, q).$$

We shall therefore treat (Φ, d) and (Φ_n, d) as metric spaces. Let J_n denote the restriction map $J_n\colon \Phi \to \Phi_n$. Then J_n is contractive.

Construction 2.4. We now describe the general construction of universal spaces to be applied in the next two sections. We shall suppose $\mathscr{P}$ is a set of F-norms on $\mathscr{F}(N)$ satisfying

(α) $J_n(\mathscr{P})$ is separable for each $n \in N$;

(β) for each $p \in \mathscr{P}$, $\bar{q} \in J_n(\mathscr{P})$ there exists $q \in \mathscr{P}$ such that $J_n(q) = \bar{q}$ and $d(p, q) = d\big(J_n(p), \bar{q}\big)$.

Pick E_1 a dense countable subset of $J_1(\mathscr{P})$. Then pick countable subsets E_n of $J_n(\mathscr{P})$ as follows: if E_{n-1} has been chosen, choose for each $p \in E_{n-1}$ a dense countable subset $\Lambda(p)$ of $J_n(\mathscr{P}) \cap J_n\big(J_{n-1}^{-1}(p)\big)$. Then $E_n = \bigcup\big(\Lambda(p)\colon p \in E_{n-1}\big)$. Then let $E = \bigcup E_n$ and define a partial order $\leqslant$ on E by $p \leqslant q$ if and only if p is a restriction of q, i.e. if $p \in E_m$ and $q \in E_n$, then $m \leqslant n$ and $p(x) = q(x)$ for $x \in R_m$. Clearly, E is treelike.

Next we define a consistent family $(\pi_p\colon p \in E)$ of F-norms. If $p \in E_m$, then $E[p] = \{p_1, p_2, \ldots, p_m\}$ where $p_1 = p|R_1, \ldots, p_m = p|R_m = p$. Define

$$\pi_p(x) = p\Big(\sum_{i=1}^m x(p_i) e_i\Big).$$

Let π be the limit F-norm on $\mathscr{F}(E)$. Then we have

(γ) *For each $p \in \mathscr{P}$ and $\varepsilon > 0$ there exists a maximal chain C in E and an order-preserving bijection $\varphi\colon C \to N$ such that the map $T\colon \mathscr{F}(N) \to \mathscr{F}(C)$ defined by $Tx(a) = x(\varphi(a))$ satisfies*

$$(1-\varepsilon)p(x) \leqslant \pi(Tx) \leqslant (1+\varepsilon)p(x).$$

Proof of (γ). Choose $\eta_n > 0$ such that $\prod_{n=1}^\infty (1+\eta_n) \leqslant 1+\varepsilon$. Let $p_0 = p$; we define a sequence $p_n \in \mathscr{P}$ such that

(i) $(1+\eta_n)^{-1} p_{n-1}(x) \leqslant p_n(x) \leqslant (1+\eta_n) p_{n-1}(x)$, $n \geqslant 1$,

(ii) $J_{n-1}(p_n) = J_{n-1}(p_{n-1})$, $n \geqslant 2$,

(iii) $J_n(p_n) \in E_n$, $n \geqslant 1$.

Suppose $\{p_k\colon k < n\}$ have been chosen. Then $J_{n-1}(p_{n-1}) \in E_{n-1}$ (this condition is vacuous if $n = 1$), and $J_n(p_{n-1}) \in J_n(\mathscr{P}) \cap J_n\big(J_{n-1}^{-1}(p_{n-1}|R_{n-1})\big)$. Pick $q \in \Lambda(p_{n-1}|R_{n-1})$ such that

$$(1+\eta_n)^{-1} p_{n-1}(x) \leqslant q(x) \leqslant (1+\eta_n) p_{n-1}(x), \qquad x \in R_n$$

and then use (β) to choose $p_n \in \mathscr{P}$ such that

$$(1+\eta_n)^{-1} p_{n-1}(x) \leqslant p_n(x) \leqslant (1+\eta_n) p_{n-1}(x), \qquad x \in \mathscr{F}(N)$$

and

$$J_n(p_n) = q.$$

By condition (ii), the sequence $\big(J_n(p_n)\big)$ is a chain in E and it is clearly maximal. Let $C = \{J_n(p_n)\colon n \in N\}$ and define $\varphi\big(J_n(p_n)\big) = n$; for $n \geqslant 1$. For $x \in R_n$

$$\pi(Tx) = p_n(x)$$

and

$$(1-\varepsilon)p(x) \leqslant p_n(x) \leqslant (1+\varepsilon)p(x),$$

since

$$(1+\eta_1)\ldots(1+\eta_n) \leqslant 1+\varepsilon.$$

3. Universal sequences in Banach and Fréchet spaces. Let $\{x_n\}$ be a sequence in an F-space X and $\{y_n\}$ a sequence in an F-space Y. Let X_0 and Y_0 be the closed linear spans of (x_n) and (y_n), respectively. Then we shall say (x_n) and (y_n) are *equivalent* if there is an isomorphism (linear homeomorphism) $T\colon X_0 \to Y_0$ such that $Tx_n = y_n$ for $n \to \boldsymbol{N}$. Let Σ be a set of sequences in F-spaces; then a sequence (w_n) in an F-space is *universal* for Σ if for every $(x_n) \in \Sigma$ there is an increasing sequence (n_k) such that (x_n) is equivalent to (w_{n_k}). Let W be the closed linear span of (w_n). If, in addition, we can for each $(x_n) \in \Sigma$ choose n_k so that there is a continuous projection $P\colon W \to W$ satisfying $Pw_{n_k} = w_{n_k}$ $(k \in \boldsymbol{N})$ and $Pw_j = 0$, $j \notin (n_k)$, we say that (w_n) is *complementably universal* for Σ.

In [13], Pełczyński showed that the families of all seminormalized bases in Banach spaces and of all seminormalized unconditional bases in Banach spaces contain complementably universal members. We shall give some related results and in particular resolve a problem posed by Pełczyński ([13], p. 266) by finding a complementably universal member of the family of all bases in Banach spaces (Theorem 3.2 below).

THEOREM 3.1. *There is a linearly independent sequence (w_n) in $C(0,1]$ which is universal for all linearly independent sequences in Banach spaces.*

Proof. Let M be the set of all norms on $\mathscr{F}(\boldsymbol{N})$. M clearly satisfies 2.4(α). For 2.4(β) note that

$$e^{-\theta}p(x) \leqslant \bar{q}(x) \leqslant e^{\theta}p(x), \quad x \in R_n,$$

where $\theta = d(J_n(p), \bar{q})$. Let

$$q(x) = \inf\{\bar{q}(y) + e^{\theta}p(x-y),\ y \in R_n\}. \tag{2}$$

Then for $x \in R_n$, it is easy to show that $q(x) = \bar{q}(x)$, and $x \in \mathscr{F}(\boldsymbol{N})$, $e^{-\theta}p(x) \leqslant q(x) \leqslant e^{\theta}p(x)$.

If $\{x_n\}$ is any linearly independent sequence in a Banach space, then its linear span is isomorphic to $(\mathscr{F}(\boldsymbol{N}), p)$ for some $p \in M$. Now construct the space $(\mathscr{F}(E), \pi)$ as in 2.4. For each $a \in E$ denote by $e_a \in \mathscr{F}(E)$, the element $e_a(a) = 1$, $e_a(b) = 0$ if $b \neq a$. Choose $\sigma\colon N \to E$ to satisfy Lemma 2.1 and write $w_n = e_{\sigma(n)}$. Then, by 2.4(γ) corresponding to (x_n) and $\varepsilon > 0$, there is an increasing sequence (n_k) such that

$$(1-\varepsilon)\Big\|\sum_{i=1}^{m} t_i x_i\Big\| \leqslant \pi\Big(\sum_{i=1}^{m} t_i w_{n_i}\Big) \leqslant (1+\varepsilon)\Big\|\sum_{i=1}^{m} t_i x_i\Big\|. \tag{3}$$

Finally, observe that $(\mathscr{F}(E), \pi)$ can be embedded isometrically in $C[0,1]$ by the Banach–Mazur theorem ([1], p. 185).

Remark. Of course (w_n) constructed above is "nearly isometrically universal" (see (3)).

THEOREM 3.2. *There is a basis of a Banach space B, which is complementably universal for the family of all bases in Banach spaces.*

Remark. By Corollary 4, p. 266 of [13], the space B is unique and isomorphic to the space constructed in [13]. One may also mimic the techniques of [13] to show that the complementably universal basis is unique up to permutation and equivalence.

Proof. The proof of Theorem 3.2 is very similar to that of Theorem 3.1. Let $\mathscr{B}$ be the family of all monotone norms on $\mathscr{F}(\boldsymbol{N})$. Again $\mathscr{B}$ satisfies 2.4(α), and for 2.4(β) we need only observe that q constructed in equation (2) is now a monotone norm. Let $(\mathscr{F}(E), \pi)$ be the constructed normed space; π is a monotone norm on $\mathscr{F}(E)$ by 2.3. Let B be the completion of $(\mathscr{F}(E), \pi)$ and choose σ by Lemma 2.1. Then $w_n = e_{\sigma(n)}$ is a monotone basis of B since for each n, $\{\sigma(1), \sigma(2)\ldots\sigma(n)\}$ is a section of E. Furthermore, for each maximal chain C in E given by $\{\sigma(k)\colon k \in \boldsymbol{N}\}$, the projection

$$P_C\Big(\sum_{i=1}^{\infty} t_i w_i\Big) = \sum_{i=1}^{\infty} t_{\sigma(i)} w_{\sigma(i)}$$

has norm one, since C is a section of E.

The theorem follows from the observation that for any basis $\{x_n\}$ of a Banach space X, there is a monotone norm p on $\mathscr{F}(\boldsymbol{N})$ such that the map

$$T\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i e_i$$

is a homeomorphism between $\operatorname{lin}\{x_n\}$ and $(\mathscr{F}(\boldsymbol{N}), p)$.

Remark. Theorem 3.2 could also be obtained by applying a technique of Zippin [27]. Let $(\mathscr{F}(E), \pi)$ be the space constructed in Theorem 3.1 and define

$$\pi^*(x) = \sup[\pi(P_A x)\colon A \text{ a section of } E].$$

Then π^* is monotone on $\mathscr{F}(E)$. Let B be the completion of $(\mathscr{F}(E), \pi^*)$. It is not difficult to show that (w_n) is now a complementably universal basis. Using either direct method or this approach we may prove Theorems 3.3 and 3.4 below.

THEOREM 3.3. *There is a complementably universal member of the family of all unconditional bases in Banach spaces.*

A sequence (x_n) in a Banach space X is semi-basic ([7]) if there is a sequence $x_n^* \in X^*$ such that (x_n, x_n^*) is *bi-orthogonal*, i.e. $x_m^*(x_n) = \delta_{mn}$. If in addition x_n^* can be chosen so that $\sup\|x_n\|\,\|x_n^*\| < \infty$, then (x_n) is *boundedly semi-basic*.

THEOREM 3.4. *There is a universal member of the set of all boundedly semi-basic sequences.*

Next we consider Fréchet spaces (i.e. locally convex F-spaces). The author does not know whether there is an analogue to Theorem 3.1 in this setting. However Theorem 3.2 can be extended:

THEOREM 3.5. *There is a basis of a Fréchet space C which is complementably universal for the family of all bases in Fréchet spaces.*

Proof. Consider the partially ordered space E which was constructed for the proof of Theorem 3.2; denote by E_n the subset of E of all norms on R_n (as in 2.4). Let λ_n denote the l_1-norm on R_n. We may suppose that $\lambda_n \in E_n$ for all n.

Now for each $m \in N$, let D_m be the set of all pairs of sequences $\{(p_n), (\alpha_n)\}$, where

(i) $p_n \in E_m$, $n \in N$,

(ii) $p_n = \lambda_m$ except for finitely many n,

(iii) α_n is a subset of $\{1, 2, \ldots, m\}$, $n \in N$,

(iv) $\alpha_n \supset \{1, 2, \ldots, \min(m, n)\}$, $n \in N$.

Then each D_m is countable and so is $D = \bigcup D_m$. We order D by $\{(p_n), (\alpha_n)\} \leqslant \{(q_n), (\beta_n)\}$ if and only if $\{(p_n), (\alpha_n)\} \in D_l$ and $\{(q_n), (\beta_n)\} \in D_m$ where $l \leqslant m$ and

(i)′ $p_n \leqslant q_n$, $n \in N$ (i.e. $p_n = q_n|R_l$),

(ii)′ $\alpha_n = \beta_n \cap \{1, 2, \ldots, l\}$, $n \in N$.

Then D is countable and treelike.

For each $m \in N$ we define a consistent family of norms on $\mathscr{F}(D)$ by

$$\pi_a^{(m)}(x) = p_m\Big(\sum_{i=1}^{k} x(a_i)e_i\Big),$$

where $a = \{(p_n), (\alpha_n)\}$ and $a_1 < a_2 < \ldots < a_k = a$ is the chain $D[a]$. Each family $(\pi_a^{(m)}\colon a \in D)$ has a limit norm $\pi^{(m)}$. Next define $A_m \subset D$ by

$$A_m = \bigcup_{k=1}^{\infty} \big\{\{(p_n), (\alpha_n)\} \in D_k\colon k \in \alpha_m\big\},$$

and

$$\varrho^{(m)}(x) = \pi^{(m)}(P_{A_m}x).$$

We have thus defined a sequence $(\varrho^{(m)})$ of monotone semi-norms on $\mathscr{F}(D)$. It is not difficult to check that they define a Hausdorff topology. Take for C the completion of $\mathscr{F}(D)$. As in the proof of Theorem 3.2 the elements $(e_a\colon a \in D)$ can be arranged in a sequence to be a basis of C, and if (a_n) is any maximal chain in D, then $\overline{\operatorname{lin}}(e_{a_n})$ is complemented in C.

It therefore remains only to show that if $\{q_n\}$ is any sequence of monotone semi-norms on $\mathscr{F}(N)$, then there is a maximal chain $\{a_n\}$ in D such that the map $T\colon \mathscr{F}(N) \to \mathscr{F}\{a_n\}$ such that $Te_i = e_{a_i}$ is a homeomorphism.

First, we may suppose each q_n is a norm on R_n; then there exist $M_n > 0$ such that $\lambda_n \leqslant M_n J_n(q_n)$. Define

$$q_n^*(x) = \inf\{\lambda_n(y) + M_n q_n(x-y)\colon y \in R_n\}.$$

Then q_n^* is equivalent to q_n and also is monotone. Furthermore, $J_m(q_n^*) = \lambda_m$ if $m \leqslant n$. The sequence $\{q_n^*\}$ defines the same topology as $\{q_n\}$. Now let

$$\gamma_n = \{k\colon q_n^*(e_k) \neq 0\}$$

and

$$q_n^{**}(x) = q_n^*(x) + \sum_{k \notin \gamma_n} |x_k|.$$

q_n^{**} is a monotone norm on $\mathscr{F}(N)$ and $J_n(q_n^{**}) = \lambda_n$. By the techniques of 2.4 there is a monotone norm p_n on $\mathscr{F}(N)$ such that $\frac{1}{2}q_n^{**} \leqslant p_n \leqslant \frac{3}{2}q_n^{**}$ and $\big(J_m(p_n)\colon m \in N\big)$ is a chain in E. We may further suppose that $J_m(p_n) = \lambda_m$ for $m \leqslant n$.

Next, let $a_m \in D_m$ be given by

$$a_m = \big\{\big(J_m(p_n)\big), (\gamma_n \cap \{1, 2, \ldots, m\})\big\}.$$

Then (a_m) is a maximal chain in D, and

$$\varrho^{(k)}\Big(\sum_{i \leqslant m} t_i e_{a_i}\Big) = \pi^{(k)}\Big(\sum_{\substack{i \in \gamma_k \\ i \leqslant m}} t_i e_{a_i}\Big) = p_k\Big(\sum_{\substack{i \in \gamma_k \\ i \leqslant m}} t_i e_i\Big)$$

and $\frac{1}{2}q_k^{**} \leqslant p_k \leqslant \frac{3}{2}q_k^{**}$. However,

$$q_k^{**}\Big(\sum_{\substack{i \in \gamma_k \\ i \leqslant m}} t_i e_i\Big) = q_k^*\Big(\sum_{i=1}^{m} t_i e_i\Big).$$

Hence the linear span of (e_{a_i}) in $\mathscr{F}(D)$ is linearly homeomorphic to $\mathscr{F}(N)$ in the topology induced by (q_n) as required.

THEOREM 3.6. *Let C_1 be any Fréchet space with a basis complementably universal for all Fréchet spaces with bases. Then $C_1 \cong C$.*

Proof. We apply the Pełczyński decomposition technique as in [13]. If $\omega(C)$ denotes the countable Cartesian product $\prod (C)_i$ of C, then $C_1 \cong \omega(C) \oplus \oplus X \cong C \oplus \omega(C) \oplus X \cong C \oplus C_1$ and similarly $C \cong C \oplus C_1$.

THEOREM 3.7. *The space C is complementably universal for any Fréchet space with the Bounded Approximation Property.*

Proof. This follows from Remark 1 of [14].

Mazur and Orlicz [12] showed that $C(-\infty, \infty)$ is universal for separable Fréchet spaces.

THEOREM 3.8. *The space C is co-universal for all separable Fréchet spaces.*

Proof. Let X be a separable Fréchet space and $\{x_n\}$ a dense sequence in X. Let S be the space of all sequences (t_n) such that $\sum t_n x_n$ converges absolutely. If (p_n) is an increasing sequence of semi-norms defining the topology of X, define p_n^* on S by

$$p_n^*(t) = \sum_{i=1}^{\infty} |t_i| p_n(x_i).$$

Then S equipped with (p_n^*) is a Fréchet space with an absolute basis, and hence is isomorphic to complemented subspace of C. Define $T: S \to X$ by $T(t) = \sum t_n x_n$. If $U = \{x: p_n^*(x) < \varepsilon\}$ is a neighbourhood of 0, then $T(U) \supset \{x_k: p_n^*(x_k) < \varepsilon\}$ and hence $\overline{T(U)}$ is a neighbourhood of 0 in X. Hence T is surjective by the Open Mapping Theorem, and X is a quotient of S and hence of C.

Finally, we remark without proof that by the Zippin technique (see [27]), we have

THEOREM 3.9. *There is a Fréchet space with a universal unconditional (absolute) basis for the family of all unconditional (absolute) bases in Fréchet spaces.*

4. Applications to non-locally convex F-spaces. It is easy to duplicate the results for Banach spaces in the setting of p-normed spaces. We omit the proof of the following:

THEOREM 4.1. (a) *For $0 < p < 1$, there is a separable p-Banach space which is universal for all separable p-Banach spaces.*

(b) *For $0 < p < 1$, there is a unique p-Banach space B_p with a basis which is complementably universal for all p-Banach spaces with a basis.*

Remark. There is no locally bounded space universal for all separable locally bounded spaces (Rolewicz [16], p. 76).

For general F-spaces the situation is slightly more complicated. We shall need the following lemma; we shall say that a metric linear space X is a *βF^*-space* if it does not contain arbitrarily short lines, i.e. $\inf_{x \neq 0} \sup_{t \in R} p(tx) > 0$ where p is any F-norm defining the topology of X.

LEMMA 4.2. *Let X be a separable linear metric space and N a separable normed space of infinite dimension. Then $X \oplus N$ contains a dense subspace which is a βF^*-space.*

Proof. (The following simple proof is due to A. Pełczyński.) Let $\{x_{mn}\}$ be a linear independent double sequence with dense linear span in N such that $\|x_{mn}\| = 1$ for $m, n \in N$. Let $\{y_n\}$ be a dense sequence in X. Let $z_{mn} \in X \oplus N$ be given by $z_{mn} = \left(y_m, \frac{1}{n} x_{mn}\right)$, and let X_0 be the linear

span of $\{z_{mn};\ m \in N, n \in N\}$. If $\{t_{mn};\ m \in N, n \in N\}$ are not all zero, then $\sum t_{mn} n^{-1} x_{mn} \neq 0$ and hence the line generated by $\sum t_{mn} z_{mn}$ has infinite length in the F-norm $\|(x, y)\| = \|x\| + \|y\|$ on $X \oplus N$. Hence X_0 is a βF^*-space. Clearly, $(y_m, 0) = \lim_{n\to\infty} z_{mn} \in \overline{X}_0$ and hence $(0, x_{mn}) = z_{mn} - (y_m, 0) \in \overline{X}_0$. It follows quickly that $\overline{X}_0 = X \oplus N$.

The next theorem answers a problem posed by Rolewicz in [16], p. 46 (Problem II. 4.1).

THEOREM 4.3. *There is a separable F-space which is universal (with respect to linear dimension) for all separable F-spaces.*

Proof. Consider the class $\mathscr{P}$ of all F-norms p on $\mathscr{F}(N)$ satisfying

(1) $\sup_{t \in R} p(tx) = 1,\ x \in \mathscr{F}(N),\ x \neq 0.$

(2) For each $n \in N$, there exists a neighbourhood U_n of 0 in R_n such that if $x \in U_n$ and $|t| \leqslant 1$, $p(tx) = |t| p(x)$.

We first verify conditions 2.4(α) and 2.4(β). For fixed $n \in N$ let $V_m = \left\{x \in R_n: \sum_{i=1}^{n} |x_i| \leqslant m^{-1}\right\}$. Let $Q(m, n)$ be the set of F-norms p on R_n satisfying

(1)′ $\sup_{t \in R} p(tx) = 1,\ x \neq 0,\ x \in R_n.$

(2)′ If $x \in V_m$, $|t| \leqslant 1$, $p(tx) = |t| p(x)$.

Then $J_n(\mathscr{P}) \subset \bigcup_{m=1}^{\infty} Q(m, n)$. We show each $Q(m, n)$ separable for the metric d of 2.4. Choose in $Q(m, n)$ a countable subset Q which is dense for the topology of uniform convergence on compact subsets of R_n. If $p \in Q(m, n)$ and $\theta > 0$, let

$$K = \{x: x \in R_n, p(x) \leqslant e^{-\frac{1}{2}\theta}\}.$$

By Lemma 3.1 of [7], K is compact as it contains no lines.

Also let

$$\gamma = \inf_{x \notin \operatorname{Int} V_m} p(x).$$

Then choose $q \in Q$ such that

$$|p(x) - q(x)| \leqslant \gamma e^{-\frac{1}{2}\theta}(1 - e^{-\frac{1}{2}\theta}), \quad x \in K.$$

Now consider three cases

(a) $x \notin K$. Then there exists $t < 1$ such that $p(tx) = e^{-\frac{1}{2}\theta}$ and $q(tx) \geqslant e^{-\frac{1}{2}\theta} - \gamma e^{-\frac{1}{2}\theta}(1 - e^{-\frac{1}{2}\theta}) \geqslant e^{-\theta}$. Then

$$\frac{p(x)}{q(x)} \leqslant \frac{1}{e^{-\theta}}$$

and

$$\frac{q(x)}{p(x)} \leqslant \frac{1}{e^{-\frac{1}{2}\theta}} < e^{\theta}.$$

(b) $x \in K - \operatorname{int} V_m$. Then

$$\frac{q(x)}{p(x)} \leqslant 1 + \frac{\gamma e^{-\frac{1}{2}\theta}(1-e^{-\frac{1}{2}\theta})}{p(x)} \leqslant 2 - e^{-\frac{1}{2}\theta} < e^{\frac{1}{2}\theta} < e^{\theta},$$

$$\frac{p(x)}{q(x)} \leqslant 1 + \frac{\gamma e^{-\frac{1}{2}\theta}(1-e^{-\frac{1}{2}\theta})}{q(x)} \leqslant 1 + \frac{e^{-\frac{1}{2}\theta}(1-e^{-\frac{1}{2}\theta})}{1-e^{-\frac{1}{2}\theta}(1-e^{-\frac{1}{2}\theta})}$$

$$= \frac{1}{1-e^{-\frac{1}{2}\theta}+e^{\theta}} < e^{\frac{1}{2}\theta} < e^{\theta}.$$

(c) $x \in \operatorname{int} V_m$. Then there exists $s > 1$ such that $sx \in \partial V_m$. Thus

$$\left|\log \frac{p(sx)}{q(sx)}\right| \leqslant \theta$$

and hence by condition (2)

$$\left|\log \frac{p(x)}{q(x)}\right| \leqslant \theta.$$

Thus $d(p, q) \leqslant \theta$ and Q is dense in $Q(m, n)$. Hence $J_m(\mathscr{P})$ is separable and 2.4(α) is established.

To prove 2.4(β), suppose $p \in \mathscr{P}$ and $\bar{q} \in J_n(\mathscr{P})$ are given such that $d(J_n(p), \bar{q}) = \theta > 0$. Let

$$p^*(x) = \min\{e^{\theta} p(x), 1\}.$$

Clearly, $p^* \in \mathscr{P}$. Then let

$$q(x) = \inf\{\bar{q}(y) + p^*(x-y) \colon y \in R_n\}.$$

We omit the proof that q is an F-norm.

If $x \in R_n$, $p^*(x-y) \geqslant \bar{q}(x-y)$ so that $q(x) = \bar{q}(x)$. Thus $J_n(q) = \bar{q}$. Also $q \leqslant e^{\theta} p$ and for any $x \in \mathscr{F}(N)$, $y \in R_n$

$$\bar{q}(y) + p^*(x-y) \geqslant e^{-\theta}\big(p(y) + p(x-y)\big) \geqslant e^{-\theta} p(x),$$

so that $q \geqslant e^{-\theta} p$. Thus $d(p, q) = \theta$.

We must show that $q \in \mathscr{P}$. Clearly, for $x \in \mathscr{F}(N)$, $x \neq 0$,

$$\sup_{t\in R} q(tx) \leqslant 1.$$

Suppose $\sup_{t\in R} q(tx) < \beta < 1$. Then, for any $t \in R$, $tx = u_t + v_t$, where $u_t \in R_n$ and $\bar{q}(u_t) \leqslant \beta$, and also $p^*(v_t) \leqslant \beta$. The set $V = \{z \in R_n \colon \bar{q}(z) \leqslant \beta\}$ is compact since it contains no lines (Lemma 3.1 of [7], again) and also so is $W = \{z \in \operatorname{lin}(R_n, x) \colon p^*(z) \leqslant \beta\}$ for the same reason. However, $\operatorname{lin}\{x\} \subset V + W$, a contradiction. Thus $\sup_{t\in R} q(tx) = 1$.

For condition (2) suppose $m \geqslant n$. Then there exists $\varepsilon_m > 0$ such that

(a) If $y \in R_n$, $\bar{q}(y) \leqslant \varepsilon_m$ and $|t| \leqslant 1$, then $\bar{q}(ty) = |t| q(y)$.

(b) If $y \in R_m$, $p^*(y) \leqslant \varepsilon_m$ and $|t| \leqslant 1$, then $p^*(ty) = |t| p^*(y)$.

Now suppose $y \in R_m$ and $q(y) < \varepsilon_m$. Suppose $|t| \leqslant 1$. If $y = u + v$, $u \in R_n$ and $\bar{q}(u) + p^*(v) < \varepsilon_m$, then $\bar{q}(tu) + p^*(tv) = |t|\big(\bar{q}(u) + p^*(v)\big)$. Hence $q(ty) \leqslant |t| q(y)$.

Conversely, if $ty = u + v$, where $\bar{q}(u) + p^*(v) < |t| q(y)$, then $\bar{q}(u) < \varepsilon_m |t|$ and $p^*(v) < \varepsilon_m |t|$. If s denotes the largest real number such that $\bar{q}(su) \leqslant \varepsilon_m$, then $\bar{q}(u) = s^{-1}\varepsilon_m$ so that $s > |t|^{-1}$; thus $\bar{q}(t^{-1}u) < \varepsilon_m$ and, similarly, $p^*(t^{-1}v) < \varepsilon_m$. Hence $\bar{q}(t^{-1}u) + p^*(t^{-1}v) < q(y)$ which is a contradiction.

Thus we have proved that

$$q(ty) = |t| q(y) \quad \text{ if } |t| \leqslant 1 \text{ and } q(y) < \varepsilon_m.$$

q is therefore homogeneous on a neighbourhood of 0 in each R_m for $m \geqslant n$. This clearly implies (2). This completes the verification of 2.4(β).

From 2.4(γ) we can conclude the existence of a countable dimensional F^*-space Z with an F-norm π on Z such that for each $p \subset \mathscr{P}$, $(\mathscr{F}(N), p)$ is linearly homeomorphic to a subspace of (Z, π).

Now let p be any F-norm on $\mathscr{F}(N)$ such that $(\mathscr{F}(N), p)$ is a βF^*-space. We show p is equivalent to some $q \in \mathscr{P}$. We may suppose that p satisfies

$$p(tx) \geqslant |t| p(x), \quad |t| \leqslant 1.$$

To see this replace p by $p^{-*}(x) = \sup_{t\geqslant 1} t^{-1} p(tx)$—for stronger results see [2].) Next replace p by p^*, where $p^*(x) = (p(x))^{1/2}$. Then p^* is equivalent to p and satisfies

$$p^*(tx) \geqslant |t|^{\frac{1}{2}} p^*(x), \qquad |t| \leqslant 1.$$

Now select any strictly increasing sequence θ_n such that $\theta_1 = 1$ and $\theta_n \to 2$. Define for $x \in \mathscr{F}(N)$,

$$p^{**}(x) = \inf\left\{\sum_{k=1}^{n} \theta_k p^*(u_k) \colon u_k \in R_k,\ n \in N,\ u_1 + \ldots + u_n = x\right\}.$$

Then $p^*(x) \leqslant p^{**}(x) \leqslant 2p^*(x)$. If $x \in R_n$, it is easy to show that

$$p^{**}(x) = \inf\left\{\sum_{k=1}^{n} \theta_k p_k^*(u) \colon u_k \in R_k,\ u_1 + \ldots + u_n = x\right\}$$

and $p^{**}(x) \leqslant \theta_n p^*(x)$.

Suppose (M_n) is an increasing sequence of positive numbers and define
$$q(x)=\inf\Big\{p^{**}(y)+\sum_{i=1}^{\infty}M_i|x(i)-y(i)|:\ y\in\mathscr{F}(N)\Big\}.$$
We claim first that if (M_n) satisfies $p^*(M_n^{-1}e_n)\leqslant 2^{-n}$, then q is equivalent to p. Clearly, $q\leqslant p^{**}\leqslant 2p^*$; now suppose $q(x_n)\to 0$. Then $x_n=u_n+v_n$, where $p^{**}(u_n)\to 0$ and $\lim_{n\to\infty}\sum M_i|v_n(i)|=0$. If $\sum M_i|v_n(i)|<1$, then $|v_n(i)|\leqslant M_i^{-1}$ for all i and hence, for any l
$$p^*(v_n)\leqslant p^*\Big(\sum_{k=1}^{l}v_n(k)e_k\Big)+2^{-l}.$$
so that $\limsup_{n\to\infty}p^*(v_n)\leqslant 2^{-l}$. Thus $p^*(v_n)\to 0$ and so $p(x_n)\to 0$.

Now we turn to the selection of $(M_n:\ n\geqslant 1)$. For a given sequence (M_n) and $k\in N$ we define an F-norm q_k on R_k by
$$q_k(x)=\inf\Big\{p^{**}(y)+\sum_{i=1}^{k}M_i|x(i)-y(i)|:\ y\in R_k\Big\}.$$
Let δ_0 be chosen so that for each m, the set $R_m\cap\{x:\ p^*(x)\leqslant\delta_0\}$ is compact (possible since $(\mathscr{F}(N),p)$ is a βF^*-space). We shall choose an increasing sequence (M_n) and a decreasing sequence (δ_n) of positive numbers so that

(A) $p^*(M_n^{-1}e_n)\leqslant 2^{-n}\ (n\in N)$,

(B) $\delta_1<\frac{1}{4}\delta_0$,

(C) if $u\in R_{n-1}$ and $p^*(u)\leqslant\frac{1}{4}\delta_0$, then $q_n(u)=q_{n-1}(u)$ $(n\geqslant 2)$,

(D) if $u\in R_n$, $p^*(u)\leqslant\delta_n$ and $|t|\leqslant 1$, then $q_n(tu)=|t|q_n(u)$.

To start the induction pick M_1 so that (A) holds and δ_1 so that (B) and (D) hold (for details of the latter condition, see below). Now suppose $M_1<M_2<\ldots<M_{n-1}$ and $\delta_1>\delta_2>\ldots>\delta_{n-1}>0$ have been chosen. The set $\{x\in R_n:\ \delta_{n-1}\leqslant p^*(x)\leqslant\delta_0\}$ is compact, and hence if λ_n denotes the l_1-norm on R_n we may choose M_n so that (A) holds and if $\delta_{n-1}\leqslant p^*(x)\leqslant\delta_0$ and $\lambda_n(y)\leqslant M_n^{-1}\delta_0$, then $p^*(x-y)\geqslant\theta_n^{-1}\theta_{n-1}p^*(x)$, and $p^*(y)\leqslant\frac{1}{4}\delta_0$.

Now suppose $x\in R_{n-1}$ and $p^*(x)\leqslant\frac{1}{4}\delta_0$; suppose $q_n(x)<q_{n-1}(x)$. It is easy to see that
$$q_n(x)=\inf\{q_{n-1}(u)+M_n\lambda_n(v)+\theta_np^*(w):\ u+v+w=x,\ u\in R_{n-1},\ v,w\in R_n\}.$$

Thus $x=u+v+w$, where
$$q_{n-1}(u)+M_n\lambda_n(v)+\theta_np^*(w)<q_{n-1}(x).$$
Hence
$$M_n\lambda_n(v)+\theta_np^*(w)<q_{n-1}(v+w).$$
Now $q_{n-1}(x)\leqslant\theta_{n-1}p^*(x)\leqslant\frac{1}{2}\delta_0$. Hence $\lambda_n(v)\leqslant M_n^{-1}\delta_0$, and
$$p^*(v+w)\leqslant p^*(v)+p^*(w)\leqslant\tfrac{1}{4}\delta_0+\tfrac{1}{2}\delta_0<\delta_0.$$
Suppose first $p^*(v+w)\geqslant\delta_{n-1}$; then $p^*(w)\geqslant\theta_n^{-1}\theta_{n-1}p^*(v+w)$, and hence
$$M_n\lambda_n(v)+\theta_np^*(w)\geqslant\theta_{n-1}p^*(v+w)\geqslant q_{n-1}(v+w)$$
which is impossible.

Next suppose $p^*(v+w)<\delta_{n-1}$, and choose $t\geqslant 1$ so that $p^*(t(v+w))=\delta_{n-1}$. Then either $\lambda_n(tv)>M_n^{-1}\delta_0$ or $p^*(tw)\geqslant\theta_{n-1}\theta_n^{-1}\delta_{n-1}$. In either case
$$M_n\lambda_n(tv)+\theta_np^*(tw)\geqslant\theta_{n-1}\delta_{n-1}$$
and hence, as $v+w\in R_{n-1}$,
$$\begin{aligned}M_n\lambda_n(v)+\theta_np^*(w)&\geqslant|t|^{-1}\theta_{n-1}\delta_{n-1}\\&\geqslant|t|^{-1}q_{n-1}(tv+tw)=q_{n-1}(v+w)\quad\text{(by (D))}\end{aligned}$$
which is again a contradiction. Thus we have proved (C).

It remains to pick δ_n to satisfy (D). Let
$$L=\max\Big(\sum_{i=1}^{n}M_i|x(i)|:\ x\in R_n,\ p^*(x)=\delta_0\Big).$$
Pick $\delta_n<\frac{1}{2}\min(\delta_{n-1},L^{-1}\delta_0^2)$. If $p^*(x)\leqslant\delta_n$, then $q_n(x)<2\delta_n$ and if $y\in R_n$, and
$$p^{**}(y)+\sum_{i=1}^{n}M_i|x(i)-y(i)|\leqslant 2\delta_n,$$
$p^*(y)\leqslant 2\delta_n$. Hence for some $t\geqslant 1$, $p^*(ty)=\delta_0$. Then $\delta_0\leqslant|t|^{\frac{1}{2}}p^*(y)$ and
$$\sum M_i|y(i)|\leqslant L|t|^{-1}\leqslant L\delta_0^{-2}(p^*(y))^2\leqslant 2\delta_nL\delta_0^{-2}p^*(y)\leqslant p^*(y).$$
Hence
$$\sum_{i=1}^{n}M_i|x(i)|\leqslant p^*(y)+\sum_{i=1}^{n}M_i|x(i)-y(i)|\leqslant p^{**}(y)+\sum_{i=1}^{n}M_i|x(i)-y(i)|;$$
It follows that $q_n(x)=\sum_{i=1}^{n}M_i|x(i)|$, whenever $p^*(x)\leqslant\delta_n$, and so $q_n(tx)=|t|q_n(x)$ $(|t|\leqslant 1,\ p^*(x)\leqslant\delta_n)$.

To conclude we observe that if $x \in R_m$, $q(x) = \lim_{n\to\infty} q_{m+n}(x)$. Hence if $p^*(x) \leqslant \delta_m$ and $|t| \leqslant 1$, $q(tx) = |t|q(x)$, by conditions (C) and (D). Thus q satisfies condition (2) of the class $\mathscr{P}$.

To ensure (1), note that as q is equivalent to p $\inf_{x\neq 0} \sup_{t\in R} q(tx) = a > 0$ and hence $\hat{q}(x) = \min\left(a^{-1}q(x), 1\right)$ belongs to $\mathscr{P}$ and is equivalent to p.

Now the space (Z, π), constructed above, is countable dimensional and universal for countable dimensional βF^*-spaces. Its completion is therefore by Lemma 4.2, universal for all separable F-spaces.

The techniques of Theorem 4.3 can be applied to bases as in § 3, but they do not yield a universal basis space. Instead we have

THEOREM 4.4. *There is a unique F-space with a basis and without arbitrarily short lines which is complementably universal for the set of all F-spaces with bases and without arbitrarily short lines.*

We omit the proof in view of its similarity to Theorem 4.3. Uniqueness can again be proved by the Pełczyński decomposition technique (cf. Theorem 3.6 — here l_1-products of F-normed spaces replace Cartesian products).

5. A co-universal space. In [16], p. 47, Rolewicz asks whether there is a separable F-space which is co-universal (i.e. universal with respect to linear co-dimension) for all separable F-spaces. In this section we construct such a space. First let us, however, point out that there is no separable F-space which is complementably universal (and hence co-universal) for all separable F-spaces. Our argument is a simple version of the argument of Johnson and Szankowski [4] for the same result in Banach spaces (where it is much deeper).

THEOREM 5.1. *There is no separable F-space containing a complemented copy of each space $L_p(0,1)$ for $0 < p < 1$.*

Proof. Suppose X is such a space and let (U_n) be a basis of neighbourhoods of 0. Suppose for each p, $0 < p < 1$, Y_p is a subspace of X isomorphic to L_p, and let $Q_p\colon X \to Y_p$ be a continuous projection.

For each p choose $m(p) \in \boldsymbol{N}$ such that $U_{m(p)} \cap Y_p \neq Y_p$ and $n(p) \in \boldsymbol{N}$ such that $Q_p(U_{n(p)}) \subset U_{m(p)}$. Then since $\{p\colon 0 < p < 1\}$ is uncountable, there exist $m_0, n_0 \in \boldsymbol{N}$ such that the set $A = \{p\colon m(p) = m_0,\ n(p) = n_0\}$ is uncountable.

For each $p \in A$ pick $y_p \in Y_p$ such that $y_p \notin U_{m_0}$. Suppose $p, r \in A$ and $p < r$. Then since $Y_p \cong L_p$ and $Y_r \cong L_r$, the map $Q_r|Y_p = 0$ by a theorem of Turpin ([22], Théorème 1). Hence $Q_r(y_r - y_p) = y_r \notin U_{m_0}$. Therefore $y_r - y_p \notin U_{n_0}$. This is impossible for every such pair (p, r) since X is separable and A is uncountable.

We now introduce a class of modular sequence spaces (cf. [25], [26]). Let Φ denote the class of continuous non-decreasing functions $\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfying (a) $\varphi(0) = 0$, (b) $\varphi(s+t) \leqslant \varphi(s) + \varphi(t)$, (c) $\varphi(s) > 0$ for $s > 0$. Then for any sequence $\varphi_n \in \Phi$ we define $l\{\varphi_n\}$ to be a space of sequences $t = \{t_n\}$ such that

$$\|t\| = \sum \varphi_n(|t_n|) < \infty.$$

$(l\{\varphi_n\}, \|\cdot\|)$ is then an F-space. The class of such spaces will be denoted by $\mathscr{A}$. $\mathscr{A}$ may also be considered as the class of F-spaces with absolute basis; a basis $\{x_n\}$ of an F-space X is *absolute* (see [18]) if there is an F-norm $\|\cdot\|$ on X defining the topology and such that

$$\Big\|\sum t_n x_n\Big\| = \sum \|t_n x_n\|.$$

PROPOSITION 5.2 (Turpin [23], 0.3.11). *Every separable F-space is the quotient of a space in class $\mathscr{A}$.*

THEOREM 5.3. *There is a space $X \in \mathscr{A}$ which is complementably universal for $\mathscr{A}$.*

Proof. Choose in Φ a countable subset Φ_0 which is dense in the topology of uniform convergence of compact subsets of $\boldsymbol{R}_+$. For each rational $a \in \boldsymbol{R}_+$, $\varphi \in \Phi_0$ let

$$\varphi^{(a)}(t) = \min\left(\varphi(t), a\right), \quad t \in \boldsymbol{R}_+.$$

Let $\{\varphi^{(a)}\colon \varphi \in \Phi_0,\ a \in Q_+\}$ be arranged in a sequence $\{\psi_n\}$ and let $X = l\{\psi_n\}$.

If $l\{\varphi_n\} \in \mathscr{A}$, let $\gamma_n = \sup_{0\leqslant t<\infty} \varphi_n(t)$ (can be $+\infty$). Pick $a_n \in Q_+$ such that $0 < a_n < \gamma_n$ and (a) $a_n = 1$ if $\gamma_n > 1$, (b) $\gamma_n - a_n < 2^{-n}$ if $\gamma_n \leqslant 1$. Then choose distinct $\theta_n \in \Phi_0$ such that

$$|\varphi_n(t) - \theta_n(t)| \leqslant 2^{-n} \quad \text{if} \quad \varphi_n(t) \leqslant a_n.$$

(Note that $\{t\colon \varphi_n(t) \leqslant a_n\}$ is compact.) Then let $\psi_{m_n} = \theta_n^{a_n}$ (the m_n are distinct). If $\varphi_n(t) \leqslant a_n$ we have

$$|\psi_{m_n}(t) - \varphi_n(t)| \leqslant 2^{-n}.$$

If $a_n < 1$ and $a_n < \varphi_n(t)$, then $a_n < \varphi_n(t) \leqslant \gamma_n \leqslant 1$, while $a_n - 2^{-n} \leqslant \psi_{m_n}(t) \leqslant a_n$. Since $\gamma_n - a_n < 2^{-n}$, we have

$$|\psi_{m_n}(t) - \varphi_n(t)| \leqslant 2\cdot 2^{-n}, \quad \varphi_n(t) \leqslant 1.$$

Thus

$$\sum \varphi_n(t_n) < \infty \Rightarrow \sum \psi_{m_n}(t_n) < \infty.$$

Conversely, if $\sum \varphi_n(t_n) = \infty$ but $\varphi_n(t_n) \to 0$, then $|\varphi_n(t_n) - \psi_{m_n}(t_n)| < 2\cdot 2^{-n}$ eventually and hence $\sum \psi_{m_n}(t_n) = \infty$. Finally, if $\sum \varphi_n(t_n) = \infty$ and $\varphi_n(t_n) \not\to 0$ choose a_n, $0 \leqslant a_n \leqslant 1$, such that $\varphi_n(a_n t_n) \to 0$ and $\sum \varphi_n(a_n t_n) = \infty$. Then $\sum \psi_{m_n}(a_n t_n) = \infty$ and hence $\sum \psi_{m_n}(t_n) = \infty$. Thus $l\{\varphi_n\} = l\{\psi_{m_n}\}$ as a sequence space, and by the Closed Graph Theorem as an F-space. Clearly, $l\{\psi_{m_n}\}$ is isomorphic to a complemented subspace of $l(\psi_n) = X$.

[Remark. The above proof is related to Proposition 2 (b) of [26]; note however that in [26] it is assumed that $\lim_{x\to\infty}\varphi(x) = \infty$ for all φ.]

THEOREM 5.4. *There is a separable F-space with an absolute basis which is co-universal for all separable F-spaces.*

Proof. By 5.3 and 5.2.

Remark. Shapiro [17], Stiles [19] and Rolewicz [16], p. 65, have shown that l_p $(0<p<1)$ is co-universal for separable p-Banach spaces.

6. Nearly convex spaces. A topological vector space X is *nearly convex* (Klee [11]) if its dual X^* is point-separating. In this case the weak topology $\sigma(X, X^*)$ is Hausdorff on X. X is said to be *weakly polar* if it has base of weakly *closed* neighbourhoods of 0; in general, a vector topology τ on X is *ϱ-polar*, where ϱ is also a vector topology, if it has a base of ϱ-closed neighbourhoods of 0 (see [7]).

In this section, we shall show that there is no separable nearly convex F-space universal for all separable nearly convex F-spaces. To do this we define an invariant of a nearly convex F-space X, an ordinal number denoted by $\eta(X)$ (cf. a similar argument in [21]).

Suppose (X,τ) is a nearly convex F-space and let θ be any ordinal whose cardinality is strictly greater than the cardinality of the set of all topologies on X. We define a transfinite sequence $(\tau_\alpha\colon \alpha<\theta)$ of Hausdorff vector topologies on X, using transfinite induction. Since X is nearly convex, we let $\tau_0 = \sigma(X, X^*)$ and τ_0 is Hausdorff. If α is a non-limit ordinal, let τ_α be the topology whose base at zero is given by all $\tau_{\alpha-1}$ closed τ-neighbourhoods of 0. If α is a limit ordinal, let $\tau_\alpha = \sup(\tau_\beta\colon \beta<\alpha)$. It is clear that $\alpha<\beta$ implies that τ_β is finer than τ_α. From the choice of θ, there exists a least $\eta<\theta$ such that $\tau_\eta = \tau_{\eta+1}$. We define $\eta(X) = \eta$.

LEMMA 6.1. *Suppose (X,τ) is a nearly convex F-space and let $\eta = \eta(X)$. Then*

(i) $\tau_\eta = \tau$;

(ii) *for $\alpha<\eta$, which is either a countable or a non-limit ordinal, τ_α is metrizable;*

(iii) *if X a p-Banach space, then each τ_α is locally p-convex and, if α is a non-limit ordinal, is also locally bounded.*

Proof. (i) The identity map $i\colon (X,\tau_\eta)\to(X,\tau)$ is almost continuous, by the definition of η, and has continuous inverse. Hence by the Closed Graph Theorem ([9], p. 213), $\tau_\eta = \tau$.

(ii) and (iii) follow easily from the observation that if α is a non-limit ordinal, then τ_α has a base $\{\overline{U}_n\colon n\in N\}$ where the closures are taken in $\tau_{\alpha-1}$ and $\{U_n\colon n\in N\}$ is a base for τ.

From 6.1(i) we observe that X is weakly polar if and only if $\eta(X)\leqslant 1$. Note also that $\eta(X) = 0$ if and only if $X\cong\omega$.

LEMMA 6.2. *Let (X,τ) be a nearly convex F-space and let Y be a closed subspace of X. If (τ_α) denotes the transfinite sequence of topologies on X and (μ_α) is the corresponding sequence on Y, then $\tau_\alpha|Y\leqslant\mu_\alpha$. In particular, $\eta(Y)\leqslant\eta(X)$.*

Proof. Clearly, $\sigma(Y, Y^*)\geqslant\sigma(X, X^*)\,|\,Y$ since the restriction of every $\varphi\in X^*$ gives a continuous linear functional on Y. Thus $\tau_0|Y\geqslant\mu_0$; we prove the result by transfinite induction. If α is a non-limit ordinal, let V be a $\tau_\alpha|Y$ neighbourhood of 0 in Y; then there exists a $\tau_{\alpha-1}$-closed τ-neighbourhood W of 0, such that $W\cap Y\subset V$. Then $W\cap Y$ is also $\mu_{\alpha-1}$ closed and hence is a μ_α-neighbourhood. Thus $\tau_\alpha|Y\leqslant\mu_\alpha$. If α is a limit ordinal, it is easy to show $\tau_\alpha|Y\leqslant\mu_\alpha$. Thus $\tau_\alpha|Y\leqslant\mu_\alpha$ for all α and hence $\eta(Y)\leqslant\eta(X)$.

LEMMA 6.3. *Suppose (X,τ) is a nearly convex F-space. Then*

(i) *if X is locally bounded, $\eta(X)$ is a non-limit ordinal;*

(ii) *if X is separable, $\eta(X)$ is countable.*

Proof. (i) Suppose η is a limit ordinal; then $\tau_\eta = \sup\{\tau_\alpha\colon \alpha<\eta\}$. Let U be a bounded neighbourhood of 0 in (X,τ); then there exists an α less than η such that $U\in\tau_\alpha$. Thus $\tau_\alpha = \tau$.

(ii) Let ω_1 be the first uncountable ordinal. Suppose U is a τ_{ω_1}-closed τ-neighbourhood of 0, then

$$U = \bigcap_{V\in\mathscr{V}}(U+V+V)$$

where $\mathscr{V}$ is the collection of all open τ_{ω_1}-neighbourhoods of 0. Thus

$$U = \bigcap_{V\in\mathscr{V}}\overline{(U+V)},$$

where the closures are taken in τ. Since $(X-U,\tau)$ is a Lindelöf space there is a countable subset V_n of $\mathscr{V}$ such that

$$U = \bigcap_{n=1}^{\infty}\overline{(U+V_n)}.$$

Each V_n is a neighbourhood of 0 in some τ_{α_n}, where $\alpha_n<\omega_1$. Hence U is closed in τ_α, where $\alpha = \sup\alpha_n<\omega_1$, so that U is a $\tau_{\alpha+1}$-neighbourhood of 0. It follows that U is a τ_{ω_1}-neighbourhood and so $\tau_{\omega_1} = \tau_{\omega_1+1} = \tau$. Now since τ has a countable base (U_n), each U_n is a τ_{β_n}-neighbourhood where $\beta_n<\omega_1$; hence $\eta = \sup\beta_n<\omega_1$.

LEMMA 6.4. *Suppose (X,τ) is a separable nearly convex F-space and let $\eta(X) = \eta$. For each $\alpha<\eta$, let $(X_\alpha,\tilde{\tau}_\alpha)$ be the completion of (X,τ_α). Then each X_α is a separable nearly convex F-space and $\eta(X_\alpha) = \alpha$.*

Proof. It is trivial that each X_α is separable, and that $(X,\tau_\alpha)^* = X^*$. Suppose X^* does not separate the points of X_α; then there is a sequence

$x_n \in X$ such that $\varphi(x_n)\to 0$ for all $\varphi \in X^*$, but x_n is τ_α-Cauchy and not convergent. Let γ be the largest ordinal less than α such that $x_n \to O(\tau_\gamma)$; it is trivial that γ exists. Clearly, $x_n \not\to O(\tau_{\gamma+1})$, but $\{x_n\}$ is a $\tau_{\gamma+1}$-Cauchy sequence. If V is a τ_γ-closed $\tau_{\gamma+1}$-neighbourhood of 0, then there exists $n \in N$ such that for $l \geqslant m \geqslant n$, $x_m - x_l \in V$. Letting $l\to\infty$, we obtain $x_m \in V$ for $m \geqslant n$ and hence $x_n \to O(\tau_{\gamma+1})$ which is a contradiction.

Now let $\{\mu_\beta\colon \beta \leqslant \eta(X_\alpha)\}$ be the transfinite sequence of topologies on $(X_\alpha, \tilde{\tau}_\alpha)$. Clearly, $\mu_0|X = \tau_0$ and we prove by transfinite induction that $\mu_\beta|X = \tau_\beta$ for $\beta \leqslant \eta(X_\alpha)$. Suppose β is a non-limit ordinal and $\mu_{\beta-1}|X = \tau_{\beta-1}$. Then if V is a $\mu_{\beta-1}$-closed μ_β-neighbourhood of 0 in X_α, $V \cap X$ is a τ_α-neighbourhood which is $\tau_{\beta-1}$-closed and hence a τ_β-neighbourhood. Conversely, if W is a $\tau_{\beta-1}$-closed τ_β-neighbourhood of 0 in X, and let V be its closure in $(X_\alpha, \mu_{\beta-1})$. Then V contains the $\tilde{\tau}_\alpha$-closure of W and hence is a τ_α-neighbourhood of 0; also V is $\mu_{\beta-1}$-closed. Hence V is a μ_β-neighbourhood and $V \cap X = W$. Thus $\mu_\beta|X = \tau_\beta$. The induction step for limit ordinals is trivial. It now follows immediately that $\eta(X_\alpha) = \alpha$.

We are now going to show that for each countable ordinal α there is a separable F-space X_α such that $\eta(X_\alpha) = \alpha$. To do this, we require a result which has some interest in its own right, and we shall therefore prove it in rather more generality than is actually required.

THEOREM 6.5. *Suppose (X, τ) is a p-Banach space where $0 < p < 1$, and suppose ϱ is a metrizable vector topology on X which is strictly weaker than τ. Then there exists a τ-closed bounded absolutely p-convex subset K of X which does not absorb its ϱ-closure.*

Proof. Denote by $\|\cdot\|$ a p-norm on X defining τ. Since $\varrho < \tau$, there is a sequence x_n such that $\|x_n\| = 1$ but $x_n \to 0(\varrho)$. By [7], 3.4 or [8], 2.1, there is a subsequence (u_n) of (x_n) which is a regular (i.e. bounded away from zero) M-basic sequence. More precisely (see the proof of Theorem 3.2 of [8]), (u_n) is *strongly regular*, i.e., if (φ_n) denotes the sequence of biorthogonal functionals defined on $Y = \overline{\mathrm{lin}}(u_n)$, then $\lim\limits_{n\to\infty}\varphi_n(x) = 0$ for $x \in Y$. Hence by the Banach–Steinhaus theorem,

$$\sup_n \|\varphi_n\| = M < \infty$$

where

$$\|\varphi_n\| = \sup_{\|x\| \leqslant 1} |\varphi_n(x)|.$$

Now let K be the τ-closure of the absolutely p-convex hull of the set $\{u_1 + u_n\colon n \geqslant 2\}$. Then K is also bounded and u_1 belongs to the ϱ-closure of K; we show K does not absorb u_1. Suppose $\delta u_1 \in K$; then for any $\varepsilon > 0$ there exist $(t_j\colon 2 \leqslant j \leqslant n)$ such that $|t_2|^p + \ldots + |t_n|^p \leqslant 1$ and

$$\Big\| \delta u_1 - \sum_{j=2}^{n} t_j (u_1 + u_j) \Big\| \leqslant \varepsilon.$$

Hence

$$\Big| \varphi_k \Big(\delta u_1 - \sum_{j=2}^{n} t_j (u_1 + u_j) \Big) \Big| \leqslant M\varepsilon^{1/p}, \quad 1 \leqslant k \leqslant n,$$

i.e.

$$|t_j| \leqslant M\varepsilon^{1/p}, \quad 2 \leqslant j \leqslant n,$$

$$\Big| \delta - \sum_{j=2}^{n} t_j \Big| \leqslant M\varepsilon^{1/p}.$$

Now

$$\Big|\sum_{j=2}^{n} t_j\Big| \leqslant \sum_{j=2}^{n} |t_j| \leqslant (\max_{2\leqslant j \leqslant n} |t_j|^{1-p}) \sum_{j=2}^{n} |t_j|^p \leqslant M^{1-p}\,\varepsilon^{(1/p-1)} \leqslant M\varepsilon^{(1/p-1)}.$$

Thus

$$|\delta| \leqslant M(\varepsilon^{1/p} + \varepsilon^{(1/p-1)}).$$

As $\varepsilon > 0$ is arbitrary, $\delta = 0$.

In the space l_p, where $0 < p < 1$, any bounded closed absolutely q-convex set, where $p < q \leqslant 1$, is compact and therefore also weakly closed (see Stiles [20] and Shapiro [18]). Of course, in a Banach space any closed convex set is weakly closed.

COROLLARY 6.6. *Suppose X is a nearly convex p-Banach space, where $0 < p < 1$, and that every bounded closed absolutely p-convex subset of X is weakly closed. Then X is finite-dimensional.*

Proof. Let μ be the Mackey topology on X, i.e. the norm topology on X whose base consists of all convex neighbourhoods of 0. Then every bounded closed p-convex subset of X is μ-closed; hence, by 6.5, μ is the original topology and X is isomorphic to a Banach space. Now we may assume the norm homogeneous. If $\dim X = \infty$, there is a net u_α such that $\|u_\alpha\| \leqslant 1$, $u_\alpha \to u$ weakly where $u \neq 0$ but $\|u - u_\alpha\| \geqslant \delta > 0$ for all α. Let $\varphi \in X^*$ be such that $\varphi(u) = 1$; then for $\varepsilon > 0$ there exists β such that, if $\alpha > \beta$, $|\varphi(u_\alpha) - 1| \leqslant \varepsilon$. Since u belongs to the closed p-convex hull of $\{u_\alpha\colon \alpha > \beta\}$, there exist $\alpha_1, \ldots, \alpha_k > \beta$ and $t_1, \ldots, t_k$ such that $|t_1|^p + \ldots + |t_k|^p \leqslant 1$ and

$$\Big\| \sum_{i=1}^{k} t_i u_{\alpha_i} - u \Big\| \leqslant \varepsilon.$$

Then, for $\varepsilon < 1/(\|\varphi\| + 1)$,

$$\Big| \sum_{i=1}^{k} t_i \varphi(u_{\alpha_i}) \Big| \geqslant 1 - \varepsilon\,\|\varphi\|$$

and hence

$$\sum_{i=1}^{k} t_i \geqslant 1 - \varepsilon\,\|\varphi\| - \varepsilon \sum_{i=1}^{k} |t_i| \geqslant 1 - \varepsilon(\|\varphi\| + 1).$$

Hence there exists $1 \leqslant l \leqslant k$,

$$t_l \geqslant \big(1-\varepsilon(1+\|\varphi\|)\big)^{1/(1-p)},$$

and then

$$\|u-u_{a_l}\| \leqslant (1-t_l)+\Big\|\sum_{i\neq l} t_i u_{a_i}\Big\|+\varepsilon \leqslant (1-t_l)+\sum_{i\neq l}|t_i|+\varepsilon \leqslant 2(1-t_l)+\varepsilon$$
$$\leqslant 2\Big(1-\big(1-\varepsilon(1+\|\varphi\|)\big)^{1/(1-p)}\Big)+\varepsilon.$$

For small enough $\varepsilon > 0$, this contradicts $\|u-u_a\| \geqslant \delta$.

THEOREM 6.7. *Suppose $0 < p < 1$. Then for a countable non-limit ordinal $a \geqslant 1$, there exists a separable p-Banach space X for which $\eta(X) = a$. For each limit ordinal a, there exists a separable locally p-convex F-space for which $\eta(X) = a$.*

Proof. Note that $\eta(l_p) = 1$. Now suppose that (X, τ) is a separable p-Banach space such that $\eta(X) = \beta$; for the special case $\beta = 1$ we take $X = l_p$. We shall construct a separable p-Banach space Y for which $\eta(Y) > \beta$.

By Lemma 6.3, $\beta = \gamma+1$ for some ordinal $\gamma \geqslant 0$; now let $(\tau_a\colon 0 \leqslant a \leqslant \beta)$ be the transfinite sequence of topologies on X. Let $\varrho = \tau_\gamma$ if $\beta > 1$, or in the special case $\beta = 1$ take ϱ to be the l_1-norm topology on l_p; thus, in general, $\varrho \geqslant \tau_\gamma$ and $\varrho < \tau$. By Theorem 6.5, there is a τ-closed bounded absolutely p-convex subset K of X and a point u in the ϱ-closure of K which is not absorbed by K.

Let U be the unit ball of (X, τ), and for each $n \in \boldsymbol{N}$ let $\|\cdot\|_n$ be the p-norm on X whose unit ball is the closed absolutely p-convex hull of $\frac{1}{n}U \cup K$. Then let Y be the space of all sequence (x_n), where $x_n \in X$ such that

$$\|(x_n)\| = \sum_{n=1}^{\infty} \|x_n\|_n < \infty.$$

Let $(\mu_a\colon a \leqslant \eta(Y))$ be the transfinite sequence of topologies on Y. Let $X_k = \{(x_n)\colon x_n = 0,\ n \neq k\} \subset Y$; $X_k \cong X$ and so $\mu_\gamma|X_k \leqslant \varrho$, by Lemma 6.2 (we identify X and X_k). Let V be the unit ball of Y and let W be its μ_γ-closure. If $\eta(Y) \leqslant \beta$, then W is bounded in Y. However, u belongs to the ϱ-closure of K and hence $(u\delta_{kn}) \in W$; thus

$$\|(u\delta_{kn})\| = \|u\|_n \leqslant M < \infty, \qquad n \in \boldsymbol{N}.$$

Hence $M^{-1/p}u \in K+\frac{2}{n}U$ and since K is closed, $M^{-1/p}u \in K$, contradicting the choice of u and K. Thus $\eta(Y) > \beta$.

Thus if B is the set of countable ordinals a such that there exists a separable p-Banach space X with $\eta(X) = a$, then $\sup B$ is a limit ordinal. If $\sup B < \omega_1$, there exists a sequence X_n of separable p-Banach spaces for which $\sup \eta(X_n) = \sup B$. Let $Y = l_1(X_n)$ be the space of sequences (x_n) such that $\|(x_n)\| = \sum \|x_n\| < \infty$. Then Y is a p-Banach space and contains a copy of each X_n; hence $\eta(Y) \geqslant \sup B$. As $\eta(Y)$ is not a limit ordinal, $\eta(Y) > \sup B$. Hence $\sup B = \omega_1$, and the result follows by Lemmas 6.1 and 6.4.

THEOREM 6.8. *For $0 < p < 1$, there is no separable nearly convex F-space X which is universal for all separable nearly convex p-Banach spaces.*

Proof. By Theorem 6.7 and Lemma 6.2, we would have $\eta(X) \geqslant \omega_1$, contradicting Lemma 6.3.

COROLLARY 6.9. *For $0 < p < 1$, there is no separable F-space which is complementably universal for all nearly convex p-Banach spaces.*

Proof. If X is such a space, let $N = \{x\colon x^*(x) = 0,\ x^* \in X^*\}$. Suppose Z is a nearly convex complemented subspace of X, then $Z \cap N = \{0\}$, since any continuous linear functional on Z can be extended to X. If $P\colon X \to Z$ is a projection of X onto Z, then for $x \in N$, $x^* \in X^*$, $x^*(Px) = (x^* \circ P)(x) = 0$ so that $P(N) \subset N$. Thus $P(N) = \{0\}$, and it follows that Z is isomorphic to a subspace of X/N which is nearly convex and universal for all nearly convex p-Banach spaces, contradicting 6.8.

7. Approximation properties. An F-space X is said to have the *Approximation Property* (AP) if the identity is in the closure of the finite-dimensional operators on X in the topology of uniform convergence on compacta. If X is separable, X has the *Bounded Approximation Property* (BAP) if there is a sequence of finite-dimensional operators converging pointwise to the identity. These approximation properties have been studied in detail only in the setting of locally convex spaces. However, it is also possible to study them in nearly convex spaces.

It follows from the Banach–Steinhaus Theorem that (BAP) implies (AP) for an F-space. Clearly, an F-space with a basis has (BAP).

An F-space X has a Schauder decomposition into subspaces $(X_n\colon n \in \boldsymbol{N})$ if there exist a sequence $Q_n\colon X \to X_n$ of continuous projections such that $Q_nQ_m = 0$ for $n \neq m$ and $x = \sum_{n=1}^{\infty} Q_n x$ for $x \in X$. An F-space with a Schauder decomposition into finite-dimensional spaces has (BAP).

The following converse results to the above remarks are proved by straightforward imitation of the methods of Pełczyński and Wojtaszczyk [15] and Pełczyński [14]; we omit the proofs.

THEOREM 7.1. *Let X be an F-space with* (BAP). *Then X is isomorphic to a complemented subspace of an F-space with a finite-dimensional Schauder* *decomposition. If X is a p-Banach space, then X is isomorphic to a complemented subspace of a p-Banach space with a basis.*

COROLLARY 7.2. *The space B_p of Theorem 4.1 is complementably universal for p-Banach spaces with* (BAP).

Our main result in this section is that B_p is universal for all weakly polar p-Banach spaces. This is based on the following result.

THEOREM 7.3. *Suppose*

(a) *(X, ϱ) is a topological vector space.*

(b) *Q is a collection of F-semi-norms on X defining the topology ϱ, such that $q, r \in Q \Rightarrow q+r \in Q$.*

(c) *For each $q \in Q$, X_q is the completion of the Hausdorff quotient of (X, q).*

(d) *Y is a subspace of X.*

(e) *τ is a vector topology on Y such that (Y, τ) is a separable F-space, $\tau \geqslant \varrho|Y$ and τ has a base of ϱ-closed neighbourhoods of 0.*

(f) *Y_0 is a dense subspace of (Y, τ) which is a βF^*-space in the subspace topology.*

Then there exists an F-space W and a Schauder decomposition (W_n) of W such that each W_n is isomorphic to the quotient of a space X_q by a finite-dimensional subspace and (Y, τ) is isomorphic to a subspace of W.

Proof. Let π be an F-norm defining τ; then for $q \in Q$ define q^* by

$$q^*(x) = \inf\{\pi(y)+q(x-y):\ y \in Y\}.$$

Then $q^* \leqslant q$, but if $q^*(x_n) \to 0$, then $x_n = y_n + (x_n - y_n)$ where $\pi(y_n) \to 0$ and $q(x_n - y_n) \to 0$. Hence $q(y_n) \to 0$ and so q^* and q are equivalent.

Next define

$$\pi^*(y) = \sup(q^*(y):\ q \in Q), \qquad y \in Y.$$

Then $\pi^* \leqslant \pi$, while if $\pi^*(y) < \delta$, $y \in \overline{\{y: \pi(y) < \delta\}}$ (the closure in ϱ). By assumption (e), π^* and π are equivalent on Y.

By (f), there exists δ, and an increasing sequence G_n of finite-dimensional subspaces of Y such that $\bigcup G_n$ is dense in Y and $\{g: \pi^*(g) \leqslant \delta\} \cap G_n$ is compact for each n. The set $\{q^*: q \in Q\}$ is directed upwards by (b) and hence by Dini's theorem we may find an increasing sequence $q_n \in Q$ such that

$$q_n^*(g) \geqslant \pi^*(g) - 2^{-n} \qquad \text{for} \quad g \in G_n,\ \pi^*(g) \leqslant \delta.$$

Now let $l_1(G_n)$ be the F-space of all sequences $g = (g_n)$, $g_n \in G_n$, such that $\sum \pi^*(g_n) < \infty$ under the F-norm $\|g\| = \sum \pi^*(g_n)$. Define $L: l_1(G_n) \to Y$ by $L(g) = \sum_{n=1}^{\infty} g_n$. Then $\overline{L\{g: \sum \pi^*(g_n) < \varepsilon\}}$ contains $\{y: \pi^*(y) < \varepsilon\}$ for all $\varepsilon > 0$ and so L is almost open. Hence L is an open map and Y is isomorphic to $l_1(G_n)/N$, where $N = \{g \in l_1(G_n):\ \sum g_n = 0\}$.

Now consider the F-space $l_1(X_{q_n}, \tilde{q}_n^*) = Z$ of all sequences $x = (x_n) \in X_{q_n}$, where

$$\|x\| = \sum_{n=1}^{\infty} \tilde{q}_n^*(x_n) < \infty.$$

($\tilde{q}_n^*$ denotes the F-norm induced on X_{q_n} by q_n^*, which is equivalent to q_n.) Define $T: l_1(G_n) \to Z$ by $T(g) = (R_n g_n)$, where $R_n: Y \to X_{q_n}$ is the quotient of the inclusion map $Y \hookrightarrow X$. Since $\tilde{q}_n^*(R_n g_n) = q_n^*(g_n) \leqslant \pi^*(g_n)$, T is well defined and $\|Tg\| \leqslant \|g\|$ for $g \in l_1(G_n)$.

Conversely, if $\|Tg\| < \delta/2$, then $q_n^*(g_n) < \delta/2$ and hence $\pi^*(g_n) \leqslant q_n^*(g_n) + 2^{-n}$. It follows that if $\|Tg^{(n)}\| \to 0$, then

$$\limsup_{n\to\infty} \|g^{(n)}\| \leqslant \limsup_{n\to\infty} \sum_{i=1}^{k} \pi^*(g_i^{(n)}) + \limsup_{n\to\infty} \sum_{i=k+1}^{\infty} \pi^*(g_i^{(n)}) \leqslant 2^{-k}$$

since $\lim \pi^*(g_i^{(n)}) = 0$ for each i. As k is arbitrary we conclude that T is an embedding. In particular $T(N)$ is closed and Y is isomorphic to a subspace of $Z/T(N) = W$. It remains to show that W has the required Schauder decomposition.

As $q_n^* \leqslant q_{n+1}^*$ we may induce a natural map $J_n: X_{q_{n+1}} \to X_{q_n}$ such that $J_n R_{n+1} = R_n$. Define $S_n: Z \to Z$ by

$$(S_n(x))_i = \begin{cases} x_i, & i < n, \\ x_n + \sum_{k>n} J_n \ldots J_{k-1} x_k, & i = n, \\ 0, & i > n. \end{cases}$$

Since $\tilde{q}_n^*(J_n x) \leqslant \tilde{q}_{n+1}^*(x)$, $S_n x$ is well defined and $S_n x \to x$ for all $x \in Z$. Furthermore, $S_m S_n = S_{\min(m,n)}$.

If $g \in N$, then

$$(S_n T(g))_i = \begin{cases} R_i g_i, & i < n, \\ R_n g_n + \sum_{k>n} J_n \ldots J_{k-1} R_k g_k = R_n\Big(\sum_{k\geqslant n} g_k\Big), & i = n, \\ 0, & i > n. \end{cases}$$

Hence

$$S_n T(g) = T\Big(g_1, g_2, \ldots, g_{n-1}, \sum_{k\geqslant n} g_k, 0, \ldots\Big) \in T(N).$$

Thus S_n factors to $\tilde{S}_n$: $W \to W$; $\tilde{S}_n \tilde{S}_m = \tilde{S}_{\min(m,n)}$ and $\tilde{S}_n w \to w$ for $w \in W$. Thus W possesses a Schauder decomposition into subspaces $W_1 = \tilde{S}_1(W)$ and $W_n = (\tilde{S}_n - \tilde{S}_{n-1})(W)$ $(n \geqslant 2)$. Now if $Z_1 = S_1(Z)$ and $Z_n = (S_n - S_{n-1})(Z)$, then since each Z_n is the range of a projection leaving N invariant, it is easy to check that $W_n \cong Z_n/N \cap Z_n$. Now

$$Z_1 = \{(x, 0, \ldots), \ x \in X_{q_1}\},$$
$$Z_n = \{(0, \ldots, -J_{n-1}x, x, 0, \ldots), \ x \in X_{q_n}\},$$

so that $Z_n \cong X_{q_n}$. Clearly, $\dim(N \cap Z_n) \leqslant \dim G_{n-1} + \dim G_n < \infty$ so that W_n is isomorphic to a finite-dimensional quotient of X_{q_n}.

Remark. If in 7.3 Q is a collection of p-semi-norms and (Y, τ) is a p-Banach space, then W is a p-Banach space as constructed above.

THEOREM 7.4. (i) *A separable F-space X is weakly polar if and only if it is isomorphic to a closed subspace of an F-space with a finite-dimensional Schauder decomposition.*

(ii) *A separable p-Banach space X is weakly polar if and only if it is isomorphic to a closed subspace of a p-Banach space with a basis.*

(iii) *B_p is universal for all weakly polar p-Banach spaces.*

Proof. First observe that if an F-space X has a finite-dimensional Schauder decomposition, there is a sequence $P_n: X \to X$ of finite-dimensional projections such that $P_n x \to x$ for $x \in X$. $\{P_n\}$ is equicontinuous by the Banach–Steinhaus Theorem. Suppose V is a closed neighbourhood of 0 in X; then $\bigcap_{n=1}^{\infty} P_n^{-1}(V) \subset V$ is also a neighbourhood and is weakly closed. For if $x_a \to x$, weakly, and $x_a \in P_n^{-1}(V)$, since P_n is weakly continuous, $P_n x_a \to P_n x$, weakly. Since $\dim P_n(X) < \infty$, $P_n x_a \to P_n x$ and so $x \in P_n^{-1}(V)$. Thus X is weakly polar and so must be any subspace of X.

Conversely for (i), note by Lemma 4.2 that it is sufficient to prove the result when X has a dense βF^*-subspace (clearly, $X \oplus N$ is weakly polar if X is). Then in Theorem 7.3 take $Y = X$ and the weak topology for ϱ with Q the family of all weakly continuous semi-norms on X. The result follows from 7.3.

For (ii), take Q to be the family of all weakly continuous p-semi-norms on X, and use the same argument, but then apply Theorem 7.1. Then (iii) follows from Theorem 4.1.

THEOREM 7.5. (i) *A separable F-space X is nearly convex if and only if it is isomorphic to a closed subspace of a separable F-space with* (AP).

(ii) *A separable p-Banach space X is nearly convex if and only if it is isomorphic to a closed subspace of a separable p-Banach space with* (AP).

Proof. (i) If $\eta(X) = 1$, this follows from Theorem 7.4. We proceed by transfinite induction. Suppose the theorem is proved for $\eta(X) < \alpha$. Then

(a) If $\eta(Y) = \alpha$, and α is a limit ordinal, consider the spaces $(Y_\beta: \beta < \alpha)$ of Lemma 6.4. If (β_n) is a sequence such that $\sup \beta_n = \alpha$, then Y_{β_n} is isomorphic to a subspace of some separable F-space X_n with (AP). Then Y is isomorphic to a subspace of the Cartesian product $\prod X$ with (AP).

(b) Suppose $\eta(Y) = \alpha$ and α is a non-limit ordinal. We must first use Lemma 4.2 to suppose that Y has a dense βF^*-subspace. To do this we clearly require that if N is a separable Banach space $\eta(Y \oplus N) = \eta(Y)$. This is proved simply by showing that $\{\mu_a\}$ and $\{\tau_a\}$ are the transfinite sequences of topologies on $Y \oplus N$ and Y, respectively, then $\alpha \geqslant 1$, μ_a is the direct sum topology of τ_a and the norm topology on N.

Suppose then that Y has a dense βF^*-subspace, and consider $Y_{\alpha-1}$. As $\eta(Y_{\alpha-1}) < \alpha$, $Y_{\alpha-1}$ can be embedded in a separable F-space X with (AP). We take for Q on X the collection $\{nq: n \in N\}$, where q is an F-norm defining the topology on X. Now apply Theorem 7.3. Y can be embedded in a space W with a Schauder decomposition (W_n), where each W_n is isomorphic to a quotient of X by a finite-dimensional subspace. As X is nearly convex (since X has (AP)) if $F \subset X$ is finite dimensional, $X \cong F \oplus G$ for some G. Thus each W_n is isomorphic to a complemented subspace of X and so has (AP). If $Q_n: W \to W$ are the natural projections onto W_n, then each Q_n is the limit of finite-dimensional operators for the topology of compact convergence. Hence so is $I = \sum_{n=1}^{\infty} Q_n$ on W and W has (AP).

(ii) The proof is virtually the same as that of (i), except in the case when $\eta(Y) = \alpha$, where $\alpha - 1$ is a limit ordinal. Then $\alpha - 1 = \sup \beta_n$, where each β_n is a non-limit ordinal and hence Y_{β_n} is a p-Banach space. Then there exists Z_n, p-Banach spaces with (AP), such that each Y_{β_n} is isomorphic to a subspace of Z_n. Thus $Y_{\alpha-1}$ is isomorphic to a subspace of $X = \prod_n Z_n$, which is locally p-convex. Take for Q on X the family of all finite sums of p-semi-norms of the form

$$q_k((z_n)) = \|z_k\|.$$

Then each X_q is isomorphic to a direct sum $Z_{n_1} \oplus \ldots \oplus Z_{n_k}$ and has (AP). Then apply Theorem 7.3 which yields the result; the space W is a p-Banach space.

COROLLARY 7.6. *If $0 < p < 1$, there is no separable universal F-space with* (AP) *for all separable p-Banach spaces with* (AP).

8. Remarks. The author has been unable to resolve some obvious questions which arise in the course of the paper.

PROBLEM 1 (see Theorem 3.4). Is there a universal Markushevich basis?

PROBLEM 2 (see Theorem 3.1). Does the analogue of Theorem 3.1 hold for Fréchet spaces? Or F-spaces?

PROBLEM 3 (see Theorem 4.4). Does Theorem 4.4 hold without the restriction on short lines?

PROBLEM 4 (see Theorem 7.4). Is every separable weakly polar F-space a subspace of an F-space with a basis?

It is not true, however, that every F-space with (BAP) is isomorphic to a complemented subspace of an F-space with a basis. We conclude by giving an example to this effect.

EXAMPLE. If V is a subset of a vector space we write V^l for $V+\ldots+V$ (l-times). Suppose for each $n\geqslant 3$ we may find in $\boldsymbol{R}^n$ a closed balanced neighbourhood V_n of 0 such that

(1) $\operatorname{co} V_n = \boldsymbol{R}^n$,

(2) $\operatorname{lin}\bigcap_{\varepsilon>0}\varepsilon V_n^{n-1}\neq \boldsymbol{R}^n$.

Then we may define an F-norm $\|\cdot\|_n$ on $\boldsymbol{R}^n$ such that $V_n=\{x\colon \|x\|_n\leqslant 1/n\}$ and $\overline{V}_n^k=\{x\colon \|x\|_n\leqslant k/n\}$, for $2\leqslant k\leqslant n$ (cf. [16]). Consider now the space $Y=l_1(\boldsymbol{R}^n, \|\cdot\|_n)$ of all sequences $x=(x_n)$, where $x_n\in\boldsymbol{R}^n$ such that $\|x\|=\sum\|x_n\|_n<\infty$. Suppose X is an F-space with a basis (u_n) with dual functionals φ_n, such that $X\supset Y$ and there is a projection P of X onto Y. Denote by Q_n the natural projection of Y onto its subspace $G_n\cong(\boldsymbol{R}^n, \|\cdot\|_n)$ (i.e. $Q_n(x_1,\ldots,x_n,\ldots)=(0,\ldots,x_n,0,\ldots)$). Then the operators $\varphi_k\otimes Q_nPu_k$ ($n\geqslant 3$, $k\geqslant 1$) are equicontinuous on X and so there exists $\varepsilon>0$ such that if $x\in Y$ and $\|x\|\leqslant\varepsilon$, then $\|\varphi_k(x)Q_nPu_k\|\leqslant\frac{1}{2}$.

For $x\in G_n$,

$$x=\sum_{k=1}^{\infty}\varphi_k(x)Q_nPu_k$$

so that $\{Q_nPu_k\colon k\in\boldsymbol{N}\ \varphi_k|G_n\neq 0\}$ spans G_n. If $\varphi_k|G_n\neq 0$, then $\varphi_k(V_n)=\boldsymbol{R}$ and hence if $n\geqslant 1/\varepsilon$, $\{tQ_nPu_k;\ t\in\boldsymbol{R}\}\subset V_n^l$, where l is any integer such that $l>\frac{1}{2}n$. Hence $Q_nPu_k\in\bigcap_{\varepsilon>0}\varepsilon V_n^{n-1}$ and fails to span G_n, a contradiction.

It remains to construct the sets V_n. For each n choose θ so that $e^{\theta}\geqslant 4n$, and then let $u_k\in\boldsymbol{R}^n$ be the sequence of points:

$$u_k(m)=\exp\big((n-m+1)k\theta\big),\qquad 1\leqslant m\leqslant n.$$

Let $W=\bigcup_{k=1}^{\infty}\{tu_k\colon |t|\leqslant 1\}$. We observe first that $\operatorname{co}W=\boldsymbol{R}^n$ since if $(a_1\ldots a_n)\neq 0$.

$$\lim_{k\to\infty}\Big|\sum_{j=1}^{n}a_j\exp\big((n-j+1)k\theta\big)\Big|=\infty.$$

Now let $L(k_1,\ldots,k_l)$ be the linear span of $u_{k_1},\ldots,u_{k_l}$, where $k_1<k_2<\ldots<k_l$ ($l<n$). Define $v_i^{(1)}=u_{k_i}$, $1\leqslant i\leqslant l$, and then, for $2\leqslant q\leqslant l-1$,

$$v_i^{(q+1)}=\begin{cases} v_i^{q}-\dfrac{v_i^{(q)}(q)}{v_{l-q+1}^{(q)}(q)}v_{l-q+1}^{(q)}, & 1\leqslant i\leqslant l-q,\\[2mm] v_i^{(q)}, & l-q+1\leqslant i\leqslant l.\end{cases}$$

(It will follow from results below that $v_{l-q+1}^{(q)}(q)>0$.) Clearly, if $j\leqslant l-q+1$ and $m<q$, $v_j^{(q)}(m)=0$. By induction we have

$$(1-2qe^{-\theta})\exp\big((n-m+1)k_j\theta\big)\leqslant v_j^{(q)}(m)\leqslant\exp\big((n-m+1)k_j\theta\big)$$

for $j\leqslant l-q+1$ and $m\geqslant q$.

In fact, assume the assertion true for q; thus $j\leqslant l-q$ and $m\geqslant q+1$,

$$\begin{aligned} v_j^{(q+1)}(m)&=v_j^{(q)}(m)-\frac{v_j^{(q)}(q)}{v_{l-q+1}^{(q)}(q)}v_{l-q+1}^{(q)}(m)\\ &\geqslant(1-2qe^{-\theta})\exp\{(n-m+1)k_j\theta\}-\\ &\quad-(1-2qe^{-\theta})^{-1}\exp\{(n-m+1)k_j\theta+(m-q)(k_j-k_{l-q+1})\theta\}\\ &\geqslant\exp\{(n-m+1)k_j\theta\}(1-2qe^{-\theta}-2e^{-\theta}).\end{aligned}$$

It follows that if $j\leqslant l-q+1$, $m\geqslant q$,

$$v_j^{(q)}(m)\geqslant\tfrac{1}{2}\exp\{(n-m+1)k_j\theta\}.$$

In particular, $v_{l-q+1}^{(q)}(m)=0$ if $m<q$, and

$$\frac{v_{l-q+1}^{(q)}(m)}{v_{l-q+1}^{(q)}(q)}\leqslant 2e^{-\theta k_{l-q+1}}\leqslant 2e^{-k_1\theta},\qquad m>q.$$

If

$$w_q=\frac{1}{v_{l-q+1}^{(q)}(q)}v_{l-q+1}^{(q)}$$

and e_q is the qth basis element, then

$$\|w_q-e_q\|=\max_m|w_q(m)-e_q(m)|\leqslant 2e^{-k_1\theta}.$$

Now suppose $x\in L(k_1\ldots k_l)$; then $(w_q\colon 1\leqslant q\leqslant l)$ is a basis of $L(k_1\ldots k_l)$ so

$$x=\sum_{q=1}^{l}c_qw_q.$$

Let

$$y=\sum_{q=1}^{l}c_qe_q.$$

Then

$$\|x-y\|\leqslant 2ne^{-k_1\theta}\max|c_q|=2ne^{-k_1\theta}\|y\|.$$

Thus

$$\|y\|\leqslant(1+2ne^{-k_1\theta})^{-1}\|x\|$$

and so

$$\|x-y\| \leqslant \frac{2ne^{-k_1\theta}}{1-2ne^{-k_1\theta}} \|x\|,$$

i.e.

$$d(x, E_l) \leqslant \frac{2ne^{-k_1\theta}}{1-2ne^{-k_1\theta}} \|x\|$$

where $E_l = \operatorname{lin}(e_1, \ldots, e_l)$.

Now let $V_n = W+U$, where U is a compact convex neighbourhood of 0. Suppose $x \in \bigcap_{\varepsilon>0} \varepsilon V_n^l \setminus \bigcap_{\varepsilon>0} \varepsilon V_n^{l-1}$; then it is easy to show that $x \in \bigcap_{\varepsilon>0} \varepsilon \overline{W^l} \setminus \bigcap_{\varepsilon>0} \varepsilon \overline{W^{l-1}}$, and hence that there is a sequence $x_m \to x$ where

$$x_m = \frac{1}{m}(t_1 u_{k(1,m)} + \ldots + t_l u_{k(l,m)})$$

and $|t_i| \leqslant 1$, $1 \leqslant i \leqslant m$ and $k(1, m) < k(2, m) < \ldots < k(l, m)$. It is easy to show that $k(1, m) \to \infty$. Thus $d(x_m, E_l) \to 0$ and $x \in E_l$. In particular,

$$\bigcap_{\varepsilon>0} \varepsilon V_n^{n-1} \subset E_{n-1} \neq \boldsymbol{R}^n.$$

This completes the construction of the example, as $\operatorname{co} V_n \supset \operatorname{co} W = \boldsymbol{R}^n$.

Addendum. The author has learned that Theorem 3.2 has been independently obtained by G. Schechtman ([28]).

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DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY COLLEGE OF SWANSEA
SINGLETON PARK, SWANSEA, U. K.

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