[7] П. И. Лизоркин, *Обобщенное лиувиллевское дифференцирование и функциональные пространства $L_p^r(E_k)$. Теоремы вложения*, Матем. сб. 60 (1963), стр. 325–353.

[8] — *Обобщенное лиувиллевское дифференцирование и метод мультипликаторов в теории вложений классов дифференцируемых функций*, Тр. Матем. инст. АН СССР 105 (1969), стр. 89–162.

[9] — *Описание пространств $L_p^r(E^n)$ в терминах разностных сингулярных интегралов*, Матем. сб. 81 (1970), стр. 79–91.

[10] — *Операторы, связанные с дробным дифференцированием и классы дифференцируемых функций*, Тр. Матем. инст. АН СССР 117 (1972), стр. 212–243.

[11] С. М. Никольский, *Приближение функций многих переменных и теоремы вложения*, „Наука", М. 1969.

[12] С. Г. Самко, *Об операторах типа потенциала*, ДАН СССР 196 (1971), стр. 299–301.

[13] — *Об интегральных уравнениях первого рода с ядром типа птенциала*, Изв. вузов. Математика 4 (1971), стр. 78–86.

[14] — *О пространстве $I^\alpha(L_p)$ дробных интегралов и об операторах типа потенцала* Изв. АН Арм. ССР 8 (1973), стр. 359–383.

[15] И. Стейн, *Сингулярные интегралы и дифференциальные свойства функций* „Мир", М. 1973.

[16] N. Aronszain, K. Smith, *Theory of Bessel potentials I*, Ann. de l'Institute Fourier 11 (1961), стр. 385–475.

[17] A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions. Partial differential equations*, Providence R. I., Amer. Math. Soc. 1961, стр. 33–49.

[18] M. Fisher, *Some generalizations of the hypersingular integral operators*, Studia Math. 47 (1973), стр. 95–121.

[19] T. M. Flett, *Temperatures, Bessel potentials and Lipschitz spaces*, Proc. London Math. Soc. 22 (1971), стр. 385–471.

[20] P. Heywood, *On the inversion of fractional integrals*, J. London Math. Soc. 3 (1971), стр. 531–538.

[21] E. M. Stein, *The characteristic of functions arizing as potentials, I*, Bull. Amer. Math. Soc. 67 (1961), стр. 102–104.

[22] W. Trebels, *Generalized Lipschitz conditions and Riesz derivatives on the space of Bessel potentials L_α^p, $0 < \alpha \leqslant 2$, I. Conditions for elements of L^p and their Riesz transforms, $0 < \alpha \leqslant 2$*, Appl. Anal. 1 (1971), стр. 75–99.

[23] — *Imbedding theorems for spaces of hypersingular integrals and Bessel potentials*, J. Approx. Theory 6 (1972), стр. 202–214.

[24] R. L. Wheeden, *On hypersingular integrals and Lebesgue spaces of differentiable functions*, Trans. Amer. Math. Soc. 134 (1968), стр. 421–436.

[25] — *On hypersingular integrals and Lebesgue spaces of differtiable functions, II*, ibid. 139 (1969), стр. 37–53.

[26] — *On hypersingular integrals and certain spaces of locally differtiable functions*, ibid. 146 (1969), стр. 211–230.

[27] — *A note on a generalized hypersingular integral*, Studia Math. 44 (1972), стр. 17–26.

[28] K. Yoshinaga, *On Liouville's differentiation*, Bull. Kyushi Inst. Technol. 11 (1964), стр. 1–17.

Received May 19, 1975 (1018)

STUDIA MATHEMATICA, T. LXI. (1977)

On the random ergodic theorem

by

TAKESHI YOSHIMOTO (Kawagoe, Japan)

Abstract. A limiting behavior of random operator averages which are subject to the accidential phenomena varying in space and time is discussed in connection with the problem of dependence on the random parameters.

In consequence, a random ergodic theorem is obtained as an extension of the Cairoli's theorem to a case of many random parameters, including the Gładysz result with the exception of weighted functions.

1. Introduction. We consider a σ-finite measure space $(Y, \mathscr{F}, \mu)$ and the product spaces:

$$X^* = X_1 \times X_2 \times \ldots, \quad \mathscr{B}^* = \mathscr{B}_1 \times \mathscr{B}_2 \times \ldots, \quad m^* = m_1 \times m_2 \times \ldots,$$
$$X_r^* = X_1 \times \ldots \times X_r, \quad \mathscr{B}_r^* = \mathscr{B}_1 \times \ldots \times \mathscr{B}_r, \quad m_r^* = m_1 \times \ldots \times m_r$$

for an arbitrarily fixed integer $r \geqslant 1$, where $(X, \mathscr{B}, m)$ is a probability space and $X_i = X$, $\mathscr{B}_i = \mathscr{B}$, $m_i = m$, $i = 1, 2, \ldots$

Suppose now that to each $x \in X_r^*$ there corresponds a positive contraction operator U_x on the usual Banach space $L_1(\mu)$, and let there be given a family $\{U_x\colon x \in X_r^*\}$ of such positive contraction operators on $L_1(\mu)$.

The family $\{U_x\colon x \in X_r^*\}$ is said to be *strongly $\mathscr{B}_r^*$-measurable* if U_x is strongly $\mathscr{B}_r^*$-measurable as an $L_1(\mu)$-operator-valued function defined on X_r^*, that is to say, for any $g \in L_1(\mu)$ there are countably $L_1(\mu)$-valued functions $h_n(x, \cdot)$ defined on X_r^* such that

$$\lim_{n\to\infty} \|h_n(x, \cdot) - (U_x g)(\cdot)\|_{L_1(\mu)} = 0 \quad m_r^*\text{-a.e.}$$

From now on, if we wish to regard $f(y, x^*)$ as a function of y defined on Y for an x^* arbitrarily fixed in X^*, we shall write $f_{(x^*)}(y)$ for $f(y, x^*)$.

In this note, on the hypothesis that the family $\{U_x\colon x \in X_r^*\}$ is strongly $\mathscr{B}_r^*$-measurable and there is a $g > 0$ $(g \in L_1(\mu))$ invariant under $\{U_x\colon x \in X_r^*\}$ for m_r^*-almost all x, we shall demonstrate that for any $f \in L_1(\mu \times m^*)$ the limit function

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U_{(x_1, \ldots, x_r)} \ldots U_{(x_k, \ldots, x_{k+r-1})} f_{(x_{k+1}, x_{k+2}, \ldots)}(y) = \tilde{f}_{(x_1, x_2, \ldots)}(y)$$
$$(\tilde{f} \in L_1(\mu \times m^*))$$

depends essentially only on the variables $y, x_1, \ldots, x_{r-1}$.

Now the limit function $\tilde{f}_{(x^*)}(y) = \tilde{f}(y, x^*)$ depends ordinarily on x^* and is, in general, not equal to a constant for almost all y. Therefore, it is an interesting problem to investigate the conditions under which the limit function $\tilde{f}(y, x^*)$ is essentially independent of the variable x^*.

Such a problem was first studied by C. Ryll-Nardzewski [3] in which the random motion is subject to a one-sided Bernoulli shift, and his result was extended by S. Gładysz [2] to a case of many random parameters. Later R. Cairoli [1] proved an operator theoretical generalization of the Ryll-Nardzewski theorem, but the obtained result does not cover the extension due to S. Gładysz. Our result is a further generalization of the Cairoli's to a case of many random parameters, including the Gładysz one without weighted functions.

The random motion we shall consider as parameters varying spatially is, of course, subject to a one-sided Bernoulli shift. It is indeed worthwhile to emphasize the importance of this fact, because the results obtained up to now along with ours have an essential aspect of the properties of Bernoulli shifts.

I wish to express my gratitude to the referee for his kind advice.

2. The main result. We consider in the sequel a strongly $\mathscr{B}_r^*$-measurable family $\{U_x\colon x \in X_r^*\}$ of positive contraction operators on $L_1(\mu)$.

For notational convenience, we denote

$$[x^*]_r = (x_1, \ldots, x_r) \in X_r^*$$

for $x^* = (x_1, \ldots, x_r, \ldots) \in X^*$ and by φ the one-sided Bernoulli shift on X^*.

Now the random ratio theorem ([4], Theorem 11) applied to $\{U_x\colon x \in X_r^*\}$ shows that if $f \in L_1(\mu \times m^*)$ and $g(y, x^*) \equiv h(y)$, $h \in L_1(\mu)$, $h \geqslant 0$, then for almost all x^*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} f_{(\varphi^k x^*)}(y)}{\sum_{k=1}^{n} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} g_{(\varphi^k x^*)}(y)}$$

exists and is finite almost everywhere on the set $E_{(x^*)}(g)$, where

$$E_{(x^*)}(g) = \Big\{y\colon \sum_{k=1}^{\infty} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} g_{(\varphi^k x^*)}(y) > 0\Big\}.$$

Thus it results from this that under the additional assumption that there is an $h > 0$, $h \in L_1(\mu)$ such that for almost all x^*

$$U_{[x^*]_r} h(y) = h(y) \qquad \mu\text{-a.e.},$$

we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} f_{(\varphi^k x^*)}(y)$$

$$= g(y, x^*) \lim_{n\to\infty} \frac{\sum_{k=1}^{n} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} f_{(\varphi^k x^*)}(y)}{\sum_{k=1}^{n} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} g_{(\varphi^k x^*)}(y)}$$

$$= \tilde{f}_{(x^*)}(y) \qquad \mu\text{-a.e.} \qquad (\tilde{f} \in L_1(\mu \times m^*))$$

for almost all $x^* \in X^*$.

As a matter of fact, this limit function $\tilde{f}_{(x^*)}(y)$ is essentially independent of the variables $x_r, x_{r+1}, \ldots$, and it is the aim of the present note to prove this fact.

The main result we shall present is now stated as follows.

THEOREM 1. *Suppose there is a $g > 0$, $g \in L_1(\mu)$ such that for almost all x^*, $U_{[x^*]_r} g(y) = g(y)$ μ-a.e. Then for every $f \in L_1(\mu \times m^*)$ there exists a function $\tilde{f} \in L_1(\mu \times m_{r-1}^*)$ such that, excepting a set of m^*-measure zero,*

$$\tilde{f}_{[x^*]_{r-1}}(y) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U_{[x^*]_r} \ldots U_{[\varphi^{k-1}x^*]_r} f_{(\varphi^k x^*)}(y)$$

almost everywhere on Y.

As the reader knows, it is very natural to assume the existence of $\{U_x\colon x \in X_r^*\}$-invariant functions. In fact, considering that many investigators of ergodic theory have succeeded in solving the problem on the existence of such invariant functions, this circumstance can easily be understood.

Before proceeding to the proof of the theorem, we deduce some consequences of our theorem just described.

If $r = 1$, then Theorem 1 reduces to the following

COROLLARY 1 (Cairoli [1], Théorème 3). *On the hypothesis of Theorem 1, let $f \in L_1(\mu \times m^*)$. Then neglecting an m^*-null set,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U_{x_1} U_{x_2} \ldots U_{x_k} f_{(x_{k+1}, x_{k+2}, \ldots)}(y) = \tilde{f}(y) \qquad (\tilde{f} \in L_1(\mu))$$

almost everywhere on Y.

If we take $\{U_x\colon x \in X_r^*\}$ to be the family of operators on $L_1(\mu)$ induced by a $\mathscr{F} \times \mathscr{B}_r^*$-measurable family $\{T_x\colon x \in X_r^*\}$ of μ-measure preserving transformations of Y, then Theorem 1 entails

Corollary 2 (cf. Gładysz [2], Satz 1). *For every $f \in L_1(\mu \times m^*)$ there exists a function $\tilde{f} \in L_1(\mu \times m^*_{r-1})$ such that the limit*

$$\tilde{f}_{[x^*]_{r-1}}(y) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_{\varphi^k x^*)}(T_{[\varphi^{k-1}x^*]_r} \cdots T_{[x^*]_r} y)$$

holds with the exception of a $\mu \times m^*$-null set.

Moreover, if μ is a finite measure,

$$\lim_{x\to\infty} \Big\| \tilde{f}_{[x^*]_{r-1}}(\cdot) - \frac{1}{n} \sum_{k=1}^{n} f_{(\varphi^k x^*)}(T_{[\varphi^{k-1}x^*]_r} \cdots T_{[x^*]_r} \cdot) \Big\|_{L_1(\mu)} = 0.$$

Applying Theorem 1 above and Theorem 4 of [4], one gets the following convergence in the operator topology:

Corollary 3. *Besides the hypothesis of Theorem 1, assume that $\|U_x\|_{L_\infty(\mu)} \leqslant 1$ for all $x \in X^*_r$ and that μ is finite. Then for any $g \in L_1(\mu)$ there exists a function $\tilde{g} \in L_1(\mu \times m^*_{r-1})$ such that*

$$\lim_{n\to\infty} \Big\| \tilde{g}_{[x^*]_{r-1}}(\cdot) - \frac{1}{n} \sum_{k=1}^{n} U_{[x^*]_r} \cdots U_{[\varphi^{k-1}x^*]_r} g(\cdot) \Big\|_{L_1(\mu)} = 0$$

almost everywhere on X^*.

3. Proof of Theorem 1. We consider the strongly $\mathscr{B}^*$-measurable positive contraction quasi semigroup $\{U^{(r)}_{(k,x^*)}: x^* \in X^*, k \geqslant 0\}$ on $L_1(\mu)$ associated with φ ([4]), which is obtained by setting

$$U^{(r)}_{(0,x^*)} = \text{identity}, \qquad U^{(r)}_{(k,x^*)} = U_{[x^*]_r} U_{[\varphi x^*]_r} \cdots U_{[\varphi^{k-1}x^*]_r}, \qquad k \geqslant 1$$

and has the quasi semigroup property

$$U^{(r)}_{(i+j,x^*)} = U^{(r)}_{(i,x^*)} U^{(r)}_{(j,\varphi^i x^*)}, \qquad i, j \geqslant 0.$$

As observed in the preceding section, we see that if, under the assumption of the theorem, $f \in L_1(\mu \times m^*)$, then there exists a function $f^* \in L_1(\mu \times m^*)$ such that except for an m^*-null set,

$$f^*_{(x^*)}(y) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^{(r)}_{(k,x^*)} f_{(\varphi^k x^*)}(y),$$

$$U^{(r)}_{(k,x^*)} f^*_{(\varphi^k x^*)}(y) = f^*_{(x^*)}(y), \qquad k \geqslant 0,$$

almost everywhere on Y. Therefore, in order to assure the assertion of the theorem, it is sufficient to show that there is a function $\tilde{f} \in L_1(\mu \times m^*_{r-1})$ with $f(y, [x^*]_{r-1}) = f^*(y, x^*)$ $\mu \times m^*$-a.e. To show this, let us now consider a strongly $\mathscr{B}$-measurable family $\{V_x: x \in X\}$ of positive contraction operators on $L_1(\mu \times m^*_{r-1})$ such that

$$V_{x_r} f_{(\varphi^r x^*)}(y, [x^*]_{r-1}) = [U_{[x^*]_r} f_{(\varphi x^*)}](y)$$

for $f \in L_1(\mu \times m^*)$, where the function $[U_{[x^*]_r} f_{(\varphi x^*)}](y)$ stands for the (y, x^*)-measurable version of the function $U_{[x^*]_r} f_{(\varphi x^*)}(y)$ (see [4], Lemma 2). Here it should be noticed that the definition of V_x is justifiable, i.e., V_{x_r} does not depend on a choice of a function f. (Note further that the function $U_{[x^*]_r} f_{(\varphi x^*)}(y)$ is not necessarily measurable with respect to (y, x^*).) If we write

$$\tilde{f}(y, x_1, \ldots, x_{r-1}) = \int_{X_r} \int_{X_{r+1}} \cdots f^*_{(x_r, x_{r+1}, \ldots)}(y, x_1, \ldots, x_{r-1})\, dm(x_r)\, dm(x_{r+1}) \ldots,$$

we have

$$\begin{aligned}
&\int_X (V_x \tilde{f})(y, x_1, \ldots, x_{r-1})\, dm(x) \\
&= \int_X \int_{X_r} \int_{X_{r+1}} \cdots (V_x f^*_{(x_r, x_{r+1}, \ldots)})(y, x_1, \ldots, x_{r-1})\, dm(x)\, dm(x_r)\, dm(x_{r+1}) \ldots \\
&= \int_{X_r} \int_{X_{r+1}} \cdots (V_{x_r} f^*_{(x_{r+1}, x_{r+2}, \ldots)})(y, x_1, \ldots, x_{r-1})\, dm(x_r)\, dm(x_{r+1}) \ldots \\
&= \int_{X_r} \int_{X_{r+1}} \cdots [U_{[x^*]_r} f^*_{(\varphi x^*)}](y)\, dm(x_r)\, dm(x_{r+1}) \ldots \\
&= \int_{X_r} \int_{X_{r+1}} \cdots f^*_{(x_r, x_{r+1}, \ldots)}(y, x_1, \ldots, x_{r-1})\, dm(x_r)\, dm(x_{r+1}) \ldots \\
&= \tilde{f}(y, x_1, \ldots, x_{r-1}) \qquad \mu \times m^*\text{-a.e.}
\end{aligned}$$

and also

$$\begin{aligned}
(V_{x_r} V_{x_{r+1}} \cdots V_{x_{r+k-1}} f^*_{(x_{r+k}, x_{r+k+1}, \ldots)})(y, x_1, \ldots, x_{r-1})
&= [U_{[x^*]_r} U_{[\varphi x^*]_r} \cdots U_{[\varphi^{k-1}x^*]_r} f^*_{(x_{k+1}, x_{k+2}, \ldots)}](y) \\
&= [U^{(r)}_{(k,x^*)} f^*_{(\varphi^k x^*)}](y) = f^*_{(x^*)}(y) \qquad \mu \times m^*\text{-a.e.} \qquad (k \geqslant 1).
\end{aligned}$$

Put

$$V^{(r)}_{(0,x^{**}_r)} = \text{identity}, \qquad V^{(r)}_{(k,x^{**}_r)} = V_{x_r} V_{x_{r+1}} \cdots V_{x_{r+k-1}}, \qquad k \geqslant 1,$$

where $x^{**}_r = (x_r, x_{r+1}, \ldots)$ for $x^* = (x_1, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots) \in X^*$, and denote by ξ the one-sided Bernoulli shift on $X_r \times X_{r+1} \times \ldots$, which is the restriction of φ. Then $\{V^{(r)}_{(k,x^{**}_r)}\}$ defines a quasi semigroup of positive linear contraction operators on $L_1(\mu \times m^*_{r-1})$ associated with ξ. And from what we have already observed the limit function f^* is invariant under the induced contraction semigroup $\{W_k, k \geqslant 0\}$ on $L_1(\mu \times m^*)$ of this quasi semigroup ([4], Theorem 1), which is such that if $f \in L_1(\mu \times m^*)$, except on a set of $m_r \times m_{r+1} \times \ldots$-measure zero,

$$(W_k f)_{(x^{**}_r)}(y, [x^*]_{r-1}) = (V^{(r)}_{(k,x^{**}_r)} f_{(\xi^k x^{**}_r)})(y, [x^*]_{r-1})$$

$\mu\times m_1\times\ldots\times m_{r-1}$-almost everywhere on $Y\times X_1\times\ldots\times X_{r-1}$, for all $k=0,1,2,\ldots$ Thus we may apply the Cairoli lemma ([1], Lemme 4) with $\{V_x\colon x\in X\}$, to obtain the following assertion

$$(V^{(r)}_{(n,x_r^{**})}\tilde f)(y,[x^*]_{r-1})=\tilde f(y,[x^*]_{r-1})\qquad \mu\times m^*\text{-a.e.}$$

for all $n=0,1,2,\ldots$ Therefore, using this and applying the Jessen theorem (cf. for example [1], [2]), we have

$$\begin{aligned}
&\tilde f(y,[x^*]_{r-1})\\
&=\lim_{n\to\infty}(V^{(r)}_{(n,x_r^{**})}\tilde f)(y,[x^*]_{r-1})\\
&=\lim_{n\to\infty}\int\limits_{X_r}\int\limits_{X_{r+1}}\ldots(V^{(r)}_{(n,x_r^{**})}f^*_{(x_r',x_{r+1}',\ldots)})(y,[x^*]_{r-1})\,dm(x_r')\,dm(x_{r+1}')\ldots\\
&=\lim_{n\to\infty}\int\limits_{X_{r+n}}\int\limits_{X_{r+n+1}}\ldots(V^{(r)}_{(n,x_r^{**})}f^*_{(x_{r+n}',x_{r+n+1}',\ldots)})(y,[x^*]_{r-1})\times\\
&\qquad\qquad\times dm(x_{r+n}')\,dm(x_{r+n+1}')\ldots\\
&=\lim_{n\to\infty}\int\limits_{X_{r+n}}\int\limits_{X_{r+n+1}}\ldots[U^{(r)}_{(n,x^*)}f^*_{(x_{n+1},\ldots,x_{n+r-1},x_{n+r}',x_{n+r+1}',\ldots)}](y)\times\\
&\qquad\qquad\times dm(x_{n+r}')\,dm(x_{n+r+1}')\ldots\\
&=\lim_{n\to\infty}\int\limits_{X_{r+n}}\int\limits_{X_{r+n+1}}\ldots f^*_{(x_r,\ldots,x_{r+n-1},x_{r+n}',x_{r+n+1}',\ldots)}(y,[x^*]_{r-1})\times\\
&\qquad\qquad\times dm(x_{r+n}')\,dm(x_{r+n+1}')\ldots\\
&=f^*_{(x_r,x_{r+1},\ldots)}(y,[x^*]_{r-1})\qquad\mu\times m^*\text{-a.e.},
\end{aligned}$$

which accomplishes our purpose. ∎

Now it follows from the proof of Theorem 1 that for any $g\in L_1(\mu\times m^*)$

$$[U^{(r)}_{(k,x^*)}g_{(\varphi^kx^*)}](y)=g_{(x^*)}(y)\qquad\mu\times m^*\text{-a.e.}$$

implies that $g_{(x_1,x_2,\ldots)}(y)$ is essentially independent of $x_r,x_{r+1},\ldots$

This fact was proved by C. Ryll-Nardzewski ([3], Theorem 1) in the special case where $r=1$ and $\{U^{(1)}_{(k,x^*)}\colon x^*\in X^*,\ k\geqslant 0\}$ is induced by $\{T^{(1)}_{(k,x^*)}\colon x^*\in X^*,\ k\geqslant 0\}$ which is the measure preserving quasi semigroup set up by $\{T_x\colon x\in X_r^*\}$.

S. Gładysz ([2], Hilfssatz 3′) proved a weighted generalization of the Ryll-Nardzewski's to a case of many random parameters in the auxiliary steps to prove the radnom ergodic theorem.

The proof of Cairoli's theorem contains an operator theoretic generalization of the result due to C. Ryll-Nardzewski but not S. Gładysz. So the result to be stated below is a further extension of the Cairoli result and the Gładysz one having no weighted functions.

Because of importance of this fact, we summarize the aspect of Bernoulli shifts observed in the proof of Theorem 1 as follows.

THEOREM 2. *Let $\{U^{(r)}_{(k,x^*)}\colon x^*\in X^*,\ k\geqslant 0\}$ be the strongly $\mathscr{B}^*$-measurable positive contraction quasi semigroup on $L_1(\mu)$ associated with the one-sided Bernoulli shift φ on X^* as in the proof of Theorem* 1. *If for any $g\in L_1(\mu\times m^*)$ and almost all x^*,*

$$(U^{(r)}_{(k,x^*)}g_{(\varphi^kx^*)})(y)=g_{(x^*)}(y)\qquad\mu\text{-a.e.},$$

then there exists a function $\tilde g\in L_1(\mu\times m^*_{r-1})$ such that

$$\tilde g(y,x_1,\ldots,x_{r-1})=g(y,x_1,x_2,\ldots)\qquad\mu\times m^*\text{-a.e.}$$

References

[1] R. Cairoli, *Sur le théorème ergodique aléatoire*, Bull. Sci. Math. 88 (1964), pp. 31–37.

[2] S. Gładysz, *Über den stochastischen Ergodensatz*, Studia Math. 15 (1956), pp. 158–173.

[3] C. Ryll-Nardzewski, *On the ergodic theorems (III) (The random ergodic theorem)*, Studia Math. 14 (1954), pp. 298–301.

[4] T. Yoshimoto, *Induced contraction semigroups and random ergodic theorems*, Dissertationes Math. (Rozprawy Mat.) 139, Warszawa 1976.

DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
TOYO UNIVERSITY
KAWAGOE, SAITAMA, JAPAN

Received August 20, 1975
Corrected version April 7, 1976
(1061)