Proof. Let $E \in \mathscr{E}^{n-1}$; then by Lemma 2 we have $p(x) \in \operatorname{conv}\{\lambda_1, \ldots, \lambda_n\}$ for some $x \in E$, $x \neq 0$. Consequently $p(x) \geqslant \lambda_n$ and so

$$M(E) = \sup_{x \in E} p(x) \geqslant \lambda_n.$$

Hence it follows that λ_n is a lower bound for the set $\{M(E): E \in \mathscr{E}^{n-1}\}$. But, according to Theorem 3, we have $\lambda_n = M(E^{n-1})$ and thus λ_n is the greatest lower bound.

THEOREM 5. *The following principle holds true*:

$$\sup_{E \in \mathscr{E}_n} \inf_{x \in E} p(x) = \lambda_n.$$

Proof. Let $E \in \mathscr{E}_n$; then there is an element $y \neq 0$ in $E \cap E^{n-1}$ and, by Theorem 3, we have

$$m(E) = \inf_{x \in E} p(x) \leqslant p(y) \leqslant \lambda_n.$$

Hence it follows that λ_n is an upper bound for the set $\{m(E): E \in \mathscr{E}_n\}$. But, according to Lemma 1, we have $m(X_n) = \lambda_n$ and thus λ_n is the least upper bound.

Remarks. 1. If the family $A = (A_1, \ldots, A_k)$ consists of compact operators, then it is easy to see that the approximative point spectrum π contains no other points but the origin, i.e., $\pi = \{0\}$.

2. Assume, for simplicity, that H is finite dimensional and let $A = (A_1, A_2)$, where A_1 has a simple spectrum:

$$\lambda_1^{(1)} > \lambda_2^{(1)} > \ldots > \lambda_n^{(1)},$$

and A_2 has the norm satisfying the estimate

$$\|A_2\| \leqslant \tfrac{1}{2} \tan \tfrac{1}{2} \varkappa \min (\lambda_i^{(1)} - \lambda_{i+1}^{(1)})$$

with some $\varkappa$, $0 < \varkappa < \pi$. Then the eigenvalues $\lambda_i = (\lambda_i^{(1)}, \lambda_i^{(2)})$ of the family A can be arranged decreasingly in the first component; and this is precisely the arrangement induced by the cone $K = \{z: |\arg z| \leqslant \frac{1}{2}\varkappa\}$.

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Some properties of functions with bounded mean oscillation

by

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Abstract. Functions with bounded mean oscillation (BMO) have been shown to be of great interest in several areas of analysis and probability. In the first part of this paper, we examine the basic properties of these functions, giving a new proof of the John–Nirenberg inequality and proving the completeness of the function space. In the second part, we discuss various examples and remarks which have arisen recently, and we give another characterization of the harmonic functions in a half-space with boundary values in BMO.

Introduction. Nearly 15 years ago Fritz John and Louis Nirenberg introduced in [6] the class of functions with bounded mean oscillation, in view of its apparent interest in real analysis as well as in partial differential equations. Ten years later, Charles Fefferman [3] gave new impetus to this subject by discovering, in his famous duality theorem, the important link between BMO and harmonic analysis in several real variables. Thus, he set the stage, in his joint work with Elias Stein [4], to several new developments and applications. For instance, references [1], [2], [5], [7], [8], [9] and [10] show a part of this outgrowth in various branches of analysis, whereas the works of Burkholder, Gundy, A. Garsia and others exemplify the new developments taking place in probability theory, stimulated by the revival of interest in BMO.

§ 1. BMO revisited. Let us consider locally integrable functions f on $\boldsymbol{R}^n$ and "regular sets" Q (such as balls, or cubes with sides parallel to the axes), and denote by f_Q the integral average

$$f_Q = |Q|^{-1} \int_Q f(x)\,dx$$

or, mean-value of f on Q. We call the function

$$Q \ni x \mapsto |f(x) - f_Q|$$

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the *oscillation of f on Q*, and shall denote by $\mu(\alpha) = \mu_Q(\alpha)$ its distribution function

(1.0) $$\mu(\alpha) = |\{x \in Q\colon\ |f(x) - f_Q| > \alpha > 0\}|.$$

Clearly, if $f \in L^\infty$ then, for all Q, $\mu(\alpha) = 0$ whenever $\alpha > 2\|f\|_\infty$.

Consider now the variance (or, mean oscillation) of f on Q:

(1.1) $$|Q|^{-1} \int_Q |f(x) - f_Q|\,dx.$$

If these L^1-averages are bounded (uniformly in Q), we say that f has *bounded mean oscillation* on $\boldsymbol{R}^n$, and we shall write $f \in \{\text{b.m.o.}\}$. Introducing the seminorm

(1.2) $$\|f\|_* = \sup\Big\{|Q|^{-1}\int_Q |f(x) - f_Q|\,dx\colon\ Q \subset \boldsymbol{R}^n\Big\},$$

we have that $\{\text{b.m.o.}\} = \{f \in L^1_{\text{loc}}(\boldsymbol{R}^n)\colon\ \|f\|_* < \infty\}$.

Suppose that there exist constants $B, c > 0$ such that for all Q and every $\alpha > 0$

(1.3) $$\mu(\alpha) \leqslant Be^{-c\alpha}|Q|.$$

Then,

$$\int_Q |f(x) - f_Q|\,dx = \int_0^\infty \mu(\alpha)\,d\alpha \leqslant (B/c)|Q|$$

so that $f \in \{\text{b.m.o.}\}$ and $\|f\|_* \leqslant B/c$. Conversely, the main result of [6] shows that any $f \in \{\text{b.m.o.}\}$ will satisfy (1.3). More precisely, John and Nirenberg proved the following:

There exist absolute constants B and b (both > 0 and depending only on n) such that, for all $f \in \{\text{b.m.o.}\}$, all cubes $Q \subset \boldsymbol{R}^n$, and all $\alpha > 0$,

(J.N.) $$\mu(\alpha) \leqslant B|Q|\exp(-b\alpha/\|f\|_*).$$

We shall give here a slightly different proof of this inequality, due to A. P. Calderón (unpublished).

Proof of (J.N.). Since for $0 < \alpha \leqslant \|f\|_*$, (J.N.) holds trivially with $B = e$ and $b = 1$, we may assume that $\alpha > \|f\|_* > 0$. Moreover, fixing a cube $Q = Q_0$ and subtracting from f its mean-value on Q_0, we may suppose that $f_{Q_0} = 0$ so that

$$\mu(\alpha) = \big|\{x \in Q_0\colon\ |f(x)| > \alpha\}\big| \equiv |E_\alpha|$$

and we must prove that

(1.3′) $$|E_\alpha| \leqslant B|Q_0|\exp(-b\alpha/\|f\|_*), \qquad \alpha > \|f\|_* > 0.$$

Since $f \in L^1(Q_0)$, for any $y \geqslant \|f\|_*$ we may apply the Calderón–Zygmund decomposition lemma to $|f|$ on Q_0. Thus, we obtain a countable *union*

D_y of disjoint dyadic cubes $C \subset Q_0$ such that

(i) $|f(x)| \leqslant y$ a.e. on $Q_0 \setminus D_y$, and

(ii) $y < |C|^{-1}\int_C |f|\,dx \leqslant 2^n y$, for all C in D_y.

By additivity, if Δ_y is any (smaller) union of cubes C in D_y, (ii) implies that

(iii) $y|\Delta_y| \leqslant \int_{\Delta_y} |f|\,dx \leqslant 2^n y|\Delta_y|$.

Next, since $f \in \{\text{b.m.o.}\}$ and $\|f\|_* \leqslant y$, we can sharpen (ii) to the following estimate:

(iv) $y < |C|^{-1}\int_C |f|\,dx \leqslant y + 2^n\|f\|_*$, for any C in D_y.

First of all, since $y \geqslant \|f\|_* \geqslant |Q_0|^{-1}\int_{Q_0} |f|\,dx$, (ii) shows that $C \neq Q_0$. This means that C was obtained by subdividing (dyadically) a bigger cube $C_0 \subset Q_0$, on which

$$m_0 \equiv |C_0|^{-1}\int_{C_0} |f|\,dx \leqslant y,$$

into 2^n subcubes of volume equal to $2^{-n}|C_0|$. Consequently,

$$|C|^{-1}\int_C |f|\,dx \leqslant |C|^{-1}\int_C |f - m_0|\,dx + m_0 \leqslant 2^n|C_0|^{-1}\int_{C_0} |f - m_0|\,dx + y$$

and (iv) follows by (1.2).

We now note that, as a result of our decomposition, if $y < y'$ then $D_{y'} \subset D_y$. In fact, for any cube C in $D_{y'}$, if $C \not\subset D_y$ we must have that $C \subset (Q_0 \setminus D_y)$, so that $y' < |C|^{-1}\int_C |f|\,dx \leqslant y$, by virtue of (ii), contradicting the hypothesis that $y < y'$.

To shorten our notation, we shall assume (as we may do) that $\|f\|_* = 1$, and we shall set $\tilde{y} = y + 2^{n+1}\|f\|_* = y + 2^{n+1}$. We claim that:

(v) $|D_{\tilde{y}}| \leqslant 2^{-n}|D_y|$.

Choose any cube C in D_y and denote by m the mean-value of $|f|$ on C. Then, (iv) shows that $m \leqslant y + 2^n\|f\|_* < \tilde{y}$. Thus, in the dyadic decomposition giving rise to $D_{\tilde{y}}$, the cube C was subdivided, and its portion $D' \equiv C \cap D_{\tilde{y}}$ (unless empty) is the union of certain disjoint cubes in $D_{\tilde{y}}$. Consequently, by the first inequality in (iii) with y replaced by $\tilde{y}$ and Δ_y by D', we get:

$$\begin{aligned}\tilde{y} &\leqslant |D'|^{-1}\int_{D'} |f|\,dx \leqslant |D'|^{-1}\int_{D'} |f - m|\,dx + m\\ &\leqslant |C|\,|D'|^{-1}\Big\{|C|^{-1}\int_C |f - m|\,dx\Big\} + m\\ &\leqslant |C|\,|D'|^{-1} + m \leqslant |C|\,|D'|^{-1} + y + 2^n\end{aligned}$$

so that

$$|C|\,|D'|^{-1} \geqslant 2^{n+1}-2^{n} = 2^{n}.$$

Therefore, for any C in D_y and with $D' = C\cap D_{\tilde{y}}$, we see that

(v′) $|D'| \leqslant 2^{-n}|C|$

which also holds (trivially) if D' is empty. Since $D_{\tilde{y}}$ is the countable union of disjoint sets D' satisfying (v′), summing first over all such sets D' and then over all C in D_y, we obtain (v).

Finally, since $a > \|f\|_* = 1$, denoting by k the integral part of $(a-1)/2^{n+1}$, it follows that the number $y \equiv 1+k2^{n+1}$ satisfies $1 \leqslant y \leqslant a$. Accordingly, with the notation

$$E_\lambda = \{x \in Q_0:\ |f(x)| > \lambda > 0\},$$

we have the inclusions $E_a \subset E_y \subset (D_y \cup Z)$ where $|Z| = 0$ by (i) above. Hence,

$$|E_a| \leqslant |E_y| = |D_y| = |D_{1+k2^{n+1}}|$$

by our choice of y. Thus, iterating estimate (v) k-times (with y replaced by $\|f\|_* = 1$ in the last step), we obtain that

$$|E_a| \leqslant 2^{-nk}|D_1| \leqslant 2^{-nk}|Q_0|$$

from which (1.3′) follows easily with the constants $B = 2^{(n/2^{n+1})+n}$, $b = (n/2^{n+1})\log 2$. Therefore, the proof of (J.N.) is complete.

Remarks. 1.1. The choice of constants f_Q in definition (1.2) is *not* essential. Indeed, given f in $L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, if there exists an $M > 0$ such that for each Q we have some constant a_Q for which

$$|Q|^{-1}\int_Q |f(x)-a_Q|\,dx \leqslant M,$$

it follows at once that $\|f\|_* \leqslant 2M$. With this in mind, it is fairly easy to see that $\log|x|$ is in {b.m.o.}.

1.2. By the standard identity (with $1 \leqslant p < \infty$)

$$\int_Q |f(x)-f_Q|^p\,dx = p\int_0^\infty a^{p-1}\mu(a)\,da$$

and estimate (J.N.), one may readily verify that replacing the L^1-means (1.1) by corresponding L^p-averages in the definition of {b.m.o} we obtain the same class of functions with an equivalent norm. In particular (see [1] and [2]) the L^2-means are often to be preferred.

1.3. Similarly, inequality (J.N.) implies that, for each $f \in$ {b.m.o.} there is some constant $c = c(f) > 0$, where $c < b/\|f\|_*$, such that

$$\int_Q \exp[c|f(x)-f_Q|]\,dx < \infty$$

or, equivalently, for any cube $Q \subset \boldsymbol{R}^n$ and any $0 < c < b/\|f\|_*$,

$$\int_Q \exp(c|f(x)|)\,dx < \infty. \tag{1.4}$$

This local, necessary condition is used in § 2 below.

1.4 (*Global necessary condition*). Using (1.2) and a certain geometric argument (see [4], Ch. 1) one obtains the property that, for each $f \in$ {b.m.o.},

$$\int_{\boldsymbol{R}^n} |f(x)|\,(1+|x|^{n+1})^{-1}\,dx < \infty. \tag{1.5}$$

This implies at once that each f in {b.m.o.} is the trace (at $t = 0$) of a harmonic function $u(x,t)$ on $\boldsymbol{R}^{n+1}_+ = \boldsymbol{R}^n\times(0,\infty)$: namely, its Poisson integral

$$u(x,t) = [P_t * f](x) = \int_{\boldsymbol{R}^n} P_t(x-y)f(y)\,dy,$$

where $P_t(x) = t^{-n}P(x/t)$ and $P(x) = c_n(1+|x|^2)^{-(n+1)/2}$.

1.5 (*Completeness*). *In view of* (1.2), $\|f\|_* = 0 \Leftrightarrow f(x) = \text{const.}$ (a.e.), *so that* $f \mapsto \|f\|_*$ *is only a seminorm on* {b.m.o.}. *We claim that, equipped with this seminorm,* {b.m.o.} *is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in {b.m.o.} and pick any cube Q. Then,

(i) $|Q|^{-1}\int_Q |[f_n(x)-(f_n)_Q]-[f_m(x)-(f_m)_Q]|\,dx \leqslant \|f_n-f_m\|_* \to 0$ as $m, n\to\infty$. Thus $\{f_n(x)-(f_n)_Q\}$ is a Cauchy sequence in $L^1(Q)$ and hence

(ii) $\{f_n(x)-(f_n)_Q\} \to g^Q(x)$, in $L^1(Q)$ norm,

for some $g^Q \in L^1(Q)$.

Likewise, on any other cube $Q_1 \supset Q$, we have that

(iii) $\{f_n(x)-(f_n)_{Q_1}\} \to g^{Q_1}(x)$ in $L^1(Q_1)$, hence also in $L^1(Q)$.

Combining (ii) and (iii), it follows that, as $n\to\infty$,

(iv) $\{(f_n)_{Q_1}-(f_n)_Q\} \to C \equiv C(Q, Q_1)$.

For any $k \in \boldsymbol{N}$, let Q_k denote the cube with center at the origin and side of length k. Since any $x \in \boldsymbol{R}^n$ belongs to some Q_k, we "define" our limit function f by the expression:

(v) $f(x) = g^{Q_k}(x) - C(Q_1, Q_k)$, for all $x \in Q_k$.

In order to see that f is well defined by (v), we must have that if $x \in Q_k \subset Q_{k'}$, then

$$g^{Q_k}(x) - C(Q_1, Q_k) = g^{Q_{k'}}(x) - C(Q_1, Q_{k'})$$

or, equivalently, that $1 < k < k'$ implies that

$$C(Q_1, Q_{k'}) = C(Q_1, Q_k) + C(Q_k, Q_{k'}).$$

But, this additivity property of the constants follows readily from their definition in (iv).

Returning to our fixed cube Q, we have $Q \subset Q_k$ for some k large enough. Then, (v) implies that

$$\int_Q |(f_n - f)(x) - (f_n - f)_Q|\,dx$$
$$= \int_Q |f_n(x) - g^{Q_k}(x) + C(Q_1, Q_k) - (f_n)_Q + f_Q|\,dx$$
$$= \int_Q |f_n(x) - (f_n)_Q - g^Q(x) + \{g^Q(x) - g^{Q_k}(x) + C(Q_1, Q_k) + f_Q\}|\,dx$$

and a simple argument shows that $\{\ldots\}$ vanishes identically on Q. Consequently, by virtue of (ii), as $n \to \infty$

$$\int_Q |(f_n - f)(x) - (f_n - f)_Q|\,dx = \int_Q |f_n(x) - (f_n)_Q - g^Q(x)|\,dx \to 0$$

and standard arguments now yield that $\|f_n - f\|_* \to 0$ and $f \in \{\text{b.m.o.}\}$.

1.6 (BMO). Introducing in {b.m.o.} the equivalence relation

$$f_1 \sim f_2 \Leftrightarrow f_1 - f_2 = \text{const. a.e.,} \tag{1.6}$$

and denoting by $\mathrm{BMO} = \mathrm{BMO}(\boldsymbol{R}^n)$ the resulting (quotient) space of equivalence classes, (1.2) defines now a norm in BMO, which is complete by Remark 1.5. Hence, BMO is Banach space.

§ 2. Examples and further remarks. We collect here certain miscellaneous observations that were made over the last two years. To begin with, let us say that an $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ belongs to $E_{\mathrm{loc}}(\boldsymbol{R}^n)$ if there exists a constant $c > 0$ such that (1.4) holds over any $Q \subset \boldsymbol{R}^n$. Since {b.m.o.} is contained in $E_{\mathrm{loc}}(\boldsymbol{R}^n)$, to show that $f \notin \{\text{b.m.o.}\}$ it suffices to show that there exists some $Q_0 \subset \boldsymbol{R}^n$ such that

$$\sum_{k=0}^{\infty} (1/k!) \Big\{ \int_{Q_0} |f(x)|^k dx \Big\} c^k = +\infty, \quad \text{for any } c > 0. \tag{2.0}$$

Example 2.1. The class {b.m.o.} is *not* a multiplicative algebra in fact $(\log |x|)^2 \notin E_{\mathrm{loc}}(\boldsymbol{R})$. Indeed, for any $c > 0$, letting $Q_0 = [0,1]$, we have

$$\sum_{k=0}^{\infty} (1/k!) \Big\{ \int_0^1 (\log x)^{2k} dx \Big\} c^k = \sum_{k=0}^{\infty} [(2k)!/k!] c^k = +\infty.$$

Example 2.2 If $p > 1$, then $|\log|x||^p \notin E_{\mathrm{loc}}(\boldsymbol{R})$. Again, we let $Q_0 = [0,1]$.

Case 1. Suppose that $p = m/k$ is rational. Then, for any $c > 0$,

$$\sum_{n=0}^{\infty} (1/n!) \Big\{ \int_0^1 |\log x|^{mn/k} dx \Big\} c^n \geqslant \sum_{j=0}^{\infty} [(mj)!/(kj)!] c^{kj}$$
$$\geqslant \sum_{j=0}^{\infty} (kj+1)^{(m-k)j} c^{kj} = +\infty.$$

Case 2. For general $p > 1$, let $p > (m/k) > 1$, for some rational as above. Note that if $0 < x < 1/e$, then $|\log x|^p \geqslant |\log x|^{m/k}$. Hence, for any $c > 0$,

$$\sum_{n=0}^{\infty} (c^n/n!) \int_0^1 |\log x|^{pn} dx \geqslant \sum_{n=0}^{\infty} (c^n/n!) \int_0^{1/e} |\log x|^{mn/k} dx.$$

Since, for all $a > 0$,

$$\int_0^{1/e} |\log x|^a dx > \int_0^1 |\log x|^a dx - 1,$$

taking $a = mn/k$ and using Case 1, the divergence of our series follows at once.

In § II of [7], we considered a certain closed subspace CMO $\subset$ BMO (where c.m.o. = *continuous mean oscillation*) whose second dual equals BMO. Here, to simplify matters a bit, we shall confine our attention to *bounded* measurable sets $E \subset \boldsymbol{R}^n$.

Definition 2.3 We say that $f \in \{\text{c.m.o.}\}(E)$ if $f \in \{\text{b.m.o.}\}(E)$ and

$$|Q|^{-1} \int_Q |f(x) - f_Q|\,dx \to 0, \quad \text{as } |Q| \to 0. \tag{2.1}$$

Example 2.4. The function $\log |x| \notin \{\text{c.m.o.}\}(E)$ if E contains any neighborhood of the origin. Indeed, it suffices to check that $\log x \notin \{\text{c.m.o.}\})([0,1])$. But, if $0 < b \leqslant 1$, we have

$$b^{-1} \int_0^b |\log x - \log b|\,dx = -b^{-1} \int_0^b \log x\,dx + \log b = 1.$$

Remark 2.5. *If $0 < p = a^{-1} < 1$, then $f(x) \equiv |\log x|^p \in \{\text{b.m.o.}\}([0,1]$ and $\|f\|_* \leqslant C_a < 1$.*

Proof. For any nonnegative a,b and any $a \geqslant 1$, recall the homogeneous inequality

(i) $\quad a^a + b^a \leqslant (a+b)^a$.

Now, with $f(x) = |\log x|^{1/a}$ and any $[a,b] \subset [0,1]$, we shall use (by Remark 1.1) the constant $f(b)$ so that, for all $x \in [a,b]$,

(ii) $\quad |f(x) - f(b)| = |\log x|^{1/a} - |\log b|^{1/a}$.

Hence, for all $\sigma > 0$,

$$S_\sigma \equiv \{x\in[a,b]: |\log x|^{1/\alpha} > \sigma + |\log b|^{1/\alpha}\}$$
$$= \{x\in[a,b]: \log x < -[\sigma + |\log b|^{1/\alpha}]^\alpha\}$$
$$= \{x\in[a,b]: x < \exp(-[\ldots]^\alpha)\}.$$

Using (i), we note that $-[\sigma + |\log b|^{1/\alpha}]^\alpha \leqslant -\sigma^\alpha + \log b$. Thus, unless S_σ is empty, $a\in S_\sigma$ and so

$$S_\sigma \subset [a, b\exp(-\sigma^\alpha)] \quad \text{for all } \sigma > 0.$$

Now, with $\mu(\sigma) = |S_\sigma|$, we have the estimate

(iii) $\mu(\sigma) < (b-a)\exp(-\sigma^\alpha)$ for all $\sigma > 0$.

Therefore,

$$(b-a)^{-1}\int_a^b |f(x)-f(b)|\,dx = (b-a)^{-1}\int_0^\infty \mu(\sigma)\,d\sigma$$
$$\leqslant \int_0^\infty \exp(-\sigma^\alpha)\,d\sigma = C_\alpha < 1$$

and the proof is complete.

Remark 2.6 *If* $0 < p = \alpha^{-1} < 1$, *then* $f(x) = |\log x|^p \in \{\text{c.m.o.}\}([0,1])$.

Proof. By the preceding remark, it suffices to show that, for any $[a,b]\subset[0,1]$,

$$(2.2)\qquad (b-a)^{-1}\int_a^b |f(x)-f(b)|\,dx \to 0 \quad \text{as} \quad (b-a)\to 0^+.$$

We distinguish two cases.

Case $a = 0$. Here, with $\mu(\sigma) = |S_\sigma| = \big|\{x\in[0,b]: |f(x)-f(b)| > \sigma\}\big|$ we readily obtain that

$$\mu(\sigma) = \exp\big(-[\sigma + f(b)]^\alpha\big) \quad \text{for all } \sigma > 0.$$

Hence,

$$b^{-1}\int_a^b |f(x)-f(b)|\,dx = b^{-1}\int_{f(b)}^\infty \exp(-s^\alpha)\,ds \equiv b^{-1}F(b),$$

where $F(b)\to 0$ as $b\to 0^+$. Therefore, by L'Hospital's rule,

$$\lim_{b\to 0^+} b^{-1}F(b) = \lim_{b\to 0^+} F'(b) = -p\lim_{b\to 0^+} |\log b|^{p-1} = 0$$

since $0 < p < 1$, and this case is settled.

Case $a > 0$. First of all, let us note that S_σ is nonempty if and only if $f(a)-f(b) > 0$; that is, $\mu(\sigma) = 0$ for all $\sigma \geqslant f(a)-f(b)$. Thus,

(i) $\quad (b-a)^{-1}\int_a^b |f(x)-f(b)|\,dx = (b-a)^{-1}\int_0^{f(a)-f(b)} \mu(\sigma)\,d\sigma,$

where now

(ii) $\quad \mu(\sigma) = \exp\big(-[\sigma + f(b)]^\alpha\big)$ for all $0 < \sigma < f(a)-f(b)$ and $0 < a < b \leqslant 1$.

Combining (i) and (ii), we easily see that

$$(b-a)^{-1}\int_a^b |f(x)-f(b)|\,dx = (b-a)^{-1}\int_{f(b)}^{f(a)} \exp(-s^\alpha)\,ds - a\,\frac{f(a)-f(b)}{b-a}$$
$$= -\frac{F(b)-F(a)}{b-a} + a\,\frac{f(b)-f(a)}{b-a},$$

where

$$F(t) = \int_0^{f(t)} \exp(-s^\alpha)\,ds \quad \text{for all } t > 0,$$

so that

(iii) $\quad F'(t) = tf'(t) = p|\log t|^{p-1}$ for all $0 < t \leqslant 1$.

With this notation,

$$(\text{iv})\qquad (b-a)^{-1}\int_a^b |f(x)-f(b)|\,dx = a\,\frac{f(b)-f(a)}{b-a} - \frac{F(b)-F(a)}{b-a}$$

and we examine various ways in which $(b-a)\to 0^+$. If $b\to a$, or else $a\to b$ in (iv), we see by (iii) that the limit is zero. If, on the other hand, $a\to t_0$ and $b\to t_0$ for some $0 < t_0 < 1$, using the fact that the "bilateral" derivative equals the ordinary derivative, we obtain the same conclusion by virtue of (iii). This completes the proof of Remark 2.6.

Remark 2.7 The space $\{\text{c.m.o.}\}([0,1])$ is *not* a multiplicative algebra. In fact, letting $f(x) = |\log x|^{2/3}$, the preceding remark implies that $f\in\{\text{c.m.o.}\}([0,1])$. However, by Example 2.2, $f^2\notin\{\text{b.m.o.}\}([0,1])$.

Remark 2.8 For any measurable $E\subset \boldsymbol{R}^n$, a simple computation shows that its characteristic function χ_E satisfies

$$(2.3)\qquad \|\chi_E\|_* = \tfrac{1}{2}.$$

However, $\chi_E\notin\{\text{c.m.o.}\}$. To see this, we may assume that $n = 1$, and we let $Q = [0,1]$. Choose any sequence of Q_n such that

(i) $\quad |Q_n|\to 0$ as $n\to\infty$,

(ii) $\quad |Q_n\cap Q| = |Q_n\setminus Q| = (1/2)|Q_n|$.

(For example, $Q_n = [1-1/n, 1+1/n]$ will do.) Denoting by χ the characteristic function of Q, it is clear that its mean-value on Q_n equals 1/2 for all $n \in \boldsymbol{N}$. Hence, for all n,

$$|Q_n|^{-1} \int_{Q_n} |\chi(x) - \chi_{Q_n}|\, dx = \tfrac{1}{2}$$

and the conclusion follows in view of (i).

Remark 2.9. By virtue of a remark in §II of [7], if we denote by $C_0(\boldsymbol{R}^n)$ the space of all continuous f such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the supremum norm, we have that

(2.4) $$C_0(\boldsymbol{R}^n) \subset \{\text{c.m.o.}\}(\boldsymbol{R}^n).$$

The next remark, and subsequent corollary, arose in a conversation with R. Jonhson.

Remark 2.10. *Let $f \in L^1_{\text{loc}}(\boldsymbol{R}^n)$. If there exists an $A > 0$ such that, for all cubes $Q \subset \boldsymbol{R}^n$,*

(*) $$\| |f| \chi_Q \|_* \leqslant A,$$

then $f \in L^\infty(\boldsymbol{R}^n)$.

Proof. Let us fix a cube Q and set $g(x) = |f(x)|\chi_Q(x)$. Then, for any other cube $Q_1 \supset Q$ such that $|Q_1| = 2|Q|$ say, we have

$$\int_{Q_1} |g(x) - g_{Q_1}|\, dx \geqslant \int_{Q_1 \setminus Q} \ldots \geqslant |Q_1 \setminus Q|\, g_{Q_1} = \tfrac{1}{2}|Q_1|\, g_{Q_1}.$$

Therefore, $g_{Q_1} \leqslant 2\|g\|_* \leqslant 2A$, by (*). On the other hand,

$$g_{Q_1} = |Q_1|^{-1} \int_Q |f|\, dx = \tfrac{1}{2}(|f|)_Q$$

so that $|f(x)| \leqslant 4A$ almost everywhere, since Q is arbitrary.

Corollary 2.11. *Let $f \in L^1_{\text{loc}}(\boldsymbol{R}^n)$. If $(f\varphi) \in \{\text{b.m.o.}\}$ for every $\varphi \in L^\infty$, then $f \in L^\infty(\boldsymbol{R}^n)$.*

Proof. We may assume that $f(x) \in \boldsymbol{R}$ and use the function $\varphi = \chi_Q(\operatorname{sgn} f)$, for any cube $Q \subset \boldsymbol{R}^n$.

The closing remarks will describe a characterization of the harmonic functions on $\boldsymbol{R}^{n+1}_+ = \boldsymbol{R}^n \times (0, \infty)$ with traces in BMO $(\boldsymbol{R}^n)$, analogous to Theorem 1.0 of [1]. For a fixed $\delta > 0$, we denote by $\Gamma(0) = \Gamma^{2\delta}_{1/2}(0)$ the truncated cone

(2.5) $$\Gamma(0) = \{(y, t) \in \boldsymbol{R}^{n+1}_+ :\ 2|y| < t < 2\delta\}$$

and let $\Gamma(x) = \Gamma^{2\delta}_{1/2}(x)$ be its translate with vertex at $x \in \boldsymbol{R}^n$. Now, let us consider the ball $B_t \subset \boldsymbol{R}^{n+1}_+$ with center at $(0,t)$, where $0 < t \leqslant \delta$, which is *tangent* to the lateral boundary of $\Gamma(0)$. Then, its radius $r(t)$ satisfies: $\gamma_n t < r(t) < t/2$, for some $\gamma_n > 0$. Therefore, for some constant $c_n > 0$,

(2.6) $$|B_t| = t^{n+1}/c_n$$

and

(2.7) $$(t/2) < s < 3t/2 \quad \text{for all } (x, s) \in B_t.$$

With any harmonic function $u(x, t)$ on $\boldsymbol{R}^{n+1}_+$ and any $\delta > 0$ we can associate the "truncated g-function" $g_\delta(x)$ on $\boldsymbol{R}^n$, where

(2.8) $$g_\delta(x) \equiv [g_\delta(u)](x) = \Big\{\int_0^\delta |\nabla u(x, t)|^2 t\, dt\Big\}^{1/2}$$

and a "truncated S-function" $S_\delta(x)$ given by

(2.9) $$S_\delta(x) \equiv [S_\delta(u)](x) = \Big\{\iint_{\Gamma(x)} |\nabla u(y, t)|^2 t^{1-n}\, dt\, dy\Big\}^{1/2}.$$

Recalling the definition of HMO $(\boldsymbol{R}^{n+1}_+)$ in § I of [1], it is easy to see that, for any $u \in \text{HMO}$, $|g_\delta(x)| < \infty$ a. e. on $\boldsymbol{R}^n$. In fact, more precisely, we have the following majorization, the proof of which is a variant of Lemma 1.1 in [1].

Lemma 2.12. *Let $u(x, t)$ be harmonic on $\boldsymbol{R}^{n+1}_+$. Then there exists an absolute constant $C > 0$ such that, for all $x \in \boldsymbol{R}^n$ and $\delta > 0$,*

(2.10) $$g_\delta(x) \leqslant C S_\delta(x).$$

Proof. For any $j \in \{1, \ldots, n+1\}$ the partial derivative $D_j u$ is again harmonic. Since, by translation invariance, it suffices to prove (2.10) at $x = 0$, we consider the ball B_t described above. Then, by harmonicity and Schwarz's inequality,

$$|D_j u(0, t)|^2 \leqslant |B_t|^{-1} \int_{B_t} |D_j u(y, s)|^2\, dy\, ds = c_n t^{-n-1} \int_{B_t} \ldots\, dy\, ds$$

so that

$$\int_0^\delta |D_j u(0, t)|^2 t\, dt \leqslant c_n \int_0^\delta t^{-n} \Big\{\int_{B_t} |D_j u|^2\, dy\, ds\Big\}\, dt.$$

Observe now that the union of all B_t, for $0 < t < \delta$, is contained in $\Gamma(0)$; moreover, for all $(y, s) \in B_t$, (2.7) implies that $(2/3)s < t < 2s$. Hence, interchanging the order of integration, the last integral is dominated by

$$\iint_{\Gamma(0)} |D_j u|^2 \Big(\int_{2s/3}^{2s} t^{-n}\, dt\Big)\, dy\, ds = C \iint_{\Gamma(0)} |D_j u|^2 s^{1-n}\, dy\, ds.$$

Therefore, adding over j, the conclusion follows at once. ■

A partial converse of (2.10) is given by the following

LEMMA 2.13. *Let $u(x,t)$ be harmonic in $\boldsymbol{R}_+^{n+1}$ and pick any ball $Q_\delta = \{x \in \boldsymbol{R}^n\colon |x-x_0| \leqslant \delta\}$. Then there exists an absolute constant $C>0$ (depending only on n) such that*

$$|Q_\delta|^{-1} \int_{Q_\delta} [S_\delta(x)]^2\,dx \leqslant C\,|Q_{2\delta}|^{-1} \int_{Q_{2\delta}} [g_{2\delta}(x)]^2\,dx. \tag{2.11}$$

In particular, for all $u \in \mathrm{HMO}(\boldsymbol{R}_+^{n+1})$,

$$|Q_\delta|^{-1} \int_{Q_\delta} [S_\delta(u)]^2(x)\,dx \leqslant C\,\|u\|_{**}^2. \tag{2.12}$$

Proof. By translation, we may take $x_0 = 0$ and denote by U_δ the union of truncated cones $\Gamma(x) = \Gamma_{1/2}^{2\delta}(x)$ for all $x \in Q_\delta$, so that $U_\delta \subset (Q_{2\delta} \times [0, 2\delta])$. Then, interchanging the order of integration, we see that

$$\begin{aligned} J_\delta &\equiv \int_{Q_\delta} \Big\{ \iint_{\Gamma(x)} |\nabla u(y,t)|^2 t^{1-n}\,dy\,dt \Big\}\,dx \\ &= \iint_{U_\delta} |\nabla u|^2 |\{x \in Q_\delta\colon (y,t) \in \Gamma(x)\}|\,dy\,dt \\ &\leqslant C2^{-n} \iint_{U_\delta} |\nabla u(y,t)|^2 t\,dt\,dy, \end{aligned}$$

since $\{x \in Q_\delta\colon (y,t) \in \Gamma(x)\} \subset \{x \in \boldsymbol{R}^n\colon |x-y| < t/2\}$. Consequently,

$$\begin{aligned} |Q_\delta|^{-1} \int_{Q_\delta} [S_\delta(x)]^2\,dx &= |Q_\delta|^{-1} J_\delta \\ &\leqslant C2^{-n} |Q_\delta|^{-1} \int_{Q_\delta} \Big\{ \int_0^{2\delta} |\nabla u|^2 t\,dt \Big\}\,dy \\ &\leqslant C_n |Q_{2\delta}|^{-1} \int_{Q_{2\delta}} [g_{2\delta}(x)]^2\,dx \end{aligned}$$

and (2.11) is established.

Estimate (2.12) follows directly from (2.11) and the definition of norm $\|u\|_{**}$ in HMO. ■

The preceding lemmas give us at once the following

COROLLARY 2.14. *Let u be harmonic on $\boldsymbol{R}_+^{n+1}$. Then $u \in \mathrm{HMO}$ if and only if*

$$\|u\|_{***} \equiv \sup \Big\{ \Big(|Q_\delta|^{-1} \int_{Q_\delta} [S_\delta(u)]^2(x)\,dx \Big)^{1/2}\colon x_0 \in \boldsymbol{R}^n,\ \delta > 0 \Big\} < \infty. \tag{2.13}$$

Moreover, $\|u\|_{***}$ is equivalent to $\|u\|_{**}$.

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