$$\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda t}P(t)f(x)\,dt = f(x)$$

a.e. for $f\in L_1(m)$. We note that $f\in L_1(\mu)$ implies $f/h\in L_1(m)$. Also

$$\int_0^\infty e^{-\lambda t}P(t)(f/h)\,dt = \Big\{\int_0^\infty e^{-\lambda t}S(t)f(x)\,dt\Big\}/h(x) \text{ a.e.}$$

and $\int_0^\infty e^{-\lambda t}S(t)f(x)\,dt = R_{\lambda+1}f(x)$. Thus

$$\lim_{\lambda\to 0}\lambda R_\lambda f(x) = \lim_{\lambda\to 0}(\lambda+1)\,R_{\lambda+1}f(x) = h(x)\lim_{\lambda\to 0}(\lambda+1)\int_0^\infty e^{-\lambda t}P(t)(f/h)\,dt$$
$$= h(x)\{f(x)/h(x)\} = f(x) \text{ a.e. } \blacksquare$$

Added in proof: Theorems 3 and 4 hold for pseudo-resolvents. The author has learned that an indirect proof of Theorem 4 for pseudo-resolvents was published in 1974 by C. Kipnis. The technique used in proving Theorem 4 may be adapted to obtain a direct proof of Kipnis' result.

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Corrigendum and addendum to the paper "In general, Bernoulli convolutions have independent powers"

Studia Math. 47 (1973), pp. 141–152

by

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Abstract. In this paper we point out an error in our earlier paper with this title and prove that with a slight modification of the definitions the results remain true. Explicitly, we show that for virtually all (in the sense of Baire category) sequences $(x_n)\in l^2$ the infinite convolution

$$\nu(x) = \mathop{\ast}_{n=1}^{\infty}\tfrac{1}{2}(\delta(-x_n)+\delta(x_n))$$

has the property that the $\sigma(L^\infty(\nu), L^1(\nu))$ closure of $\{e^{int}:\ n\in\mathbf{Z}\}$ contains all constants in $[-1, 1]$.

1. Corrigendum. We are indebted to Professor S. Saeki for pointing out to us that Remark 4 on p. 142 of [1] is false. In addition, we have subsequently found an error in the proof of the main theorem of [1]. The error arises in the final paragraph of the proof of Lemma 4 because the sets $M_i^{-1}U_i$ are not necessarily open in the relative topology of B. Nevertheless the main theorem of [1] remains true as stated and an appropriate variant of Remark 4 is obtained when, for example, the set F is replaced by the set F' defined by

$$F' = \Big\{(b_n):\ \sum_{n=1}^{\infty} b_n \leqslant \xi,\ b_n \geqslant 0\ (n = 1, 2, 3, \ldots)\Big\},$$

where ξ is any irrational number in $[0, 1]$.

Since generalizations of the theorems stated in [1] will appear with full proofs in the forthcoming paper of Lin and Saeki [2], we refrain from giving the details of the corrections needed in our original arguments. Instead we wish to state and prove a variant of the main theorem of [1] which admits a simple direct proof and which yields a more natural interpretation of the title result of that paper.

2. Addendum. For any sequence $(x_n)_{n=1}^{\infty}$ of real numbers consider the (formal) Bernoulli convolution

$$\nu(x) = \mathop{\ast}_{n=1}^{\infty}\tfrac{1}{2}\big(\delta(-x_n)+\delta(x_n)\big), \tag{1}$$

where $\delta(x)$ denotes the probability atom at the point x. Interpreting the limit in the sense of weak* convergence, one sees that the appropriate condition for (1) to define a probability measure in $M(\boldsymbol{R})$ is that $\boldsymbol{x}$ belongs to l^2. In [1], measures were regarded as elements of $M(\boldsymbol{T})$ (the circle group $\boldsymbol{T}$ being realized as $[0, 1[$ with addition modulo one) and attention was restricted to the set B of those sequences $\boldsymbol{x}$ with the properties that

$0 \leqslant x_n \leqslant 1$, for $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} x_n \leqslant 1$.

B was then regarded as a subset of the metric space $[0, 1]^{\aleph_0}$ with the relative topology and the phrases "virtually all Bernoulli convolutions", "in general, Bernoulli convolutions" were taken to refer to subsets of B which are residual in the sense of Baire's category theorem. The fact (which was overlooked in [1]) that the subset of all $\boldsymbol{x}$ in B such that $\sum_{n=1}^{\infty} x_n^2 = 1$ is already residual makes such a use of language appear artificial.

Moreover, if one regards measures as elements of $M(\boldsymbol{R})$ it is entirely natural simply to interpret "virtually all Bernoulli convolutions" to mean all measures of the form $\nu(\boldsymbol{x})$ as $\boldsymbol{x}$ ranges over a residual subset of the metric space $l^2(\boldsymbol{R})$. The object of this addendum is to demonstrate that the statements of [1] reinterpreted in this way admit simple proofs. However, it is necessary to pay for this simplification by making an explicit discussion of the method for transferring the results from the line to the circle.

As a prerequisite we require a distinguished embedding of the integers in the maximal ideal space $\Delta M(\boldsymbol{R})$ of the measure algebra of the line. To this end observe that each element τ of the dual group $\boldsymbol{R}$ is induced by an element t of $\boldsymbol{R}$ according to the formula

$$\tau(x) = \exp(2\pi i t x) \quad (x \in \boldsymbol{R}),$$

and regard the integers embedded in $\boldsymbol{R}$ by restriction of the correspondence $t \to \tau$. Since $\boldsymbol{R}\hat{}$ can be embedded in $\Delta M(\boldsymbol{R})$ by the formula

$$\tau(\mu) = \int \tau(x)\, d\mu(x) \quad (\mu \in M(\boldsymbol{R})),$$

we have an embedding of $\boldsymbol{Z}$ in $\Delta M(\boldsymbol{R})$. Let us denote the image of $\boldsymbol{Z}$ in $\Delta M(\boldsymbol{R})$ by Φ. Defining the quotient map $\theta\colon \boldsymbol{R} \to \boldsymbol{T}$ by $\theta(x) \in [0, 1[$, $\theta(x) \equiv x \pmod 1$, we obtain induced maps $\theta^*\colon M(\boldsymbol{R}) \to M(\boldsymbol{T})$, $\theta^{**}\colon \Delta M(\boldsymbol{T}) \to \Delta M(\boldsymbol{R})$. Observe that Φ is the image under θ^{**} of (the canonical image of $\boldsymbol{T}\hat{}$) in $\Delta M(\boldsymbol{T})$.

For μ in $M(\boldsymbol{R})$, we define $S(\mu)$ to be those constants which arise as μ-coordinates of generalized characters in the Gelfand closure of Φ — thus

$$S(\mu) = \{a \in C\colon\ a = \varphi\mu(x) \ \ (\mu \text{ a.e. } x) \ \text{ for some } \varphi \text{ in } \operatorname{cl}\Phi\}.$$

We shall be concerned with results which describe $S(\mu)$ for all μ in certain subsets A of $M(\boldsymbol{R})$. It is now clear that such a result may be transferred to a result about $M(\boldsymbol{T})$, which describes properties of the μ-coordinates of generalized characters in $\operatorname{cl}\boldsymbol{T}\hat{}$ in $\Delta M(\boldsymbol{T})$ as μ ranges through $\theta^*(A)$. With this point clarified we confine the discussion of the new theorem to the case of the real line.

THEOREM. *There is a dense G_δ-subset G of l^2 such that for each $\boldsymbol{x}$ in G, the measure $\nu(\boldsymbol{x}) = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}\big(\delta(-x_n) + \delta(x_n)\big)$ has the property that $S\big(\nu(\boldsymbol{x})\big) = [-1, 1]$.*

There is a dense G_δ-subset, G', of l^1 such that for each $\boldsymbol{x}$ in G', the measure $\omega(\boldsymbol{x}) = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}\big(\delta(0) + \delta(x_n)\big)$ has the property that $S\big(\omega(\boldsymbol{x})\big)$ is the unit disc.

Proof. Enumerate the rationals in $[0, 1]$ as $(p_n)_{n=1}^{\infty}$. For positive integers, q, r, s, t let $G(q, r, s, t)$ be the open subset of all $\boldsymbol{x}$ in l^2 such that

$$\theta(tx_n) < q^{-1}s^{-1}, \quad 1 \leqslant n \leqslant s-1; \tag{2}$$

$$\theta(tx_s - p_r) < q^{-1}; \tag{3}$$

$$(\|\boldsymbol{x}\|_2)^2 < \sum_{n=1}^{s} |x_n|^2 + t^{-2}q^{-1}. \tag{4}$$

(Here $\|\ \|_2$ denotes the l^2-norm and $\theta(\ )$ denotes residue modulo one.) We define G by

$$G = \bigcap_{q,r,u,v=1}^{\infty} \Big(\bigcup_{t=v}^{\infty} \bigcup_{s=u}^{\infty} G(q, r, s, t)\Big). \tag{5}$$

G is evidently a G_δ-subset of l^2. Now fix q, r, u, v, and some neighbourhood V in l^2. Choose $s \geqslant u$ and $\boldsymbol{x}$ in V such that $x_1, x_2, \ldots, x_s$ are rationally independent and such that $x_n = 0$ whenever $n > s$. It is now possible to choose $t \geqslant v$ such that (2) and (3) hold; meanwhile (4) is trivially true.

Thus we have verified that G is dense and it remains to check that $S\big(\nu(\boldsymbol{x})\big) = [-1, 1]$. Because $S\big(\nu(\boldsymbol{x})\big)$ is clearly a closed semigroup, it will suffice to check that $\cos(2\pi p_r) \in S\big(\nu(\boldsymbol{x})\big)$ for each r. Comparing with Lemma 1 of [1], we see that this, in turn, amounts to checking that there is a sequence $\big(t(k)\big)_{k=1}^{\infty}$ of positive integers such that

$$\nu(\boldsymbol{x})\hat{}\big(t(k)\big) \to \cos(2\pi i p_r); \tag{6}$$

$$\exp\big(2\pi i t(k) x_n\big) \to 1, \quad \text{for each } n. \tag{7}$$

But for each k, we choose $q = v = u = k$ and (5) guarantees the existence of $t = t(k)$, $s = s(k)$ such that $\boldsymbol{x} \in G(q, r, s, t)$. For definiteness we choose the least such $t(k)$ (which is, of course, no smaller than k). It is evident from (2) that (7) holds for this choice of $\big(t(k)\big)_{k=1}^{\infty}$. Moreover (2), (3), (4) give rise to the inequalities

$$\Big|1-\prod_{n=1}^{s-1}\cos\big(2\pi t(k)x_n\big)\Big| < 2\pi k^{-1},$$

$$|\cos(2\pi p_r)-\cos\big(2\pi t(k)x_s\big)| < 2\pi k^{-1},$$

$$\Big|1-\prod_{n=s+1}^{\infty}\cos\big(2\pi t(k)x_s\big)\Big| < 2\pi k^{-1},$$

and these combine to give (6). This completes the proof of the first assertion of the theorem.

For the second part we proceed in a similar way but replace (4) by (4'),

$$\|x\|_1 < \sum_{n=1}^{s}|x_n| + t^{-2}q^{-1}, \tag{4'}$$

and work with $\frac{1}{2}\big(1+\exp(2\pi i p_r)\big)$ in place of $\cos(2\pi p_r)$.

Remark. The statements of Corollaries 1, 2, 3 of Theorem 1 of [1] remain valid when the phrase "virtually all" is interpreted in the sense of "residual in l^2".

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Semi-stable probability measures on $\boldsymbol{R}^N$

by

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Abstract. Let $\{\xi_n\}$ be a sequence of $\boldsymbol{R}^N$-valued, independent and identically distributed random variables. Consider the sums

$$A_n(\xi_1+\ldots+\xi_{k_n})+c_n, \tag{0}$$

where A_n are non-singular linear operators in $\boldsymbol{R}^N$, $c_n \in \boldsymbol{R}^N$ and $k_n^{-1}k_{n+1}\to\gamma$. The limit law for sums of the form (0) is called semi-stable. The aim of this paper is to describe the class of all full semi-stable measures in $\boldsymbol{R}^N$.

1. Introduction and notation. We begin with some notation. By M we denote the set of all Borel probability measures on the real Euclidean space $\boldsymbol{R}^N$. We regard M as an Abelian topological semigroup with the convolution as a semigroup operation and the topology of weak convergence of measures.

We denote the convolution of two measures μ and ν by $\mu*\nu$. Throughout, the power μ^n is taken in the sense of the convolution. Moreover, by $\delta(x)$ we denote the probability measure concentrated at the point $x\in\boldsymbol{R}^N$. The characteristic function (Fourier transform) $\hat{\mu}$ of a measure $\mu\in M$ is defined by the formula

$$\hat{\mu}(y) = \int_{\boldsymbol{R}^N}\exp i(x, y)\,\mu(dx).$$

The group of all non-singular linear operators acting in $\boldsymbol{R}^N$ will be denoted by G.

For a Borel mapping $F\colon \boldsymbol{R}^N\to\boldsymbol{R}^N$ and a measure μ from M we denote by $F\mu$ the measure defined by the formula

$$F\mu(Z) = \mu(F^{-1}Z)$$

for any Borel subset Z of the space $\boldsymbol{R}^N$. In particular, it is easy to verify that the mapping $(A,\mu)\to A\mu$ from $G\times M$ onto M is jointly continuous and the formulas

$$A(\mu*\nu) = A\mu*A\nu, \qquad \widehat{A\mu}(y) = \hat{\mu}(A^*y)$$

hold, here A^* denotes the adjoint operator. Given $\mu\in M$, we define $\bar{\mu}$ putting $\bar{\mu}(Z) = \mu(-Z)$, where $-Z = \{-y\colon y\in Z\}$.