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Nonseparable "James tree" analogues of the continuous functions on the Cantor set

by

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Abstract. For every cardinal number $\mathfrak{m}$ with $\mathfrak{m}^{\aleph_0} < 2^{\mathfrak{m}}$, there exists a Banach space X of dimension $\mathfrak{m}$ such that X^* contains an isomorph of $l^1_{2^{\mathfrak{m}}}$, but such that X contains no isomorph of $l^1(\Lambda)$ for any uncountable set Λ. This space X can be taken to be the continuous functions on a compact Hausdorff space Ω. Topological properties of Ω are investigated.

Introduction. The purpose of this paper is twofold. First, we complete the study of this problem: *Are there infinite cardinal numbers $\mathfrak{m}$ with the property that if X is any Banach space of dimension ($=$ density character) $\mathfrak{m}$ such that X^* contains a subspace isomorphic to $l^1(\Gamma)$ with the cardinality of Γ greater than $\mathfrak{m}$, then X contains a subspace isomorphic to $l^1(\Lambda)$ for some uncountable set Λ?* (The answer is no.) Second, in the course of the study of the problem above, we are led to consider a compact Hausdorff space Ω so that $C(\Omega)$ shares many of the properties of weakly compactly generated Banach spaces without being one itself. The space Ω has interesting topological properties which are discussed in Theorem 2 below.

We accomplish these ends by constructing a nonseparable Banach space using the general techniques introduced by R. C. James in [6]. The compact space Ω is a certain weak* compact subset of X^*.

Our main results are:

THEOREM 1. *Let $\mathfrak{m}$ be a cardinal number, satisfying $\mathfrak{m}^{\aleph_0} < 2^{\mathfrak{m}}$. Then there exists a Banach space X satisfying the following properties:*

(1) *The dimension of X is $\mathfrak{m}$;*

(2) *X^* contains a subspace isometrically isomorphic to $l^1(\Gamma)$ where Γ has cardinality $2^{\mathfrak{m}}$;*

(3) *X does not contain a subspace isomorphic to $l^1(\Lambda)$ for any uncountable set Λ;*

(4) *X^* does not contain a subspace isomorphic to $L^1\{0,1\}^{\mathfrak{n}}$ for any uncountable cardinal number $\mathfrak{n}$;*

(5) *X is not weakly compactly generated.*

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THEOREM 2. *Let* $\mathfrak{m}$ *be a cardinal number satisfying* $\mathfrak{m}^{\aleph_0} < 2^{\mathfrak{m}}$. *Then there exists a compact Hausdorff space* Ω *satisfying the following*:

(1) *The topological weight of* Ω *is* $\mathfrak{m}$;

(2) *The cardinality of* Ω *is* $2^{\mathfrak{m}}$;

(3) $C(\Omega)$ *does not contain a subspace isomorphic to* $l^1(\Lambda)$ *for any uncountable set* Λ;

(4) *For any regular Borel measure* μ *on* Ω, $L^1(\mu)$ *is separable*;

(5) $C(\Omega)$ *is not weakly compactly generated*;

(6) Ω *is sequentially compact*;

(7) Ω *has a dense, dense in itself subset* Γ *such that no countable subset of* Γ *has an accumulation point in* Γ.

We remark that (1)–(5) of Theorem 2 imply (1)–(5) of Theorem 1. However, we state two separate results because we first produce a Banach space X satisfying Theorem 1, then use the space X to construct and to analyze the space Ω of Theorem 2. Given a cardinal number $\mathfrak{m}$, the $\mathfrak{m}$-dimensional space X we consider is a space of functions defined on a dyadic tree $\mathscr{T}$. When $\mathfrak{m} = \aleph_0$, the space we obtain is isomorphic to the continuous functions on the Cantor set. (All of this will be made precise later.)

Let us briefly mention the history of the problem which motivated this study. In [3] (cf. also [9]) it is proved that if X^* contains an isomorph of $l^1(\Gamma)$ and the dimension of X is less than the cardinality of Γ, then X contains an isomorph of l_1. It is also shown in [3] that if $\mathfrak{m}$ is a cardinal number satisfying $\mathfrak{m}^{\aleph_0} = 2^{\mathfrak{m}}$ then there exists an $\mathfrak{m}$-dimensional Banach space X (in fact, a space of continuous functions on a compact Hausdorff space) satisfying the following: X^* contains a subspace isometrically isomorphic to $l^1(\Gamma)$, where Γ has cardinality $2^{\mathfrak{m}}$, but X contains no subspace isomorphic to $l^1(\Lambda)$ for any uncountable set Λ. (In this case, the spaces X are weakly compactly generated.)

The questions remaining from [3], then, involve cardinal numbers $\mathfrak{m}$ for which $\mathfrak{m}^{\aleph_0} < 2^{\mathfrak{m}}$. In some sense, the answer for these cardinals (which include the cardinal number $\mathfrak{c}$, of course) is more interesting and the examples are less artificial (depending more on Banach space properties than on cardinality relations) than the examples in [3].

Remarks. (1) There are cases where known conditions imply that an $\mathfrak{m}$-dimensional Banach space X contains an isomorph of $l^1(\Lambda)$ for some uncountable set Λ. For example, in [4] it is shown that any $\mathfrak{m}$-dimensional subspace ($\mathfrak{m} > \aleph_0$) of $C\{0,1\}^{\mathfrak{n}}$ has this property.

(2) Another application of the technique due to James of building Banach spaces of functions defined on trees is given in [5], where a separable, hereditarily c_0 Banach space with nonseparable dual is constructed.

Let us briefly indicate the organization of the remainder of this paper. In Section 1 we introduce the dyadic tree and notions relevant to it and define the $\mathfrak{m}$-dimensional spaces which satisfy Theorem 1. An easy but important result (Lemma 3) on the behavior of a sequence of nodes of the tree is included at the end of this section. We also discuss here the analogy between these spaces and the continuous functions on the Cantor set.

Section 2 contains the proofs of the main results. Proposition 4 is a proof of the (certainly known) result that if X is a weakly compactly generated Banach space, then $\dim X^* \leqslant (\dim X)^{\aleph_0}$ where $\dim X$ denotes the dimension of the Banach space X. Lemma 5 gives an explicit characterization of the space Ω and is crucial in the proof of Theorem 2. Finally, it follows from Proposition 3.3 of [9] that (3) is a consequence of (4) in both the main theorems. However, we give a direct proof of (3) and then indicate the minor changes needed to prove (4). (It is not known if (3) and (4) are equivalent.)

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1. Preliminaries. For the most part, our Banach space notation and terminology are standard, or can be found in References [1], [2] or [3]. All Banach spaces will be real Banach spaces.

The Banach spaces we shall construct are spaces of functions on "dyadic trees", which we now define. Let β be a limit ordinal. The *dyadic tree of height* β is the set $\mathscr{T}_\beta = \bigcup_{\alpha<\beta} \{0,1\}^\alpha$ together with the partial order described below. Elements $\varphi \in \mathscr{T}_\beta$ are called *nodes*. If φ is a node of $\mathscr{T}_\beta$ and $\varphi \in \{0,1\}^\alpha$, then we write $|\varphi| = \alpha$. In this case we write $\varphi = (\varepsilon_\tau\colon \tau \leqslant \alpha)$ if α is not a limit ordinal and $(\varepsilon_\tau\colon \tau < \alpha)$ if α is a limit ordinal. (Of course, $\varepsilon_\tau = 0$ or 1 for each τ.) Now let ψ be a node with $|\psi| \geqslant \alpha$. If $\psi = (\delta_\tau\colon \tau \leqslant |\psi|)$ (or $(\delta_\tau\colon \tau < |\psi|)$ if $|\psi|$ is a limit ordinal), then we say that $\psi \geqslant \varphi$ if $\delta_\tau = \varepsilon_\tau$ for all $\tau \leqslant \alpha$ ($\tau < \alpha$ if α is a limit ordinal). If $\psi \geqslant \varphi$ and $|\psi| > |\varphi|$, then we write $\psi > \varphi$. Given two nodes $\psi, \varphi \in \mathscr{T}_\beta$ such that neither $\psi \geqslant \varphi$ nor $\varphi \geqslant \psi$, then φ and ψ are *incomparable*.

A *segment* S is a subset of $\mathscr{T}_\beta$ such that there exists an ordinal interval $\alpha \leqslant \tau \leqslant \sigma$ or $\alpha \leqslant \tau < \sigma$ (with $\sigma \leqslant \beta$) such that

(i) for each τ in this interval there exists exactly one $\varphi \in S$ with $|\varphi| = \tau$;

(ii) if τ_1 and τ_2 are this in interval with $\tau_1 < \tau_2$ and if $\varphi, \psi \in S$, $|\varphi| = \tau_1$, $|\psi| = \tau_2$, then $\psi > \varphi$.

In case S is a segment and $\varphi \in S$, we say that S *passes through* φ. It is clear from the definitions that there is a segment S passing through nodes φ and ψ if and only if $\varphi \geqslant \psi$ or $\psi \geqslant \varphi$.

Let S be a segment. Then we define $\sup\{\psi: \psi \in S\} = \varphi$ if $|\varphi| = \sup\{|\psi|: |\psi| \in S\}$ and $\varphi \geqslant \psi$ for all $\psi \in S$. (It is possible that $\sup\{|\psi|: \psi \in S\} = \beta$, so $\varphi \notin \mathscr{T}_\beta$, but this definition still makes sense.)

A *branch* B is a maximal segment, i. e., a segment of $\mathscr{T}_\beta$ so that for every ordinal $\alpha < \beta$, there exists a $\varphi \in \mathscr{T}_\beta$ with $|\varphi| = \alpha$. By identifying a branch B with the sequence $\sup_{\varphi \in B} \varphi \in \{0,1\}^\beta$, we clearly have a one-one correspondence between branches of $\mathscr{T}_\beta$ and $\{0,1\}^\beta$. We will say that $\varphi \in \{0,1\}^\beta$ *determines a branch* B if B is the unique branch for which $\sup_{\psi \in B} \psi = \varphi$.

Remark. Reference to the obvious pictorial representation of the tree, its partial order, the segments and branches, etc., will greatly facilitate understanding of this paper.

Now, let $x: \mathscr{T}_\beta \to \mathbf{R}$ be a function. We will denote $x = \{t_\varphi: \varphi \in \mathscr{T}_\beta\}$ where $t_\varphi = x(\varphi)$ for all $\varphi \in \mathscr{T}_\beta$. For a node φ of $\mathscr{T}_\beta$, e_φ denotes the function satisfying

$$e_\varphi(\psi) = \begin{cases} 1 & \text{if } \varphi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Let us define some (algebraic) linear functionals and projections on the vector space of finitely non-zero functions on $\mathscr{T}_\beta$. Fix such an $x = \{t_\varphi: \varphi \in \mathscr{T}_\beta\}$. If S is a segment, then we define $S^*(x) = \sum_{\varphi \in S} t_\varphi$. (If $S = \{\varphi\}$, then we write φ^* instead of $\{\varphi\}^*$.) If B is a branch, then we define $B^*(x) = \sum_{\varphi \in B} t_\varphi$. We now define the projections. For $\psi \in \mathscr{T}_\beta$, define P_ψ by

$$P_\psi(x) = \begin{cases} t_\varphi & \text{if } \varphi \geqslant \psi, \\ 0 & \text{otherwise.} \end{cases}$$

For any $\alpha < \beta$, we define P_α by

$$P_\alpha(x) = \begin{cases} t_\varphi & \text{if } |\varphi| \geqslant \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to define our $\mathfrak{m}$-dimensional Banach spaces. Let $\mathfrak{m}$ be an infinite cardinal number and β the initial ordinal corresponding to the cardinal $\mathfrak{m}$. For a finitely non-zero function x on $\mathscr{T}_\beta$, define $\|x\| = \max |S^*(x)|$, where the max is taken over all segments S of $\mathscr{T}_\beta$. Let $X_\mathfrak{m}$ be the completion of the vector space of finitely non-zero functions in the above norm. It is clear that the projections and functionals defined above all are of norm one.

For the moment, let us put $\beta = \omega$ (the first ordinal) and $\mathfrak{m} = \aleph_0$. We denote $X_{\aleph_0}$ by X_0. Let $\Delta = \{0,1\}^\omega$ denote the Cantor set, $C(\Delta)$ the space of continuous real-valued functions on Δ, and the sets $U_\varphi = \{(\varepsilon_i: i \in N): (\varepsilon_1, \ldots, \varepsilon_n) = \varphi, |\varphi| = n\}$ the "natural" base of closed open subsets of Δ. (Of course, $\varepsilon_i = 0$ or 1 for each i.) Let h_φ denote the characteristic function of the corresponding set U_φ.

Define an operator $R: X_0 \to C(\Delta)$ by $Re_\varphi = h_\varphi$. It is easy to see that R is continuous and is in fact a quotient map. If we put $k_\varphi = e_\varphi - e_{\varphi,0} - e_{\varphi,1}$, then routine computations show that the kernel K of the operator R is the closed linear span of the set $\{k_\varphi: \varphi \in \mathscr{T}_\omega\}$ and that this set is equivalent to the usual basis of c_0. (If $\varphi = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$ and $\varepsilon = 0$ or 1, then $\varphi,\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n, \varepsilon) \in \{0,1\}^{n+1}$.) Since K (being isomorphic to c_0) is complemented in the (separable) space X_0 (cf. [1], p. 96), we have that $X_0 \sim C(\Delta) \oplus c_0 \sim C(\Delta)$ by the decomposition method (cf. [8]).

We shall not prove any of these assertions about X_0. We point them out to show that for cardinals $\mathfrak{m} > \aleph_0$, the space X is a natural non-separable analog of X_0, which in turn is isomorphic to $C(\Delta)$.

Many of the proofs involve the selection of subsequences of given sequences. As the constructions can involve several parameters at one time, we adopt the following conventions: We will denote sequences of vectors by $x(1), x(2), \ldots$ and sequences of nodes by $\varphi(1), \varphi(2), \ldots$ (Sometimes, these sequences may be indexed as functions of two or more variables.) There will be no confusion between $x(n)$, the nth term of a sequence, and $x(\varphi)$, the value of the function x at the node $\varphi \in \mathscr{T}_\beta$. Whenever possible, we index sequences by infinite subsets of the positive integers $\mathbf{N}$. If M is an infinite subset of $\mathbf{N}$, we will consider M as a subsequence of $\mathbf{N}$. On the other hand, we will index sequences of scalars by the traditional subscripts, e.g., a sequence of scalars $t_1, t_2, \ldots$ or $\{t_j: j \in N\}$, sequences of ordinals by $\alpha_1, \alpha_2, \ldots$ or $\{\alpha_j: j \in \mathbf{N}\}$, and sequences of integers by $n_1, n_2, \ldots$ or $\{n_i: j \in \mathbf{N}\}$.

We prove here one result concerning the behavior of a sequence of nodes.

Lemma 3. *Let $\{\varphi(n): n \in N\}$ be a sequence in $\mathscr{T}_\beta$. Then there exists a subsequence $N' \subset \mathbf{N}$ satisfying one of the following alternatives:*

(i) *For all $m, n \in N'$, $\varphi(m) = \varphi(n)$.*

(ii) *There exists an $\alpha < \beta$ such that $|\varphi(n)| = \alpha$ for all $n \in N'$.*

(iii) *If $m, n \in N'$ and $m > n$, then $\varphi(m)$ and $\varphi(n)$ are incomparable and $|\varphi(m)| > |\varphi(n)|$.*

(iv) *If $m, n \in N'$ and $m > n$, then $\varphi(m) > \varphi(n)$.*

Proof. Assume that neither (i) nor (ii) hold for any subsequence of $\mathbf{N}$. Then it is straightforward to produce a subsequence N_1 of $\mathbf{N}$ such that if $m, n \in N_1$ and $m > n$, then $|\varphi(m)| > |\varphi(n)|$.

Once this is done, the proof is completed either by a direct application of Ramsey's theorem or by translating the proof of Lemma 4 of [8] into the appropriate terminology. ■

2. Proofs of the main results. Let $\mathfrak{m}$ be an infinite cardinal number satisfying $\mathfrak{m}^{\aleph_0} < 2^{\mathfrak{m}}$ and let $X = X_{\mathfrak{m}}$. We show that X satisfies Theorem 1 and that a certain weak* compact subset of X^* satisfies Theorem 2. For the remainder of this paper, let $\mathscr{T} = \mathscr{T}_\beta$ where β is the initial ordinal corresponding to $\mathfrak{m}$.

Proof of Theorem 1. To prove (1) observe that the set $\{e_\varphi: \varphi \in \mathscr{T}\}$ has cardinality $\mathfrak{m}$, so finite linear combinations of the e_φ's with rational coefficients are dense in X.

To prove (2), observe that there are $2^{\mathfrak{m}}$ branches B of $\mathscr{T}$. Let $B_1 B_2, \ldots, B_n$ be distinct branches, and let scalars $t_1, \ldots, t_n$ be given. Pick an ordinal $\alpha < \beta$ such that if $\{\varphi(i)\} = \{0,1\}^\alpha \cap B_i$ for $i = 1, \ldots, n$, then $\varphi(i) \neq \varphi(j)$ if $i \neq j$.

Let $x = \sum_{j=1}^{n} \operatorname{sgn}(t_j) e_{\varphi(j)}$, where $\operatorname{sgn}(t_j) = 1$ if $t_j \geqslant 0$ and -1 if $t_j < 0$. Then $\|x\| = 1$ and

$$\Big|\sum_{i=1}^{n} t_i B_i^* \Big(\sum_{j=1}^{n} \operatorname{sgn}(t_j) e_{\varphi(j)}\Big)\Big| = \sum_{i=1}^{n} t_i \operatorname{sgn}(t_i) B_i^*(e_{\varphi(i)}) = \sum_{i=1}^{n} |t_i|.$$

which shows that the set $\Gamma = \{B^*: B \text{ is a branch of } \mathscr{T}\}$ is isometrically equivalent to the usual basis of $l^1(\Gamma)$.

To prove (3) assume that there exists in X a set of norm one vectors $\{z(\lambda): \lambda \in \Lambda\}$ (where Λ is uncountable) and a $\delta > 0$ such that

$$\Big\|\sum_{i=1}^{n} t_i z(\lambda_i)\Big\| \geqslant \delta \sum_{i=1}^{n} |t_i|$$

for all $n, \lambda_1, \ldots, \lambda_n \in \Lambda$, and scalars $t_1, \ldots, t_n$.

Since finite linear combinations of the e_φ's with rational coefficients are dense in X, we may select for each $\lambda \in \Lambda$ a finite linear combination

$$x(\lambda) = \sum_{i=1}^{k(\lambda)} a(\lambda, i) e_{\varphi(\lambda, i)}$$

such that $a(\lambda, i)$ is rational and $|a(\lambda, i)| \leqslant 1$ for each i, and such that $\|z(\lambda) - x(\lambda)\| < \delta/2$. Since Λ is uncountable, there is an uncountable subset $\Lambda_1 \subset \Lambda$ such that, for all $\lambda \in \Lambda_1$, $k(\lambda) = k$. But now, since there are only a countable number of k-tuples of rational numbers, there is an uncountable subset $\Lambda_2 \subset \Lambda_1$ and rational numbers $a_1, \ldots, a_k$ such that, for any $\lambda \in \Lambda_2$ and $i = 1, \ldots, k$, $a(\lambda, i) = a_i$. In other words for $\lambda \in \Lambda_2$ we have

$$x(\lambda) = \sum_{i=1}^{k} a_i e_{\varphi(\lambda, i)}.$$

Now, let $\lambda_1, \lambda_2, \ldots$ be any distinct sequence of elements of Λ_2. Writing $x(n) = x(\lambda_n)$ and $\varphi(n, i) = \varphi(\lambda_n, i)$ for these elements, we have that

$$x(n) = \sum_{i=1}^{k} a_i e_{\varphi(n,i)}.$$

For $i = 1$, select a subsequence $N_1 \subset N$ such that $\{\varphi(n, 1): n \in N_1\}$ satisfies one of (i)–(iv) of Lemma 3. Proceeding inductively from $i = 1, \ldots, k$, select subsequences $N \supset N_1 \supset \ldots \supset N_k$ such that N_i is a subsequence of N_{i-1} and so that $\{\varphi(n, i): n \in N_i\}$ satisfies one of (i)–(iv) of Lemma 3. Observe that for any $i = 1, \ldots, k$, $\{\varphi(n, i): n \in N_k\}$ satisfies one of (i)–(iv) of Lemma 3.

Out of N_k, let us pick a sequence $m_1 < n_1 < m_2 < \ldots$ Then for fixed i, if $\{\varphi(n, i): n \in N_k\}$ satisfies (ii), (iii), or (iv) of Lemma 3, then $\{e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}: j \in N\}$ is equivalent to the usual basis of c_0. To see this, let s and scalars $t_1, \ldots, t_s$ be given. If $\{\varphi(n, i): n \in N_k\}$ satisfies (ii) or (iii) of Lemma 3, then no segment can pass through more than one $\varphi(n, i)$, $n \in N_k$. Hence, for any segment S,

$$\Big|S^* \sum_{j=1}^{s} t_j (e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)})\Big| \leqslant \max_j |t_j|.$$

On the other hand, if $|t_p| = \max_j |t_j|$, then

$$\Big|\varphi(m_p, i)^* \sum_{j=1}^{S} t_j (e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)})\Big| = |t_p|.$$

These computations combine to show that

$$\Big\|\sum_{j=1}^{s} t_j (e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)})\Big\| = \max_j |t_j|.$$

If $\{\varphi(n, i): n \in N_k\}$ satisfies (iv) of Lemma 3, then for each j, $\varphi(n_j, i) > \varphi(m_j, i)$. So if S is any segment, then for at most two j's is $S^*(e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}) \neq 0$, since if S passes through both $\varphi(m_j, i)$ and $\varphi(n_j, i)$, then $S^*(e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}) = 1 - 1 = 0$. Hence,

$$\Big|S^* \sum_{j=1} t_j (e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)})\Big| \leqslant 2 \max_j |t_j|,$$

and it follows as in the other cases that

$$\max_j |t_j| \leqslant \Big\|\sum_{j=1}^{s} t_j (e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)})\Big\| \leqslant 2 \max_j |t_j|.$$

Finally, if $\{\varphi(n, i): n \in N_k\}$ satisfies (i) of Lemma 3, we have $e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)} = 0$ for each j.

In each of the above cases, we can conclude that for every s and scalars $t_1, \dots, t_s$,

$$\Big\|\sum_{j=1}^{s} t_j\big(e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}\big)\Big\| \leqslant 2\max_j |t_j|.$$

For each j, put $y(j) = x(m_j) - x(n_j) = \sum_{i=1}^{k} a_i\big(e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}\big)$. Fix an integer M. Then

$$\Big\|\sum_{j=1}^{M}\frac{1}{M}y(j)\Big\| = \frac{1}{M}\Big\|\sum_{j=1}^{M}\sum_{i=1}^{k} a_i\big(e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}\big)\Big\|$$

$$\leqslant \frac{1}{M}\sum_{i=1}^{k}|a_i|\Big\|\sum_{j=1}^{M}\big(e_{\varphi(m_j,i)} - e_{\varphi(n_j,i)}\big)\Big\| \leqslant 2kM^{-1}.$$

On the other hand, if the set $\{z(n)\colon n\in N_k\}$ is δ-equivalent to the usual basis of l^1, we have

$$\Big\|\sum_{j=1}^{M}\frac{1}{M}y(j)\Big\| \geqslant \Big\|\sum_{j=1}^{M}\frac{1}{M}\big(z(m_j) - z(n_j)\big)\Big\| -$$

$$-\sum_{j=1}^{M}\frac{1}{M}\big(\|z(m_j) - x(m_j)\| + \|z(n_j) - x(n_j)\|\big) \geqslant 2\delta - \delta \geqslant \delta.$$

If we pick M so that $2kM^{-1} < \delta$, we have a contradiction. Thus, X does not contain isomorph of $l^1(\Lambda)$ for any uncountable set Λ.

The proof of (4) is similar to that of (3). Let $\mathfrak{n} > \aleph_0$ be a cardinal number. Let $I\colon H \to L^1\{0,1\}^{\mathfrak{n}}$ be an isomorphism of $\mathfrak{n}$-dimensional Hilbert space H into $L^1\{0,1\}^{\mathfrak{n}}$ and let $\{r(\alpha)\colon \alpha\in\mathscr{A}\}$ be an orthonormal set in H, where $\mathscr{A}$ has cardinality $\mathfrak{n}$. Let $\{r(\alpha)^*\colon \alpha\in\mathscr{A}\}$ in H^* be the functionals biorthogonal to the $r(\alpha)$'s.

Assume that there exists an isomorphism $U\colon L^1\{0,1\}^{\mathfrak{n}} \to X^*$. Then $I^*U^*\colon X^{**} \to H^*$ is onto. Since by Goldstine's theorem [1] the weak* closure of the unit ball of X is the unit ball of X^{**}, since U^* is weak* continuous and I^* is weak*-weak continuous, it follows that $I^*U^*|_X$ is onto. Thus, there exists a constant K such that, for each $\alpha\in\mathscr{A}$, there exists $z(\alpha)\in X$, $\|z(\alpha)\| \leqslant K$, such that $I^*U^*\big(z(\alpha)\big) = r(\alpha)^*$. For each α, pick a finite linear combination $x(\alpha)$ of the e_φ's with rational coefficients such that $\|x(\alpha) - z(\alpha)\| \leqslant (2\|I^*U^*\|)^{-1}$. Then if $\alpha_1, \alpha_2\in\mathscr{A}$ and $\alpha_1 \neq \alpha_2$, we have $\big\|I^*U^*\big(x(\alpha_1) - x(\alpha_2)\big)\big\| > \sqrt{2} - 1$.

Since the operator I^*U^* factors through $L^\infty\{0,1\}^{\mathfrak{n}}$, and since $L^\infty\{0,1\}^{\mathfrak{n}}$ satisfies the Dunford–Pettis property (cf. [10], for example) it follows that no sequence chosen from the set $\{x(\alpha)\colon \alpha\in\mathscr{A})\}$ can have a weak Cauchy

subsequence. Hence, by the results of [11], every sequence chosen from the set $\{x(\alpha)\colon \alpha\in\mathscr{A}\}$ has a subsequence equivalent to the usual basis of l^1. But our construction in (3) shows that the set $\{x(\alpha)\colon \alpha\in\mathscr{A}\}$ has a sequence with *no* subsequence equivalent to the usual basis of l^1. This contradiction proves (4).

Next, since X has dimension $\mathfrak{m}$, X^* has dimension $2^{\mathfrak{m}}$ and $\mathfrak{m}^{\aleph_0} < 2^{\mathfrak{m}}$, (5) follows from the following easy result. The idea for the proof comes from [7] and [10].

PROPOSITION 4. *Let X be a weakly compactly generated Banach space, $\mathfrak{m}$ the dimension of X. Then the dimension of X^* is at most $\mathfrak{m}^{\aleph_0}$.*

Proof. We first show that if Y is a bounded convex subset of X^*, then Y is weak* sequentially dense in $\mathrm{cl}^*(Y)$ (where $\mathrm{cl}^*(\cdot)$ denotes weak* closure). Let K be a weakly compact subset of X which generates X. Let $T\colon X^* \to C(K)$ be defined by $Tx^*(k) = x^*(k)$ for $x^*\in X^*$, $k\in K$. It is clear that T is one to one. Also, $T|_{\mathrm{cl}^*(Y)}$ is weak*-weak continuous (cf., for example, the proof of Corollary 3.4 of [12]). Thus, $\overline{T(Y)}^{w} \supset T\big(\mathrm{cl}^*(Y)\big)$ and since Y is convex, $\overline{T(Y)}^{w} = \overline{T(Y)}$ ($^{-w}$ denotes weak closure, $\overline{}$ norm closure). Hence $\overline{T(Y)} = T\big(\mathrm{cl}^*(Y)\big)$.

Let $f\in\mathrm{cl}^*(Y)$ be given. For each n, pick $f(n)\in Y$ such that $\|Tf(n) - Tf\| < 1/n$. Since $\mathrm{cl}^*(Y)$ is homeomorphic to weakly compact set in a Banach space, it follows from the Eberlein–Smulian theorem (cf. [2]) that $\mathrm{cl}^*(Y)$ is sequentially compact in the weak* topology. Hence there exists a subsequence $f(n_k)$ of $f(n)$ and a $g\in\mathrm{cl}^*(Y)$ such that $f(n_k) \to g$ weak*. Therefore $Tf(n_k) \to Tg$ weakly and since $Tf(n_k) \to Tf$ in norm, it follows from the Hahn–Banach theorem that $Tf = Tg$. Since T is one to one, $f = g$. Therefore $f(n_k) \to f$ weak*.

Now, let Y be a convex subset of the unit ball B of X^* such that $\mathrm{cl}^*(Y) = B$ and let $\{y(\alpha)\colon \alpha\in\mathscr{A}\}$ be a norm dense set in Y. Since Y is weak* sequentially dense in B, it follows that

$$\dim(X^*) \leqslant \mathrm{card}(B) \leqslant \big(\mathrm{card}(\mathscr{A})\big)^{\aleph_0}.$$

Now let $\{x(\alpha)\colon \alpha\in\mathscr{A}\}$ be a dense set in the unit ball of X with $\mathrm{card}(\mathscr{A}) = \dim(X)$. For each $\alpha\in\mathscr{A}$ pick $f(\alpha)\in B$ such that $f(\alpha)\big(x(\alpha)\big) = \|x(\alpha)\|$. Then by the Hahn–Banach theorem, the convex hull Y of the set $\{f(\alpha)\colon \alpha\in\mathscr{A}\}$ is such that $\mathrm{cl}^*(Y) = B$. Since the norm density character of Y is $\mathfrak{m}\cdot\aleph_0 = \mathfrak{m}$, we have that $\mathfrak{m}^{\aleph_0} \geqslant \dim(X^*)$, proving the lemma. ■

To describe the compact set Ω, we introduce more notation. Let $\alpha < \beta$ be a limit ordinal and for $\varphi\in\mathscr{T}$, $|\varphi| = \alpha$, we define the segment S_φ by $S_\varphi = \{\psi\in\mathscr{T}\colon \psi < \varphi\}$. Let $\Gamma = \{B^*\colon B$ is a branch of $\mathscr{T}\}$. Define $\Omega = \mathrm{cl}^*(\Gamma)$.

To prove Theorem 2, it is useful to have an explicit characterization of those functionals in X^* which belong to Ω.

Lemma 5. *Let Ω be defined as above. Then $\Omega = \Gamma \cup \{S_\varphi^*\colon |\varphi| = \alpha$, α a limit ordinal $< \beta\}$.*

Proof. Let f be weak* accumulation point of Γ. We show that $f = S_\varphi^*$ for some φ, $|\varphi| = \alpha$, α a limit ordinal. Since $B^*(e_\varphi) = 1$ or 0 for every branch B of $\mathscr{T}$ and $\varphi \in \mathscr{T}$, we must have $f(e_\varphi) = 1$ or 0 for every $f \in \Omega$. In particular, since $B^*(e_\emptyset) = 1$ for every $B^* \in \Gamma$ ($\emptyset$ is the "null sequence" in $\{0,1\}^0$) we must have $f(e_\emptyset) = 1$ for all $f \in \Omega$. Thus, $\|f\| = 1$ for all $f \in \Omega$.

We claim that $S = \{\psi \in \mathscr{T}\colon f(e_\psi) = 1\}$ forms a segment. First, if $\psi(1), \psi(2) \in S$ and $\psi(1)$ and $\psi(2)$ are incomparable, then $\|e_{\psi(1)} + e_{\psi(2)}\| = 1$ but $f(e_{\psi(1)} + e_{\psi(2)}) = 2$ which is impossible since $\|f\| = 1$.

Now, let $\varphi = \sup\{\psi\colon \psi \in S\}$. Assume that for some $\psi < \varphi$, $f(e_\psi) = 0$. Pick $\varepsilon = 0$ or 1 such that ψ,ε and φ are incomparable and pick $\varrho \in S$ with $\psi < \varrho$. Let $x = e_\psi - e_\varrho - e_{\psi,\varepsilon}$. Then $f(x) = -f(e_\varrho) = -1$, but if B is any branch, then $B^*(x) = 0$ if B does not pass through ψ, if B passes through ψ and ψ,ε, or if B passes through ψ and ϱ. Also, $B^*(x) = 1$ if B passes through ψ and not ψ,ε or ϱ. Since this exhausts all possible casses, it follows that $\{g \in X^*\colon |g(x) - f(x)| < \frac{1}{2}\}$ contains no $B^* \in \Gamma$, so f is not an accumulation point of Γ. This shows that S is either the segment $\{\psi\colon \psi < \varphi\}$ or $\{\psi\colon \psi \leqslant \varphi\}$. We show that the latter case is impossible. Observe that if B is any branch, then $B^*(e_\varphi - e_{\varphi,0} - e_{\varphi,1}) = 0$. On the other hand, if $\varphi \in S$, then $f(e_\varphi - e_{\varphi,0} - e_{\varphi,1}) = f(e_\varphi) = 1$, which is impossible if f is an accumulation point of Γ. Thus, $S = S_\varphi$, and it is clear from the above that $|\varphi|$ is a limit ordinal. Since $f(e_\psi) = S_\varphi^*(e_\psi)$ for all $\psi \in \mathscr{T}$, we must have $f = S_\varphi^*$. To complete the proof of the lemma we show that every such S_φ^* is in the weak* closure of the set Γ. Let $x(1), \ldots, x(n)$ be finitely non-zero elements in X. Then there exists an $\alpha < |\varphi|$ such that for all $\psi \in \mathscr{T}$ with $\alpha \leqslant |\psi| < |\varphi|$, $\psi^*(x(i)) = 0$ for $i = 1, \ldots, n$. Let us pick and fix one such $\psi \in S_\varphi$. Then there are an uncountable number of $\psi' \in \mathscr{T}$ with $\psi' > \psi$ and $|\psi'| = |\varphi|$. Since the $x(i)$'s are finitely non-zero, it follows that there exists at least one branch B passing through ψ for which $P_\psi^* B^*(x(i)) = 0$ for $i = 1, \ldots, n$. In particular, since $\psi \in S_\varphi$, we have that $B^*(x(i)) = S_\varphi^*(x(i))$ for $i = 1, \ldots, n$.

Now let $z(1), \ldots, z(n) \in X$ and $\varepsilon > 0$ be given. Find finitely non-zero $x(1), \ldots, x(n)$ in X such that $\|x(i) - z(i)\| < \varepsilon/2$ for $i = 1, \ldots, n$ and find B and ψ satisfying the conclusions of the paragraph above for $x(1), \ldots, x(n)$. Then, for each i,

$$\begin{aligned}\big|(B^* - S_\varphi^*)(z(i))\big| &\leqslant \big|B^*(z(i) - x(i))\big| + \big|(B^* - S_\varphi^*)(x(i))\big| + S_\varphi^*\big(z(i) - x(i)\big)\big| \\ &\leqslant 2\,\|z(i) - x(i)\| < \varepsilon\end{aligned}$$

so $B^* \in \{g \in X^*\colon \big|g(z(i)) - S_\varphi^*(z(i))\big| < \varepsilon$ for $i = 1, \ldots, n\}$. This completes the proof. ■

Proof of Theorem 2. We show first that the evaluation operator $R\colon X \to C(\Omega)$ defined by $Rx(f) = f(x)$ for $x \in X$, $f \in \Omega$, has dense range. Observe that for $\varphi, \psi \in \mathscr{T}$.

$$(Re_\varphi)(Re_\psi) = \begin{cases} Re_\psi & \text{if} \quad \psi \geqslant \varphi, \\ Re_\varphi & \text{if} \quad \varphi \geqslant \psi, \\ 0 & \text{if} \quad \varphi \text{ and } \psi \text{ are incomparable.}\end{cases}$$

From this, it follows that $\{Rx\colon x \in X$ is finitely non zero$\}$ is a subalgebra of $C(\Omega)$. Clearly, $Re_\emptyset = 1$, where $\emptyset \in \{0,1\}^0$, and $\{Re_\varphi\colon \varphi \in \mathscr{T}\}$ separates the points of Ω. By the Stone–Weierstrass theorem, $R(X)$ is dense in $C(\Omega)$.

To prove (1), observe that since X is $\mathfrak{m}$-dimensional and there is a map from X into $C(\Omega)$ with dense range, the dimension of $C(\Omega)$ is at most $\mathfrak{m}$. Since the cardinality of Ω trivially seen to be $2^{\mathfrak{m}}$ (establishing (2)), a standard argument shows that the dimension of $C(\Omega)$ is at least $\mathfrak{m}$. Thus, the dimension of $C(\Omega)$ is $\mathfrak{m}$. Since the topological weight of Ω is equal to the dimension of $C(\Omega)$, we have established (1). This discussion combined with Proposition 4 establishes (5).

Next, since the operator $R\colon X \to C(\Omega)$ has dense range, it follows from (3) of Theorem 1 and Lemma 3.1 of [9] that $C(\Omega)$ does not contain an isomorph of $l^1(\Lambda)$ for any uncountable set Λ. This establishes (3).

Implication (4) is proved in a fashion similar to that of (4) of Theorem 1, so we just sketch the proof. Let $\mathfrak{n}$, $H, \{r_\alpha, r_\alpha^*\colon \alpha \in \mathscr{A}\}$ and $I\colon H \to L^1\{0,1\}^{\mathfrak{n}}$ be as in the proof of (4) of Theorem 1, $U\colon L^1\{0,1\}^{\mathfrak{n}} \to C(\Omega)^*$ an isomorphism, and $R\colon X \to C(\Omega)$ as above. By dualizing the diagram

$$H \xrightarrow{I} L^1\{0,1\}^{\mathfrak{n}} \xrightarrow{U} C(\Omega)^* \xrightarrow{R^*} X^*,$$

we have

$$\begin{array}{ccccccc} X^{**} & \xrightarrow{R^{**}} & C(\Omega)^{**} & \xrightarrow{U^*} & L^\infty\{0,1\}^{\mathfrak{n}} & \xrightarrow{I^*} & H^* \\ \cup & & \cup & & & & \\ X & \xrightarrow{R} & C(\Omega). & & & & \end{array}$$

As in (4) of Theorem 1, $I^* U^*|_{C(\Omega)}$ is onto. So there exists, for each $\alpha \in \mathscr{A}$, $g(\alpha) \in C(\Omega)$ such that $I^* U^* g(\alpha) = r(\alpha)^*$.

Since R has dense range, there exists for each α an $x(\alpha) \in X$ finitely non-zero such that $\|Rx(\alpha) - g(\alpha)\| < (2\|I^* U^*\|)^{-1}$. We then have $\|I^* U^* x(\alpha) - r(\alpha)^*\| < 2^{-1}$, so $\|I^* U^* R(x(\alpha_1) - x(\alpha_2))\| \geqslant \sqrt{2} - 1$ for distinct $\alpha_1, \alpha_2 \in \mathscr{A}$.

Since $\mathscr{A}$ is uncountable, there exists an uncountable subset $\mathscr{A}' \subset \mathscr{A}$ and an $L < \infty$ such that $\|x(\alpha)\| \leqslant L$ for all $\alpha \in \mathscr{A}'$. Since $C(\Omega)$ has the Dunford–Pettis property (cf. [10], for example), no sequence chosen out of the set $\{x(\alpha)\colon \alpha \in \mathscr{A}'\}$ can be a weak Cauchy sequence. However, the proof of (3) of Theorem 1 shows that there exists a sequence in $\{x(\alpha): \alpha\in\mathscr{A}'\}$ that has *no* subsequence equivalent to the usual basis of l^1. But this contradicts the main result of [11] and the proof of (4) is complete.

We prove (6) as follows. Let $\{f(n): n\in N\}$ be a sequence in Ω. Each $f(n)$ corresponds either to summing along a branch or a segment. In the latter case let $\varphi(n)\in\mathscr{T}$ be such that $f(n) = S^*_{\varphi(n)}$. In the former case let $\varphi(n)\in\{0,1\}^\beta$ be such that if B is the branch determined by $\varphi(n)$, then $f(n) = B^*$.

Now let $N_0 = N$ and let α_1 be the smallest ordinal such that there exists a $\psi\in\mathscr{T}$ with $|\psi| = \alpha_1$ and a *proper* infinite $M\subset N$ such that for all $n\in M$, $\varphi(n)\geqslant\psi$ and for all $n\in N\setminus M$, $\varphi(n)$ and ψ are incomparable. Let $\psi(1)$ be one such ψ and N_1 an infinite proper subset of N_0 so that $\psi(1)$ and N_1 satisfy the above property.

Inductively pick ordinals $\alpha_1<\alpha_2<\ldots$, elements $\psi(j)\in\mathscr{T}$ and infinite subsets $N_0, N_1, N_2, \ldots$ such that (1) $\psi(1)<\psi(2)<\ldots$; (2) α_j is the smallest ordinal $>\alpha_{j-1}$ such that there exists a $\psi(j)>\psi(j-1)$ with $|\psi(j)| = \alpha_j$ and a *proper* infinite subset $N_j\subset N_{j-1}$ such that $\varphi(n)\geqslant\psi(j)$ for all $n\in N_j$ and $\varphi(n)$ and $\psi(j)$ are incomparable for all $n\in N_{j-1}\setminus N_j$.

Let $\psi = \sup_j \psi(j)$ and $f = S^*_\psi\in\Omega$. We claim that if we pick $n_j\in N_{j-1}\setminus N_j$, then $f(n_j)\to f$ weak*. To see this, we first let y be a finitely non-zero element of X. Then if $\{R(j): j\in N\}$ is any sequence of pairwise disjoint segments, there must exist a $\tilde{k}$ such that if $j\geqslant\tilde{k}$, $R(j)^*(y) = 0$.

Now let x, $\|x\| = 1$, and $\varepsilon>0$ be given. Pick a finitely non-zero $y\in X$ such that $\|y\| = 1$ and $\|y-x\|<\varepsilon/2$. Since $\alpha = |\psi|$ is a limit ordinal, there exists a $\tau<\alpha$ such that $(P_\tau - P_\alpha)y = 0$. Also, since $\sup_j|\psi(j)| = \alpha$, there exists a k_0 such that $|\psi(j)|>\tau$ for $j\geqslant k_0$. Thus, for $j\geqslant k_0$, $S_{\varphi(n_j)}$ passes through $\psi(k_0)$, so $f(n_j)(y-P_\alpha y) = S^*_{\varphi(n_j)}(y-P_\alpha y) = f(y)$.

By the construction and the observation above, there exists a $\tilde{k}_0\geqslant k_0$ such that, for $j\geqslant\tilde{k}_0$, $P^*_\alpha f(n_j)(y) = 0$. Thus, if $j\geqslant\tilde{k}_0$,

$$|\big(f(n_j)-f\big)(x)|\leqslant|f(n_j)(x-y)|+|f(x-y)|+\big|\big(f(n_j)-f\big)(y)\big|<\varepsilon.$$

This proves that $f(n_j)\to f$ weak*.

Finally, we prove (7). To see that Γ is dense in itself, let $B^*\in\Gamma$ and let $x(1), x(2), \ldots, x(n)\in X$ and $\varepsilon>0$ be given. Since $x(1), \ldots, x(n)$ are all at most countably non-zero, there must exist an $\alpha<\beta$ such that $P^*_\alpha x(i) = 0$ for $i = 1, \ldots, n$. Let $\psi\in B$, $|\psi| = \alpha$. Then for any branch B_1 passing through ψ, we have $B_1^*\big(x(i)\big) = B^*\big(x(i)\big)$. This proves that Γ is dense in itself.

Finally, let $B_1, B_2, \ldots$ be a countable set of distinct branches of $\mathscr{T}$. For fixed i, there exists a $\psi\in B_i\setminus\bigcup_{j\neq i}B_j$. Thus, $B_i^*(e_\psi) = 1$ and $B_j^*(e_\psi) = 0$ if $j\neq i$. Therefore,

$$\{g\in X^*: |g(e_\psi)-1|<\tfrac{1}{2}\}\cap\{B_j^*: j\in N\} = \{B_i\}$$

which proves that the set $\{B_j^*: j\in N\}$ is discrete. (In fact, this argument shows that even though Γ is dense in itself and Ω is sequentially compact, no countable set in Γ has an accumulation point in Γ.) ■

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