STUDIA MATHEMATICA, T. LXI. (1977)

Normalized weakly null sequence with no unconditional subsequence*

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Abstract. Examples are given of sequences of norm-one elements of certain Banach spaces which tend weakly to zero, yet have no unconditional subsequence.

§ 1. Introduction. Let (b_n) be a finite or infinite sequence of non-zero elements of a real Banach space B and K a positive number. (b_n) is said to be *normalized* if $\|b_n\| = 1$ for all n; *weakly null* if (b_n) is an infinite sequence which tends weakly to zero as n tends to infinity; *K-unconditional* if for all n, $F \subset \{1, \ldots, n\}$, and scalars $a_1, \ldots, a_n$, $\|\sum_{j\in F} a_j b_j\| \leqslant K \|\sum_{j=1}^{n} a_j b_j\|$; *unconditional* if it is K-unconditional for some $K < \infty$.

It is a famous open question if every infinite dimensional Banach space contains an infinite unconditional basic sequence. We show that the following related question, stated in 1958 in [2], has a negative answer: *Does every normalized weakly null sequence in a Banach space have an infinite unconditional subsequence?*

Here are some related positive results. As shown in [3], if a Banach space has an unconditional basis (i.e. an unconditional sequence with dense linear span), then every normalized weakly null sequence in the space does have an unconditional subsequence. It is proved in [9], using Ramsey's theorem, that *if (b_n) is a normalized weakly null sequence in a Banach space and $\varepsilon > 0$ and k a positive integer are given, then there is a subsequence (b'_n) of (b_n) so that* for all m, *sets $F \subset \{1, \ldots, m\}$ with $|F| \leqslant k$ ($|F|$ denotes the cardinality of F) and scalars $a_1, \ldots, a_m$, $\|\sum_{i\in F} a_i b'_i\| \leqslant (1+\varepsilon) \|\sum_{i=1}^{m} a_i b'_i\|$.*

In particular, every subsequence of (b'_i) with k-elements is $1+\varepsilon$-unconditional, a previously known result which can also be deduced from the results of [5]. It is shown in [9] that if S is a set and (A_n) is a sequence of non-empty subsets of S with (χ_{A_n}) weakly null in $l^\infty(S)$, then (χ_{A_n}) has a 1-unconditional subsequence. ($l^\infty(S)$ denotes the Banach space of

* The research for this paper was partially supported by NSF Grant GP-28577.

all bounded real-valued functions on S under the sup norm; χ_A denotes the characteristic function of the set A). We prove in Theorem 3.4 that every normalized weakly null sequence in $C(\omega^\omega+1)$ has an unconditional subsequence. (An ordinal α is identified with the set of ordinals preceding it; $C(K)$ denotes the Banach space of real-valued continuous functions on the space K.) (It is known that $C(\omega^\omega+1)$ has no unconditional basis.)

Finally, we note that if an infinite-dimensional Banach space has no normalized weakly null sequence, then it contains an infinite unconditional basic sequence, in fact it contains a subspace isomorphic to l^1, the Banach space of all absolutely converging series (this is an immediate consequence of the results in [7]).

It is convenient to introduce the following terminology in order to describe some of the features of our example.

Let Se denote the Banach space of all converging series of real numbers. For any (c_n) with $\sum c_n$ convergent, put $\|(c_n)\|_{Se} = \sup_k \left|\sum_{j=1}^{k} c_j\right|$. Let $e_1, e_2, \ldots$ be the unit-vector-basis for Se, i. e. $(e_j)_k = \delta_{jk}$ for all j and k. We call (e_j) the *summing basis*. It is evident and well known that (e_j) is not unconditional. Indeed, by considering the expansion $e_1-2e_2+2e_3+ + \ldots + (-1)^{n+1}2e_n$, one sees that the unconditional constant of $(e_1, \ldots, e_n)$ equals n for all n. (The unconditional constant of a sequence is by definition the smallest K so that it is K-unconditional.)

Given K, a positive number, and sequences (x_n), (y_n) in Banach spaces X and Y, respectively, we say that (x_n) *is K-block-represented in* (y_n) if (x_n) is K-equivalent to a block-basis of (y_n); that is, there exists a sequence (z_j) in Y, a sequence $F_1, F_2, \ldots$ of finite subsets of N, the positive integers, and scalars (c_j) so that for all n, $\max F_n < \min F_{n+1}$, $z_n = \sum_{j\in F_n} c_j y_j$ and

$$\Big\|\sum_{j=1}^{n} a_j x_j\Big\| \leqslant \Big\|\sum_{j=1}^{n} a_j z_j\Big\| \leqslant K\Big\|\sum_{j=1}^{n} a_j x_j\Big\|$$

for all scalars $a_1, \ldots, a_n$. We say that (x_n) *is K-finitely block-represented in* (y_n) if (x_n) is infinite and every finite-subsequence of (x_n) is K-block-represented in (y_n).

Finally, we say that (x_n) *is block-represented* (resp. *finitely block-represented*) *in* (y_n) if (x_n) is K-block-represented (resp. K-finitely block-represented) in (y_n) for some $K < \infty$. It is trivial that if (x_n) is block-represented in (y_n), (x_n) is finitely block-represented in (y_n); it is easily seen that:

If (y_n) is an unconditional sequence and (x_n) is finitely block-represented in (y_n), then (x_n) is an unconditional sequence.

In Example 1 of § 2 we construct a normalized weakly null sequence (b_n) in a Banach space B so that the summing basis is block-represented in every subsequence of (b_n). (These properties are proved in Theorem 1.) We note that B^* is necessarily unseparable. [A normalized weakly null sequence in a space with a separable dual has a shrinking subsequence (see e.g. [6]) and every block-basis of a shrinking sequence is shrinking; the summing basis is not shrinking.]

In Example 2, we construct a normalized weakly null sequence (b_n) in a Banach space B which isometrically imbeds in $C(\omega^{\omega^2}+1)$ so that the summing basis is finitely block-represented in every subsequence of (b_n). (The proof that B imbeds in $C(\omega^{\omega^2}+1)$ is given in Theorem 3.2.) Thus B^* is separable.

In Example 3 we show that for every K there exists a Banach space B isomorphic to Hilbert space (i.e. l^2) and a sequence (b_n) in B equivalent to the usual basis for Hilbert space so that every subsequence of (b_n) has unconditional constant at least K. (This example is due to W. B. Johnson.)

Finally, in Example 4, we construct a normalized weakly null sequence in a uniformly convex Banach space B which has no unconditional subsequence. It follows that B does not imbed in a space with an unconditional basis; this seems to be the first example of a space with this property. (L. Tzafriri has recently shown that this B does not imbed in a uniformly convex Banach lattice; see [11].)

The construction of this example uses the technique of interpolation due to Lions–Peetre and a theorem of Beauzamy (see [1]).

Our work suggests the following open questions:

1. Is there an infinite-dimensional reflexive Banach space B such that the summing basis is finitely block-represented in every normalized weakly null sequence in B? (Such a space would have no infinite unconditional sequence.)

2. Is there a normalized weakly null sequence in some Banach space so that no constant-coefficient block basis of a subsequence is unconditional?

3. Does every normalized weakly null sequence (f_n) in $L^1[0,1]$ have an unconditional subsequence? (By the results of [8], the answer is affirmative if the closed linear space of the f_n's is reflexive.)

§ 2. Motivation and construction of the examples. We shall first give a pictorial description of the construction of the examples. Later we shall pass to a careful analytical description, less intuitive but easier to work with.

We may think of any Banach space as a subspace of $l^\infty(S)$ for some set S; a normalized weakly null sequence may then be realized as a sequence of functions (f_n) in $l^\infty(S)$ with $\|f_n\|_\infty = 1$ for all n and the f_n's satisfying a stronger condition than $f_n \to 0$ pointwise on S. (Precisely, the condition is that if $\delta > 0$ and $n_1 < n_2 < \ldots$ is any increasing sequence of indices, then there is a k so that $\bigcap_{i=1}^{k} \{s\colon |f_{n_i}(s)| > \delta\} = \emptyset$; see [9]).

In order to obtain such an (f_n) with the summing basis finitely block-represented in every subsequence of (f_n) it is useful to have the following sufficient criterion for a sequence (g_j) in $l^\infty(S)$ to be isometrically equivalent to the summing basis:

The Summing-Basis Criterion. *Let (g_j) be a finite or infinite sequence in $l^\infty(S)$ so that for all j for which g_j exists and $s \in S$, $0 \leqslant g_j(s) \leqslant 1$, $\{s \in S\colon g_j(s) = 1\}$ is non-empty and $\{s \in S\colon g_j(s) \neq 0\} \subsetneqq \{s \in S\colon g_{j-1}(s) = 1\}$ if $j > 1$. Then for all n, $(g_1, \ldots, g_n)$ is isometrically equivalent to the first n terms of the summing basis (if g_n exists).*

We note that if each g_j is zero-or-one-valued, the criterion simply reduces to the assertion that (A_j) is a strictly decreasing sequence of sets, where $A_j = \{s \in S\colon g_j(s) = 1\}$ for all j.

To see the criterion, let n be fixed and scalars $c_1, \ldots, c_n$ be given. For any $k \leqslant n$ choose t so that $g_k(t) = 1$ and $g_{k+1}(t) = 0$ (if $k < n$). Then $g_j(t) = 1$ for all $j \leqslant k$ and $g_r(t) = 0$ all $r > k$, hence $\left|\sum c_j g_j(t)\right| = \left|\sum_{j=1}^{k} c_j\right|$. Thus $\left\|\sum c_j g_j\right\|_\infty \geqslant \left\|\sum c_j e_j\right\|_{S_e}$.

Now suppose $t \in S$; choose $k \leqslant n$ so that $g_k(t) = 1$ and $g_{k+1}(t) \neq 1$ (if $k < n$). Then $g_j(t) = 1$ for all $j \leqslant k$ and $g_r(t) = 0$ all $r > k+1$ (if $k < n$), so of $k = n$,

$$\left|\sum c_j g_j(t)\right| = \left|\sum_{j=1}^{n} c_j\right|$$

while if $k < n$,

$$\begin{aligned}\left|\sum_{j=1}^{n} c_j g_j(t)\right| &= |c_1 + \ldots + c_k + g_{k+1}(t) c_{k+1}| \\ &= \left|\big(1 - g_{k+1}(t)\big)(c_1 + \ldots + c_k) + g_{k+1}(t)(c_1 + \ldots + c_k + c_{k+1})\right| \\ &\leqslant \max\{|c_1 + \ldots + c_k|, |c_1 + \ldots + c_{k+1}|\} \leqslant \left\|\sum c_j e_j\right\|_{S_e}.\end{aligned}$$

Now by standard arguments, if a sequence $(\tilde{g}_j)$ is a small perturbation of a sequence (g_j) satisfying the criterion, then $(\tilde{g}_j)$ is still equivalent to the summing basis. We shall first describe the construction of normalized weakly sequence (f_n) so that (e_1, e_2) is $1+\varepsilon$-block-represented in every subsequence of (f_n) for every $\varepsilon > 0$.

Let $2 \leqslant k_1 < k_2 < \ldots$ be a strictly increasing sequence of positive integers with $\lim_{j\to\infty} \dfrac{k_j}{k_{j+1}} = 0$. Let $A_1, A_2, \ldots$ be a sequence of infinite disjoint subsets of a set S. Each A_j shall again be decomposed into a sequence of disjoint subsets with a strange enumeration. For each j, let $\mathscr{B}_j = \{F \subset N\colon |F| = k_j \text{ and } j < \min F\}$. Now let $\{A_{j,F}\colon F \in \mathscr{B}_j\}$ be a family of infinite subsets of A_j with $A_{j,F} \cap A_{j,F'} = \emptyset$ if $F \neq F'$. Fix j, $F \in \mathscr{B}_j$ and $x \in A_{j,F}$. We define $f_n(x)$ as follows:

$$f_n(x) = \begin{cases} |F|^{-1/2} & \text{if } n \in F, \\ 1 & \text{if } n = j, \\ 0 & \text{if } n \text{ is not as above.} \end{cases}$$

Now if $x \in A_{j,F}$ for some j and $F \in \mathscr{B}_j$, and $n_1, \ldots, n_k$ are such that $f_{n_i}(x) \neq 0$ for all $1 \leqslant i \leqslant k$, then $k \leqslant k_j + 1$ and $\sum_{i=1}^{k} f_{n_i}(x) \leqslant k/\sqrt{k_j} + 1 \leqslant 2\sqrt{k}$ for k sufficiently large. Hence, there is a constant C so that for all finite sets G, $\left\|\sum_{n\in G} f_n(x)\right\| \leqslant C|G|^{1/2}$, which easily yields that $f_n \to 0$ weakly in $l^\infty(S)$. Trivially $\|f_n\|_{l^\infty(S)} = 1$ for all n.

Now fix j and $F \in \mathscr{B}_j$. Put $g_1 = f_j$, $g_2 = |F|^{-1/2} \sum_{n\in F} f_n$, $\tilde{g}_1 = g_1$ and $\tilde{g}_2 = g_2 \cdot \chi_{A_j}$. If $x \in A_{j,F}$, then $\tilde{g}_2(x) = 1$; if $x \in A_{j,F'}$ where $F' \in \mathscr{B}_j$, then $\tilde{g}_2(x) = |F \cap F'|\,|F|^{-1} \leqslant 1$ and, particular, $\tilde{g}_2(x) = 0$ if $F \cap F' = \emptyset$. Hence $(\tilde{g}_1, \tilde{g}_2)$ satisfies the Summing-Basis Criterion; to complete the discussion of this example, it suffices to show that $\|g_2 - \tilde{g}_2\|$ can be made arbitrarily small if j is large enough. For then given M an infinite subset of N and $\varepsilon > 0$, choosing j a large element of M and $F \in \mathscr{B}_j$ with $F \subset M$ yields (g_1, g_2) $1+\varepsilon$-equivalent to (e_1, e_2). (It follows incidentally that the unconditional constant of any subsequence of (f_n) is greater than or equal to 2).

Since $k_j/k_{j+1} \to 0$, given $\varepsilon > 0$ we may choose J so that $j \geqslant J$ implies $1/k_j^{1/2} + \min\sqrt{k_j/k_{j'}, k_{j'}/k_j} < \varepsilon$ for all $j' \neq j$.

Suppose $j \geqslant J$, $j' \neq j$, $F' \in \mathscr{B}_{j'}$, and $x \in A_{j',F'}$. If $j' < j$,

$$\begin{aligned} g_2(x) &= |F|^{-1/2}|F'|^{-1/2}|F \cap F'| \\ &\leqslant |F|^{-1/2}|F'|^{-1/2} \min\{|F|, |F'|\} = \sqrt{k_{j'}/k_j} < \varepsilon. \end{aligned}$$

If $j < j'$,

$$g_2(x) \leqslant \frac{1}{k_j^{1/2}} + \sqrt{k_j/k_{j'}} < \varepsilon.$$

To obtain a sequence (f_n) such that the first three terms of the summing basis are $1+\varepsilon$-block-represented in every subsequence of (f_n) for every $\varepsilon > 0$, we essentially repeat the above process, thinking of the $A_{j,F}$'s as a denumerable family of disjoint sets. We let M be a lacunary subset of the positive integers; again if $m_1 < m_2 < \ldots$ is an increasing enumeration of M, it is sufficient to assume that $m_j/m_{j+1} \to 0$. Now assume that $\{k_j, k_{j,F}\colon j = 1,2,\dots$ and $F \in \mathscr{B}_j\}$ is a set of distinct elements of M. For each j and $F \in \mathscr{B}_j$, let $\mathscr{B}_{j,F} = \{G \subset N\colon |G| = k_{j,F}$ and $\max F < \min G\}$. Let $\{A_{j,F,G}\colon G \in \mathscr{B}_{j,F}\}$ be a family of infinite subsets of $A_{j,F}$ with $A_{j,F,G} \cap A_{j,F,G'} = \emptyset$ for all $G \neq G'$. Define a new sequence (f_n) on S as follows: Fix j, $F \in \mathscr{B}_j$, $G \in \mathscr{B}_{j,F}$, and $x \in A_{j,F,G}$:

$$f_n(x) = \begin{cases} |G|^{-1/2} & \text{if } n \in G, \\ |F|^{-1/2} & \text{if } n \in F, \\ 1 & \text{if } n = j, \\ 0 & \text{if } n \text{ is not as above.} \end{cases}$$

It may again be verified that if $\varepsilon > 0$ is given and M' is an infinite subset of N, then for $j \in M'$, j large enough; choosing $F \subset M'$ with $F \in \mathscr{B}_j$ and then $G \subset M'$ with $G \in \mathscr{B}_{j,F}$, that (g_1, g_2, g_3) is $1+\varepsilon$-equivalent to the first three terms of the summing basis, where $g_1 = f_j$, $g_2 = |F|^{-1/2}\sum_{n\in F} f_n$, and $g_3 = |G|^{-1/2}\sum_{n\in G} f_n$. (In fact, if $\tilde g_1 = g_1$, $\tilde g_2 = g_2\cdot\chi_{A_j}$, and $\tilde g_3 = g_3\cdot\chi_{A_{j,F}}$, then $(\tilde g_1, \tilde g_2, \tilde g_3)$ satisfies the Summing-Basis Criterion and is a small perturbation of (g_1, g_2, g_3) for j large enough.)

To obtain a weakly null normalized sequence so that the summing basis is finitely block-represented in every subsequence, we consider sets of the form $A_{j,F_2,\dots,F_k}$ where $1 \leqslant k \leqslant j$, $\max F_{r-1} < \min F_r$ for all r, and $A_{j,F_2,\dots,F_{k-1},F}$ is a family of disjoint subsets of $A_{j,F_2\dots F_{k-1}}$ with cardinalities equal to a certain function of $(j, F_2, \dots, F_{k-1})$. As long as these cardinalities are sufficiently lacunary and disjoint for $(j', F'_2, \dots, F'_l) \neq (j, F_2, \dots, F_{k-1})$, the functions (f_n) defined by $f_n(x) = |F_k|^{-1/2}$ if $X \in A_{j,F_2,\dots,F_k}$ and $n \in F_k$, have the desired properties (where $F_1 = \{j\}$ by definition).

We wish now to give purely analytical expressions for the examples. From now on, we let (f_n) denote the usual coordinate functions on the positive integers; that is $f_n(m) = 1$ if $n = m$, $f_n(m) = 0$ if $n \neq m$. Let c_{00} denote the linear span of the f_n's, i. e. the space of all functions on the positive integers with finite support.

For f in c_{00} and g any real-valued function on N, we let $\langle f, g\rangle$ denote the usual inner product of f and g;

$$\langle f, g\rangle = \sum_{n=1}^{\infty} f(n)g(n).$$

Now the first example that we described, as a Banach space, is isometric to the completion of c_{00} under the norm $\|f\| = \sup|\langle f, \chi_{\{j\}} + |F|^{-1/2}\chi_F\rangle|$, the supremum taken over all j and sets F with $|F| = k_j$ and $\min F > j$. Put another way, the map which sends the "f'_n's" of the example to our coordinate f_n's is a linear isometry. We wish to point out in passing that certain features of this example are similar to those of an old example of Schreier [10]. Schreier's space has the property that there is a weakly null sequence (b_n) in it so that for no subsequence $(b_{n_1}, b_{n_2}, \dots)$ does one have $\lim_{k\to\infty}\frac{1}{k}\|\sum_{i=1}^{k} b_{n_i}\| = 0$. Schreier's space is essentially the completion of c_{00} under the norm $\|f\| = \sup_{|F|=\min F}|\langle f, \chi_F\rangle|$ (where "F" ranges over finite subsets of N).

EXAMPLE 1. Let $\mathscr{S}$ be the family of all finite sequences of finite subsets of N;

$$\mathscr{S} = \{(F_1, \dots, F_k)\colon F_i \subset N,\ |F_i| < \infty,\ 1 \leqslant i \leqslant k,\ k = 1, 2, \dots\}.$$

Let $M = \{4^{j^2}\colon j = 1,2,\dots\}$. Let $\psi\colon \mathscr{S} \to M$ be a one-one map. Let $\mathfrak{F}$ be the family of all infinite sequences $(F_1, F_2, \dots)$ of finite subsets of N so that for all $k > 1$,

(a) $|F_1| = 1$;

(b) $\max F_{k-1} < \min F_k$;

(c) $|F_k| = \psi\big((F_1, \dots, F_{k-1})\big)$.

Now define $\|\cdot\|$ on c_{00} by

$$\|f\| = \sup_{(F_j)\in\mathfrak{F}}\Big|\Big\langle f, \sum_{j=1}^{\infty}|F_j|^{-1/2}\chi_{F_j}\Big\rangle\Big|$$

(where for $(F_j)\in\mathfrak{F}$, $\sum_{j=1}^{\infty}|F_j|^{-1/2}\chi_{F_j}$ denotes the function g so that $g(n) = |F_j|^{-1/2}$ if $n \in F_j$ and $g(n) = 0$ if $n \notin \bigcup_{j=1}^{\infty} F_j$).

THEOREM 2.1. *Let B denote the completion of c_{00} under the above norm. Then (f_n) is a normalized weakly null sequence in B so that the summing basis is block-represented in every subsequence of (f_n).*

It is convenient to summarize the properties of M and $\mathfrak{F}$ that we use.

PROPOSITION 2.2. *Let $h(j, k) = \min\{\sqrt{j/k}, \sqrt{k/j}\}$ for j and k positive integers. Then setting $m_j = 4^{j^2}$,*

$$(*) \qquad \sum_{i=1}^{\infty}\sum_{1\leqslant j\neq i} h(m_j, m_i) = c < \tfrac{1}{2}.$$

For any $(F_n)\in\mathfrak{F}$,

(1) *for all k,* (a) *and* (b) *above hold and $|F_{k+1}| \in M$ (by* (c)*);*

(2) *for all k and n, there exists a (G_j) in $\mathfrak{F}$ so that $F_j = G_j$ for all $j \leqslant k$ and $n < \min G_{k+1}$;*

(3) *for any* (G_n) *in* $\mathfrak{F}$ *and any* j *and* k, *if* $|F_{j+1}| = |G_{k+1}|$ *then* $j = k$ *and* $F_i = G_i$ *for all* $i \leqslant k$;

(4) *for any infinite subset* N' *of* N *and* j *in* N', *there exists an* $(F_n) \in \mathfrak{F}$ *with* $\{j\} = F_1$ *and* $F_n \subset N'$ *for all* n.

Proof. We leave the simple verification of all but (4) to the reader. (4) is really also evident; given N' and j in N' define $F_1 = \{j\}$; having defined $F_1, \ldots, F_k$, let F_{k+1} be any finite subset of N' with $|F_{k+1}| = \psi\big((F_1, \ldots, F_k)\big)$ and $\min F_{k+1} > \max F_k$. The sequence (F_n) thus defined by induction is the desired member of $\mathfrak{F}$.

The following simple estimate will be useful in proving Theorem 2.1.

$$(1)\qquad \langle |F|^{-\frac12}\chi_F, |G|^{-\frac12}\chi_G\rangle = |F|^{-\frac12}|G|^{-\frac12}|F\cap G| \leqslant |F|^{-\frac12}|G|^{-\frac12}\min\{|F|, |G|\} = h(|F|, |G|)$$

for any finite non-empty subsets F and G of N (where h is defined in Proposition 2.2).

LEMMA 2.3. *For any* $(F_j) \in \mathfrak{F}$, *any* n, *and any scalars* $c_1, \ldots, c_n$,

$$\sup_k \Big|\sum_{j=1}^k c_j\Big| \leqslant \Big\|\sum_{i=1}^n c_i |F_i|^{-\frac12}\chi_{F_i}\Big\| \leqslant (1+c)\sup_k \Big|\sum_{j=1}^k c_j\Big|.$$

It follows that the summing basis is block-represented in every subsequence of (f_n). For let $(f_n)_{n\in N'}$ be a subsequence. By (4) of Proposition 2.2, we may choose $(F_n) \in \mathfrak{F}$ with $F_n \subset N'$ for all n. The lemma asserts that the block basis $(|F_n|^{-\frac12}\chi_{F_n})$ of $(f_n)_{n\in N'}$ is equivalent to the summing basis.

To prove the lemma, let $k \leqslant n$ and choose by (2) of Proposition 2.2 a (G_j) in $\mathfrak{F}$ satisfying its statement. Then

$$\Big\|\sum_{j=1}^n c_j|F_j|^{-1/2}\chi_{F_j}\Big\| \geqslant \Big|\Big\langle \sum_{j=1}^n c_j|F_j|^{-1/2}\chi_{F_j}, \sum_{j=1}^\infty |G_j|^{-1/2}\chi_{G_j}\Big\rangle\Big| = \Big|\sum_{j=1}^k c_j\Big|.$$

This proves the left inequality of Lemma 2.3.

Now let (G_j) be an arbitrary element of $\mathfrak{F}$ and let s be the largest integer less than or equal to n such that $|F_s| = |G_s|$. It follows from (3) of Proposition 2.2 that then $F_j = G_j$ for all $1 \leqslant j < s$ and moreover if for any $j \leqslant n$ and k arbitrary one has $|F_j| = |G_k|$ then $j = k \leqslant s$. (Also for $j \geqslant s$, $k > s$, the $|F_j|$'s and $|G_k|$'s are distinct members of M.) Hence

$$(2)\qquad \sum_{\substack{j\geqslant s\\ k>s}} h(|F_j|, |G_k|) \leqslant c/2$$

by (1) of Proposition 2.2, since the sum in (2) is a partial sum of the series $(*)$ in Proposition 2.2. (Note that each term $h(m_j, m_i)$ appears twice in $(*)$ and once in (2).) Now

$$\Big|\Big\langle \sum c_j|F_j|^{-1/2}\chi_{F_j}, \sum |G_k|^{-1/2}\chi_{G_k}\Big\rangle\Big| \leqslant \Big|\sum_{j=1}^{s-1} c_j + c_s|F_s|^{-1/2}|G_s|^{-1/2}\langle\chi_{F_s}, \chi_{G_s}\rangle\Big| + \sup_m |c_m| \sum_{\substack{j\geqslant s\\ k>s}} h(|F_j|, |G_k|) \leqslant (1+c)\sup_k \Big|\sum_{j=1}^k c_j\Big|$$

by (1), (2), and a simple inequality in our earlier analysis of the summing basis. This completes the proof of Lemma 2.3.

It remains to show that (f_n) is a normalized weakly null sequence. Now its trivial that $\|f_j\| \leqslant 1$ for any j. Fixing j and choosing (F_n) in $\mathfrak{F}$ (with $F_1 = \{j\}$ by (4) of Proposition 2.2), we have that $|\langle f_j, \sum |F_n|^{-\frac12}\chi_{F_n}\rangle| = 1$, so (f_n) is normalized. The fact that (f_n) is weakly null is an immediate consequence of the following result:

LEMMA 2.4. *For any finite set* $F \subset N$,

$$\|\chi_F\| \leqslant (2+c)\,|F|^{1/2}.$$

Proof. Let $k = |F|$, (G_j) in $\mathfrak{F}$, $j_1, \ldots, j_r$ the integers j so that $G_j \cap F \neq \emptyset$, $s_i = |G_{j_i} \cap F|$ and $a_i = |G_{j_i}|$ for all $1 \leqslant i \leqslant r$. We thus have that the a_i's are distinct members of M, $\sum_{i=1}^r s_i \leqslant k$, $1 \leqslant s_i \leqslant a_i$ for all i, and

$$\Big\langle \chi_F, \sum |G_j|^{-1/2}\chi_{G_j}\Big\rangle = \sum_{i=1}^r a_i^{-1/2}s_i.$$

Let A be the set of i's such that $a_i \leqslant k$. Then if A is non-empty,

$$\sum_{i\in A} a_i^{-1/2}s_i \leqslant \sum_{i\in A} a_i^{1/2} \leqslant (1+c)\max_{i\in A} a_i^{1/2} \quad (\text{by } (*)) \leqslant (1+c)\sqrt{k},$$

by the definition of A. If $\{1, \ldots, r\} \sim A$ is non-empty, then

$$\sum_{i\notin A} a_i^{-1/2}s_i \leqslant \max_{i\notin A} a_i^{-1/2}\Big(\sum s_i\Big) \leqslant \frac{1}{\sqrt{k}}\cdot k = \sqrt{k}.$$

Hence $\sum_{i=1}^r a_i^{-1/2}s_i \leqslant (2+c)\sqrt{k}$. This completes the proof of Lemma 2.4, and thus of Theorem 2.1. (We are indebted to L. Dor for an illuminating discussion concerning the last lemma. Our original proof that (f_n) is weakly null did not contain the more precise information, due to Dor, contained in the lemma. In fact, Dor obtained a more refined analysis which shows that

$$\overline{\lim_{|F|\to\infty}} \| |F|^{-1/2}\chi_F\| = 1.)$$

EXAMPLE 2. Let $\mathfrak{F}$ be as in Example 1. Let $\tilde{\mathfrak{F}}$ be the family of all finite sequences (F_n) of finite subsets of N such that $(F_n) = (F_1, \ldots, F_j)$ where $\{j\} = F_1$ and there exists an $(\tilde{F}_n)\in\mathfrak{F}$ with $F_n = \tilde{F}_n$ for all $1 \leqslant n \leqslant j$. Define a norm $|||\cdot|||$ on c_{00} by $|||f||| = \sup_{(F_n)\in\tilde{\mathfrak{F}}} \left|\left\langle f, \sum |F_n|^{-1/2}\chi_{F_n}\right\rangle\right|$.

Let $\tilde{B}$ denote the completion of c_{00} under the norm $|||\cdot|||$. Then our proof of Theorem 2.1 yields that (f_n) *is a normalized weakly null sequence in* $\tilde{B}$ *so that the summing basis is finitely block-represented in every subsequence of* (f_n). Indeed, Lemma 2.4 holds for the norm $|||\cdot|||$ as well and, of course, the f_n's all have norm one. Given N' an ininite subset of N and k a positive integer, choose $k \leqslant j$ and $(F_1, \ldots, F_j)\in\tilde{\mathfrak{F}}$ with $F_i \subset N'$ for all i and $F_1 = \{j\}$. The proof of Lemma 2.3 shows that $(|F_i|^{-1/2}\chi_{F_i})_{i=1}^{j}$ is a block basis of $(f_n)_{n\in N}$ which is $1+c$-equivalent to the first j terms of the summing basis. It can be demonstrated that $\tilde{B}$ is isometric to a subspace of $C(\omega^{\omega^2}+1)$.

EXAMPLE $2k$. Fix k an integer larger than one and let $\mathfrak{F}_k = \{(F_1, \ldots$ $\ldots, F_k)\colon (F_j)_{j=1}^{\infty}\in\mathfrak{F}\}$. Define $\|\cdot\|^k$ on c_{00} by

$$\|f\|^k = \sup_{(F_n)\in\mathfrak{F}_k} \left|\left\langle f, \sum_{n=1}^{k} |F_n|^{-1/2}\chi_{F_n}\right\rangle\right|.$$

Let B_k be the completion of c_{00} under $\|\cdot\|^k$. Then (f_n) is a normalized weakly null sequence in B_k such that the first k-terms of the summing basis are $1+\varepsilon$-block-represented in every subsequence of (f_n) for every $\varepsilon > 0$. ($\varepsilon > 0$ given, we simply choose N so that $\sum_{i=N}^{\infty}\sum_{0\leqslant j\neq i} h(m_j, m_i) < \varepsilon$ and n_0 such that $\psi(n) \geqslant m_N$ for $n \geqslant n_0$, where h and (m_j) are as in Proposition 2.2. It follows that $(|F_n|^{-1/2}\chi_{F_n})_{n=1}^{k}$ is $1+\varepsilon$-equivalent to $(e_1, \ldots$ $\ldots, e_k)$ in B_k as long as $\min F_1 \geqslant n_0$. This remark also shows that the summing basis is $1+\varepsilon$-block-represented (resp. $1+\varepsilon$-finitely block-represented) in every sequence of (f_n) in B (resp. $\tilde{B}$) for every $\varepsilon > 0$). We shall prove in Theorem 3.2 that B_{k+1} isometrically imbeds in $C(\omega^{\omega\cdot k}+1)$

We note that the particular form of the representability of the first two terms of the summing basis shows that the bimonotone constant of any subsequence of (f_n) in the B_2-norm is equal to 2. That is, for any infinite subset N' of N,

$$\sup_{n, g\in c_{00}(N')} \frac{\|\chi_{[n,\infty[}g\|^2}{\|g\|^2} = 2.$$

EXAMPLE 2'. Define $\|\cdot\|'$ on c_{00} by $\|g\|' = \sup_k \|\chi_{[k,\infty)}g\|^{k+1}$ and let B' be the completion of c_{00} under $\|\cdot\|'$. We shall prove in Theorem 3.2 that B' isometrically imbeds in $C(\omega^{\omega^2}+1)$ and that (f_n) is a normalized weakly null sequence in B' such that the summing basis is finitely block-represented in every subsequence of (f_n).

EXAMPLE 3 (due to W. B. Johnson). Fix $k > 1$, let $\|f\|_2 = \left(\sum |f(n)|^2\right)^{1/2}$ for $f\in c_{00}$, and let $|||f|||^k = \max(\|f\|_2, \|f\|^k)$ for all $f\in c_{00}$ (where $\|\cdot\|^k$ is defined in Example $2k$).

PROPOSITION 2.5. *Let H_k be the completion of c_{00} in the norm $|||\cdot|||^k$. Then H_k is isomorphic to Hilbert space; (f_n) is a normalized weakly null sequence in H_k which is in fact $\sqrt{k}$-equivalent to the usual l^2-basis; yet every subsequence (f_n') of (f_n) has unconditional constant at least as large as $\sqrt{k}/2$.*

Indeed, for any $f\in c_{00}$, $(F_j)\in\mathfrak{F}_k$,

$$\left|\left\langle f, \sum_{j=1}^{k} |F_j|^{-1/2}\chi_{F_j}\right\rangle\right| \leqslant \|f\|_2 \left\|\sum_{j=1}^{k} |F_j|^{-1/2}\chi_{F_j}\right\|_2 = \sqrt{k}\|f\|_2;$$

$$\text{hence} \quad \|f\|^k \leqslant \sqrt{k}\|f\|_2.$$

But then $\|f\|_2 \leqslant |||f|||^k \leqslant \sqrt{k}\,\|f\|_2$. Now let N' be an infinite subset of N and $(F_n)_{n=1}^{k}\in\mathfrak{F}_k$ be such that $F_n \subset N'$ for all n, put $x_k = \sum_{n=1}^{k} |F_n|^{-1/2}\chi_{F_n}$ and $y_k = \sum_{n=1}^{k}(-1)^n|F_n|^{-1/2}\chi_{F_n}$. The inequalities in Lemma 2.3 hold and imply that $\|x_k\|^k \geqslant k$ and $\|y_k\|^k \leqslant 1+c$. Since $\|x_k\|_2 = \sqrt{k} = \|y_k\|_2$, we obtain that $|||x_k|||^k \geqslant k$ and $|||y_k|||^k = \sqrt{k}$. It follows that the unconditional constant of $(|F_n|^{-1/2}\chi_{F_n})_{n=1}^{k}$ and hence of $(f_n)_{n\in N'}$ is at least as large as $\sqrt{k}/2$.

EXAMPLE 4. For a norm $|\cdot|$ on c_{00} we let $|\cdot|^*$ be the dual norm on c_{00} defined by $|f|^* = \sup|\langle f, x\rangle|$, the supremum taken over all x in c_{00} with $|x| \leqslant 1$. Now let $|\cdot|_0$ be the norm on c_{00} given as $\|\cdot\|$ in Example 1 and $|\cdot|_2 = \|\cdot\|_2$, the usual l^2-norm.

Then let $|\cdot|$ be a uniformly convex norm on c_{00} satisfying

$$|f| \leqslant (|f|_0\,|f|_2)^{1/2} \quad \textit{and} \quad |f|^* \leqslant (|f|_0^*\,|f_2|^*)^{1/2}$$

for all $f\in c_{00}$. (Such a norm is obtained using Lions–Peetre interpolation between the norms $|\cdot|_0$ and $|\cdot|_2$; the fact that it is uniformly convex is due to Beauzamy [1].)

THEOREM 2.6. *Let U be the completion of c_{00} under the norm $|\cdot|$ of Example 4. Then U is a uniformly convex Banach space and (f_n) is a normalized weakly null sequence in U having no unconditional subsequence.*

Proof. Since $|f_n| \leqslant 1$ and $|f_n|^* \leqslant 1$, $|f_n| = 1$ for all n. For any finite set F, $|\chi_F| \leqslant \big((2+c)\,|F|^{1/2}|F|^{1/2}\big)^{1/2} \leqslant 2\,|F|$ by Lemma 2.4, hence (f_n) is weakly null. Let N' be an infinite subset of N and choose $(F_n) \in \mathfrak{F}$ with $F_n \subset N'$ for all n. Fix k and define x_k and y_k as in the proof of Proposition 2.5.

Again by Lemma 2.3 we have that $|y_k|_0 \leqslant 1+c$; of course, $|y_k|_2 = \sqrt{k}$, hence $|y_k| \leqslant k^{1/4}\sqrt{1+c}$. Now the definition of $|\cdot|_0$ yields that $|x_k|_0^* \leqslant 1$ and, of course, $|x_k|_2^* = |x_k|_2 = \sqrt{k}$, so $|x_k|^* \leqslant k^{1/4}$. But

$$k = \langle x_k, x_k \rangle \leqslant |x_k|\;|x_k|^* \leqslant k^{1/4}|x_k|;$$

hence $|x_k| \geqslant k^{3/4}$. Thus $|x_k|/|y_k| \geqslant k^{1/2}(1+c)^{-1/2}$; this implies that $(|F_0|^{-1/2}\chi_{F_j})_{j=1}^k$ has an unconditional constant of at least $k^{1/2}(1+c)^{-1/2}2^{-1}$. Since k is arbitrary, $(f_n)_{n \in N'}$ is not unconditional.

Remark. We indicate briefly the following observation of Tzafriri [11]: *No subsequence of (f_n) in U has a closed linear span imbedding into a uniformly convex Banach lattice, or more generally, into a Banach lattice with σ-complete σ-order continuous norm.*

Tzafriri shows that if (g_n) is a normalized sequence in a σ-complete σ-order continuous Banach lattice so that (g_n) has no unconditional subsequence, then

$$\limsup_{n\to\infty} \sup_{\pm} n^{-1/2}\Big\|\sum_{j=1}^{n} \pm g_j\Big\| > 0.$$

Now let $\|\cdot\|$ be the norm of Example 1, and let (s_j) be a sequence of natural numbers so that $\lim_{j\to\infty} m_j/s_j = \lim_{j\to\infty} s_j/m_{j+1} = 0$, where $m_j = 4^{j^2}$ for all j. (For example $(s_j = j\cdot 4^{j^2})$ has this property.) It follows from the proof of Lemma 2.4 that if (F_j) is a sequence of finite sets with $|F_j| = s_j$ for all j, then $\lim_{j\to\infty} |F_j|^{-\frac12}\|\chi_{F_j}\| = 0$. Indeed, let $\varepsilon > 0$; choose J so that $m_j/s_j < \varepsilon^2$ and $s_j/m_{j+1} < \varepsilon^2$ all $j \geqslant J$, and fix F with $k = s_j = |F|$ for some $j \geqslant J$. Now adhering to the notation and terms introduced in the proof of Lemma 2.4,

$$\max_{i\in A} a_i^{1/2} \leqslant m_j^{1/2} < \varepsilon k^{1/2} \quad\text{ and }\quad \max_{i\notin A} a_i^{-1/2} \leqslant m_{j+1}^{-1/2} < \varepsilon k^{-1/2};$$

hence

$$\|\chi_F\| \leqslant 2\varepsilon k^{\frac12} = 2\varepsilon\,|F|^{1/2}.$$

Of course, if $F = \{j_1, \ldots, j_k\}$, then $\sup_{\pm}\Big\|\sum_{i=1}^{k} \pm f_{j_i}\Big\| = \|\chi_F\|$. Hence letting $|\cdot|$ be the norm in U, $\sup_{\pm}\Big|\sum_{i=1}^{k} \pm f_{j_i}\Big| \leqslant \sqrt{2\varepsilon}k^{1/2}$. Thus if $g_n = f_n'$ for some subsequence (f_n') of $|f_n|$,

$$\limsup_{n\to\infty} \sup_{\pm} s_n^{-1/2}\Big|\sum_{j=1}^{s_n} \pm g_j\Big| = 0.$$

§ 3. Normalized weakly null sequences in $C(\alpha)$ for ordinal α. We show in this section that certain of our examples imbed in $C(\omega^{\omega^2}+1)$ while in $C(\omega^{\omega}+1)$ every normalized weakly null sequence has an unconditional subsequence.

We need the following technical result:

LEMMA 3.1 *Let $1 \leqslant n < \infty$.*

(a) *There exists a sequence (A_j) of clopen (closed and open) subsets of ω^n+1 so that*

(i) *any n of the A_j's intersect in a singleton: i. e. for all F with $|F| = n$, $\Big|\bigcap_{j\in F} A_j\Big| = 1$;*

(ii) *no $n+1$ of the A_j's have a common point;*

(iii) *for every isolated point x of ω^n+1, there exists an F with $|F| = n$ and $\bigcap_{j\in F} A_j = \{x\}$.*

(b) *For any sequence (A_j) of non-empty clopen subsets of ω^n+1 with $\chi_{A_j} \to 0$ weakly in $C(\omega^n+1)$ there exists a subsequence (A_j') of (A_j) and a $1 \leqslant k \leqslant n$ so that any k of the A_j''s have a common point, yet no $k+1$ of the A_j''s have a common point.*

Proof (a). We establish this by induction on n. The case $n = 1$ is trivial. Suppose (a) is proved for n. Let $B_1, B_2, \ldots,$ be a sequence of disjoint compact open subsets of ω^{n+1} so that B_j is homeomorphic to ω^n+1 for all j and $\omega^{n+1} = \bigcup_{j=1}^{\infty} B_j$. For each m, let $(A_k^m)_{k=m+1}^{\infty}$ be clopen subsets of B_m satisfying (i)–(iii) (for "A_j" = "A_{m+j}^m" and "ω^n+1" = "B_m"). For each $m = 1, 2, \ldots$, put $A_m = B_m \cup \bigcup_{j=1}^{m-1} A_m^j$.

We claim that (A_m) satisfies (i)–(iii) for $n+1$. It is evident that the A_m's are clopen subsets of $\omega^{n+1}+1$. Let $|F| = n+1$ with $F \doteq m_1, \ldots, m_{n+1}$ and $m_1 < m_j$ for $2 \leqslant j \leqslant n+1$. Then

$$\Big|\bigcup_{i=1}^{n+1} A_{m_i} \cap B_{m_1}\Big| = \Big|\bigcap_{j=2}^{n+1} A_{m_j}^{m_1}\Big| = 1.$$

If $m > m_1$, $A_{m_1} \cap B_m = \emptyset$. If $m < m_1$, then

$$\bigcap_{i=1}^{n+1} A_{m_i} \cap B_m = \bigcap_{i=1}^{n+1} A_{m_i}^m = \emptyset$$

by (ii). Hence $\Big|\bigcap_{i=1}^{n+1} A_{m_i}\Big| = 1$. For any isolated point x of ω^{n+1}, we may choose m_1 with $x \in B_{m_1}$. By (iii) we may choose $m_2, \ldots, m_{n+1}$ with $m_1 < m_j$ for $2 \leqslant j \leqslant n+1$ and $\bigcap_{i=2}^{n+1} A_{m_i}^m = \{x\}$. Thus $\bigcap_{i=1}^{n+1} A_{m_i} = \{x\}$ and (iii) is established. It is also easily seen that (ii) holds for $n+1$.

Remark. It is possible to introduce an ordering $\leqslant$ on $\mathcal{S}^n \overset{\mathrm{df}}{=} \{F \subset N: |F| = n\}$ in such a way that $\mathcal{S}^n$ in the order topology is homeomorphic to the set of isolated points of ω^n+1 in the relative topology. We then put $A_j = \{F \in \mathcal{S}^n: j \in F\}$. It can be shown that with this ordering, the A_j's carry over to an appropriate sequence of clopen subsets of ω^n+1 satisfying (i)–(iii).

Proof. (b) It suffices to prove (by induction) that there is a subsequence (A_j') of the A_j's so that no $n+1$ of the A_j''s have a common point. For then let k be the smallest integer such that there exists a subsequence (A_j'') of (A_j') so that no $k+1$ of the A_j'''s intersect; Choose (A_j'') for this k; if $k = 1$ we are done. Otherwise let C be the family of all $F \subset N$ with $|F| = k$ and $\bigcap_{j\in F} A_j'' \neq \emptyset$. By Ramsey's theorem there exists an infinite subset N' of N so that $F \in C$ for all $F \subset N'$ with $|F| = k$; or so that no $F \in C$ if $F \subset N'$ with $|F| = k$. But the latter is impossible since then $(A_j'')_{j\in N'}$ has the property that no k of the terms of this sequence intersect, contradicting the definition of k.

Suppose now $n = 1$. Since $\chi_{A_j}(\omega) \to 0$, there exists an n_1 so that $\omega \notin A_j$ for all $j \geqslant n_1$, which implies that A_j is finite for all $j \geqslant n_1$. Suppose $n_1 < n_2 < \ldots < n_k$ have been chosen with $A_{n_i} \cap A_{n_{i'}} = \emptyset$ all $1 \leqslant i, i' \leqslant k$, $i \neq i'$. Since $\bigcup_{i=1}^k A_{n_i}$ is finite, there must be infinitely many m's with $A_m \cap \left(\bigcup_{i=1}^k A_{n_i}\right) = \emptyset$; so simply choose n_{k+1} to be one of these m's with $n_{k+1} > n_k$.

Suppose (b) has been proved for n. We may choose $B_1, B_2, \ldots$ compact open subsets of ω^{n+1} so that for each j, $\sup B_j < \inf B_{j+1}$ and B_j is homeomorphic to ω^n, with $\omega^{n+1} = \bigcup_{j=1}^\infty B_j$. Since $\chi_{A_j}(\omega^{n+1}) \to 0$, there exists an n_1 so that $\omega^{n+1} \notin A_j$ for all $j \geqslant n_1$, which implies that each A_j intersects finitely many B_k's for all $j \geqslant n_1$. Let $F = \{k: B_k \cap A_{n_1} \neq \emptyset\}$. Then $A_{n_1} = \bigcup_{k\in F} A_{n_1} \cap B_k$. By applying our induction hypothesis $|F|$ times successively, we may choose an infinite subset M_1 of N with $n_1 < \inf M_1$ so that for each $k \in F_1$, no $n+1$ of the sets $(B_k \cap A_m)_{m\in M_1}$ have a common point. It follows that no $n+1$ of the sets $(A_{n_1} \cap A_m)_{m\in M_1}$ have a common point. It is now evident that we may choose n_j's and infinite subsets M_j of N with $n_k < \inf M_k = n_{k+1}$ and $M_k \supset M_{k+1}$ for all k so that for each k, no $n+1$ of the sets $(A_{n_k} \cap A_m)_{m\in M_k}$ have a common point. It follows that no $n+2$ of the sets $(A_{n_j})_{j=1}^\infty$ have a common point, completing the proof.

Remarks. Choose disjoint compact open subsets B_j of ω^ω with $\omega^\omega = \bigcup_{j=1}^\infty B_j$ and B_j homeomorphic to ω^j+1 for all j. In each B_j choose clopen subsets $(A_k^j)_{k=1}^\infty$ so that any k of these sets have a common point, no $k+1$ have a common point and define $A_m = B_m \cup \bigcap_{j=1}^{m-1} A_m^j$ for all m. Then it is fairly easy to see that the A_m's are compact open subsets of ω^ω with the property that $\chi_{A_m} \to 0$ pointwise (hence (χ_{A_m}) is weakly null in $C(\omega^\omega+1)$, yet $\bigcup_{i=1}^k A_{j_i} \neq \emptyset$ for any $j_1 < \ldots < j_k$ with $j_1 \geqslant k$). In turn, it can be seen that then for any infinite subsequence $j_1 < j_2 < \ldots$,

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{i=1}^n \chi_{A_{j_i}} \right\| > 0.$$

The closed linear span of the χ_{A_m}'s is essentially the same space as the on of Schreier's mentioned earlier; this combinatorial description motivated certain of the constructions which follow.

THEOREM 3.2.

(a) *Let $n \geqslant 1$. There exists a normalized weakly null sequence (f_j) in $C(\omega^{\omega\cdot n}+1)$ such that the first $n+1$ terms of the summing basis are $1+\varepsilon$-block represented in every subsequence of (f_j) for every $\varepsilon > 0$.*

(b) *There exists a normalized weakly null sequence (f_j) in $C(\omega^{\omega^2}+1)$ so that summing basis is finitely block-represented in every subsequence of (f_j).*

Remark. Fix n and let (f_j) be the sequence from part (a). It follows that the unconditional constant of every subsequence of (f_j) is at least equal to $n+1$. We show in Theorem 3.4 that this is the best possible result (up to an arbitrary $\varepsilon > 0$) for $n = 1$. It is known that $C(\omega^{\omega\cdot n}+1)$ is isomorphic to $C(\omega^\omega+1)$ [4]; in fact there is an absolute constant A so that $C(\omega^{\omega\cdot n}+1)$ is $A\cdot n$ isomorphic to $C(\omega^\omega+1)$. Thus by Theorem 3.4, every normalized weakly null sequence (f_j) in $C(\omega^{\omega\cdot n}+1)$ has a subsequence (f_j') which is $A'n$ unconditional for some other absolute constant A'. Thus for general n, (a) yields the best order of magnitude for such an example. Now let (f_n) be the sequence in (b). Then (f_n) has no unconditional subsequence. Since $C(\alpha+1)$ is isomorphic to a subspace of $C(\omega^\omega+1)$ for every $\alpha < \omega^{\omega^2}$ [4], we have that ω^{ω^2} is the least ordinal α for which there exists a normalized weakly null sequence (f_j) in $C(\alpha+1)$ with no unconditional subsequence.

Proof of Theorem 3.2. (a) It suffices to show that the Banach space B_{n+1} constructed as Example $2(n+1)$, isometrically imbeds in $C(\omega^{\omega\cdot n}+1)$. We prove this by induction, beginning with $n = 1$.

Choose $(D_j)_{j=1}^\infty$ a sequence of compact open subsets of ω^ω with $\sup D_j < \inf D_{j+1}$ and D_j homeomorphic to $\omega^{K_j}+1$ for all j, with ω^ω

$= \bigcup_{j=1}^{\infty} D_j$, where $K_j = \psi(\{j\})$ for all j and ψ is defined in Example 1. For each k choose a sequence $(A_j^k)_{j=K_k+1}^{\infty}$ of clopen subsets of D_k satisfying (i)–(iii) of Lemma 3.1 for "n" $= K_k$, "A_j" $= A^k_{K_k+j}$ and "ω^n+1" $= D_k$.

Now define functions f_m on $\omega^\omega+1$ as follows: fix k and $x \in B_k$; put

$$f_m(x) = \begin{cases} K_k^{-1/2} & \text{if } x \in A_m^k \text{ where } m > k; \\ 1 & \text{if } m = k; \\ 0 & \text{otherwise.} \end{cases}$$

It is evident that the f_m's thus defined are continuous functions on $\omega^\omega+1$. Suppose that $g = \sum_{j=1}^{m} c_j f_j$ for some m and scalars $c_1, \ldots, c_m$ and x is an isolated point of $\omega^\omega+1$. Then there exists a k and a set F with $|F| = K_k$ and $k < \min F$ so that $\{x\} = \bigcap_{j\in F} A_j^k$. Putting $F_1 = \{k\}$ and $F_2 = F$, it follows that $g(x) = \sum_{j\in F_1} c_j + |F_2|^{-1/2} \sum_{j\in F_2} c_j$. Since the norm of g equals its supremum on the set of isolated points of $\omega^\omega+1$ and $\bigcap_{j\in F} A_j^k$ is non-empty for any such k and F, this proves that B_2 is isometric to the closed linear span of the f_m's in $C(\omega^\omega+1)$. Let now $f_j^1 = f_j$ for all j.

Now suppose a sequence $(f_j^n)_{j=1}^{\infty}$ of continuous functions on $\omega^{\omega\cdot n}+1$ has been constructed so that there is a one-one correspondence τ between a subset I_n of the isolated points of $\omega^{\omega\cdot n}+1$ and $\mathfrak{F}_{n+1}$ satisfying the following condition: for each $g = \sum_{j=1}^{m} c_j f_j^n$ in the linear span of the f_j^n's and $x \in I_n$, $g(x) = \sum_{k=1}^{n+1} |F_k|^{-1/2} \sum_{j\in F_k} c_j$ where $\tau x = (F_1, \ldots, F_{n+1})$. (These conditions imply that B_{n+1} is isometric to the closed linear span of the f_j^n's in $C(\bar{I}_n)$. Since $C(\bar{I}_n)$ isometrically imbeds in $C(\omega^{\omega\cdot n}+1)$, it follows that B_{n+1} also does. We note that our construction of the f_j^1's satisfies these conditions for I_1 equal to the set of isolated points of $\omega^\omega+1$.)

Let $\{D_y : y \in I_n\}$ be a family of compact open subsets of ω^ω so that for all $y \neq y'$, $D_y \cap D_{y'} = \emptyset$ and for each $y \in I_n$, D_y is homeomorphic to $\omega^{\psi\tau y}+1$.

For each y, let $(A_j^y)_{j=\max F_{n+1}+1}^{\infty}$ be a sequence of clopen subsets of D_y satisfying (i)–(iii) of Lemma 3.1 for "n" $= \psi\tau y$, "A_j" $= A^y_{\max F_{n+1}+j}$ and "ω^n+1" $= D_y$, where $\tau y = (F_1, \ldots, F_{n+1})$.

We now identify $\omega^{\omega\cdot(n+1)}+1$ with $(\omega^\omega+1)\times(\omega^{\omega\cdot n}+1)$ with the cartesian product endowed with the reverse-lexicographic order topology. We define functions f_m^{n+1}, $m = 1, 2, \ldots$ as follows: Let $(x, y) \in \omega^{\omega\cdot(n+1)}+1$. If $y \in I_n$, $(F_1, \ldots, F_{n+1}) = \tau y$, $x \in A_m^y$ and $\max F_{n+1} < m$, let $f_m^{n+1}\big((x, y)\big) = (\psi\tau y)^{-1/2}$; if $y \in I_n$, $\max F_{n+1} < m$, and $x \notin A_m^y$, let $f_m^{n+1}\big((x, y)\big) = 0$; if $y \in I_n$ and $\max F_{n+1} \geq m$ or if $y \notin I_n$, let $f_m^{n+1}\big((x, y)\big) = f_m^n(y)$.

The functions thus defined are continuous; indeed, fix m and let

$$V_m = \{y \in I_n : \max F_{n+1} < m \text{ where } \tau y = (F_1, \ldots, F_{n+1})\}.$$

Then V_m is a finite set and $A_m^y \times \{y\}$ is a clopen subset of $\omega^{\omega\cdot(n+1)}+1$ for each $y \in V_m$. An alternate description of f_m^{n+1} is that $f_m^{n+1} = \sum_{y\in V_m} (\psi\tau y)^{-1/2} \chi_{A_m^y \times \{y\}}$ on $(\omega^\omega+1)\times V_m$ while $f_m^{n+1}\big((x, y)\big) = f_m^{n+1}(y)$ for all $(x, y) \in (\omega^\omega+1)\times(\sim V_m)$.

We now let $I_{n+1} = \{(x, y) : y \in I_n$ and x is an isolated point of $D_y\}$. Let $(x, y) \in I_{n+1}$, $(F_1, \ldots, F_{n+1}) = \tau y$, m and scalars $c_1, \ldots, c_m$ be given and put $g = \sum_{j=1}^{m} c_j f_j^{n+1}$, $\tilde{g} = \sum_{j=1}^{m} c_j f_j^n$. We may choose an F_{n+2} depending uniquely on (x, y) with $|F_{n+2}| = \psi\tau y$ and $\max F_{n+1} < \min F_{n+2}$ such that $\{x\} = \bigcap_{j\in F_{n+1}} A_j^y$. Then $f_j^n(y) = 0$ if $j > \max F_{n+1}$ and $f_j^{n+1}\big((x, y)\big) = 0$ unless $j \in F_{n+2}$ (since $j \notin F_{n+2}$ implies $x \notin A_j^y$), in which case $f_j^{n+1}\big((x, y)\big) = |F_{n+2}|^{-1/2}$. For $j \leq \max F_{n+1}$, $f_j^{n+1}\big((x, y)\big) = f_j^n(y)$, hence

$$g\big((x, y)\big) = \tilde{g}(y) + |F_{n+2}|^{-1/2} \sum_{\substack{j\in F_{n+2} \\ 1\leq j\leq m}} c_j = \sum_{k=1}^{n+2} |F_k|^{-1/2} \sum_{\substack{j\in F_k \\ 1\leq j\leq m}} c_j.$$

For any $(F_1, \ldots, F_{n+2}) \in \mathfrak{F}_{n+1}$ there is a unique (x, y) in I_{n+1} with $\tau y = (F_1, \ldots, F_{n+1})$ and $\{x\} = \bigcap_{j\in F_{n+2}} A_j^y$, thus establishing the desired 1-1 correspondence between I_{n+1} and $\mathfrak{F}_{n+2}$ with the appropriate properties.

To prove (b), let $A_1, A_2, \ldots$ be disjoint compact open subsets of ω^{ω^2} with A_j homeomorphic to $\omega^{\omega\cdot j}+1$ for all j. By part (a), we may choose for each j a sequence $(f_m^j)_{m=1}^{\infty}$ of continuous functions on A_j which is isometrically equivalent in $C(A_j)$ to the coordinate functions (f_m) on N endowed with the $\|\cdot\|^{j+1}$ norm. Now for each n define $\bar{f}_n$ on $\omega^{\omega^2}+1$ by $\bar{f}_n(x) = f_n^j(x)$ if $x \in A_j$ for some $1 \leq j \leq n$ and $\bar{f}_n(x) = 0$ otherwise. It is trivial that the $\bar{f}_n$'s are continuous functions on $\omega^{\omega^2}+1$. Fix n and scalars (c_j) with $c_j = 0$ for all $j > n$; put $\bar{g} = \sum_{j=1}^{\infty} c_j \bar{f}_j$ and $g = \sum_{j=1}^{\infty} c_j f_j$. For an arbitrary k,

$$\|\bar{g}|A_k\|_\infty = \Big\|\sum_{j=k}^{\infty} c_j f_j^k\Big\|_{C(A_k)} = \|\chi_{[k,\infty)}\, g\|^{k+1},$$

the latter equality holding by part (a). Since $\|\bar{g}\| = \sup_k \|\bar{g}|A_k\|_\infty$, we have established that the closed linear span of the $\bar{f}_n$'s in $C(\omega^{\omega^2}+1)$ is isometric to the space B' defined in Example 2′ of § 2.

We thus need only prove the properties of B' asserted in the statement of this example. We shall need the fact that (f_n) is a monotone basis for B_k for all k; i. e. for all $g \in c_{00}$, all k, and all n, $\|\chi_{\{1,\ldots,n\}}g\|^k \leqslant \|g\|^k$, hence

$$\|\chi_{[n,\infty)}g\|^k \leqslant 2\,\|g\|^k.$$

This fact is easily deduced from the following property of $\mathfrak{F}$, which is slightly stronger than (2) of Proposition 2.2: for all k,n,m and $(F_j) \in \mathfrak{F}$ there exists a (G_j) in $\mathfrak{F}$ with $F_j = G_j$ for all $j \leqslant k$, $G_{k+1} \cap \{1, \ldots, m\} = F_{k+1} \cap \{1, \ldots, m\}$ and $n < \min G_{k+1} \cap \{m+1, m+2, \ldots\}$.

Now its easily seen that $\|f_n\|' = 1$ for all n and the proof of Lemma 2.4 shows that $\|\chi_F\|^k \leqslant (2+c)\,|F|^{1/2}$ for all finite subsets F of N, whence $\|\chi_F\|' \leqslant (4+2c)\,|F|^{1/2}$ for all such F. Thus (f_n) is a normalized weakly null sequence in B'.

We now need the following technical result:

Lemma 3.3. *There is an absolute constant K so that for all $n > 1$ and (F_l) in $\mathfrak{F}$,*

$$\sum_{l=n+1}^{\infty} \|\,|F_l|^{-1/2}\chi_{F_l}\|^n \leqslant K.$$

Proof. Let $(F_l) \in \mathfrak{F}$, fix n and $l > n$, let $F = F_l$, put $k = |F_l|$, choose i so that $k = m_i = 4^{i^2}$ and let $(G_j)_{j=1}^n$ in $\mathfrak{F}_n$. Now $k \notin \psi(\mathfrak{F}_{n-1})$. Thus adhering to the notation of the proof of Lemma 2.4, and observing that $a_j \leqslant m_{i-}$ or $a_j \geqslant m_{i+1}$ for all j,

$$\sum_{j\in A} a_j^{-1/2}s_j \leqslant (1+c)\max_{j\in A} a_j^{1/2} \leqslant (1+c)m_{i-1}^{1/2}$$

(using (*) of Proposition 2.2) and

$$\sum_{j\notin A} a_j^{-1/2}s_j \leqslant \max_{j\notin A} a_j^{-1/2}\Big(\sum_j s_j\Big) \leqslant m_{i+1}^{-1/2}m_i.$$

Thus

$$\|\,|F|^{-1/2}\chi_F\|^n \leqslant (1+c)\frac{m_{i-1}^{1/2}}{m_i^{1/2}} + \frac{m_i^{1/2}}{m_{i+1}^{1/2}}.$$

Since $|F_l| \neq |F_{l'}|$ if $l \neq l'$, the result follows from (*) of Proposition 2.2 if we simply put $K = c^2+2c$.

We are now prepared to complete the proof of Theorem 3.2. Let N' be an infinite subset of N, n an integer larger than one, and choose (F_l) in $\mathfrak{F}$ so that $F_l \subset N'$ for all l and $j \geqslant n$ where $\{j\} = F_1$. It follows from the proof of Lemma 2.3 that $(|F_l|^{-1/2}\chi_{F_l})_{l=1}^n$ is $1+c$-equivalent to the firs n terms of the summing basis in the $\|\cdot\|^k$-norm for all $k \geqslant n$. Let now scalars $c_1, \ldots, c_n$ be given with $\sup_k \big|\sum_{l=1}^k c_j\big| = 1$ and put $g = \sum_{l=1}^n c_l|F_l|^{-1/2}\chi_{F_l}$. It follows by our choice of (F_l) that

$$\|g\|' \geqslant \|\chi_{[n-1,\infty)}g\|^n = \|g\|^n \geqslant 1.$$

Using the monotonicity of the basis (f_n) in $\|\cdot\|^k$ and the above-mentioned summing-basis-equivalence,

$$\|\chi_{[k-1,\infty[}g\|^k \leqslant 2\|g\|^k \leqslant 2+2c \quad \text{for all} \quad k \geqslant n.$$

Finally let $1 < k < n$. Again using Lemma 2.3 and also Lemma 3.3,

$$\|\chi_{[k-1,\infty)}g\|^k \leqslant 2(1+c)\sup_{l\leqslant k}\Big|\sum_{j\leqslant 1}^{l} c_j\Big| + \sup|c_l|\sum_{l>k}\|\,|F_l|^{-1/2}\chi_{F_l}\|^k \leqslant 2(1+c)+2K.$$

Thus $\|g\|' \leqslant 2(1+c)+2K$, so $(|F_l|^{-1/2}\chi_{F_l})_{l=1}^n$ is $2(1+c)+2K$-equivalent to the first n terms of the summing basis in B'. This completes the proof of Theorem 3.2.

Our final result gives positive results for normalized weakly null sequences in $C(\alpha+1)$ for small ordinals α.

Theorem 3.4. *Let α be an ordinal, let (f_n) be a normalized weakly null sequence in $C(\alpha+1)$ and let $\varepsilon > 0$.*

(a) *If $\alpha < \omega^\omega$, (f_n) has a subsequence (f_n') with unconditional constant at most $1+\varepsilon$.*

(b) *If $\alpha = \omega^\omega$, (f_n) has a subsequence (f_n') with unconditional constant at most $2+\varepsilon$.*

Proof. From now one we deal with infinite sequences. We require the following standard

Perturbation result (cf. [3]): *Let $K < \infty$ and (f_n) and (g_n) be infinite sequences in a Banach space with (f_n) semi-normalized, (g_n) K-unconditional and $\sum\|f_n - g_n\| < \infty$. Then for every $\varepsilon > 0$ there exists an m so that $(f_n)_{n=m}^\infty$ is $(K+\varepsilon)$-unconditional.*

Now let (f_n) be as in Theorem 3.4. Since $\alpha+1$ is a countable set and (f_n) tends to zero pointwise, there exists an increasing sequence (λ_n) of positive numbers with $\lambda_n \to \infty$ so that $(\lambda_n f_n)$ tends to zero pointwise. By passing to a subsequence if necessary we may assume that

(1) $$\sum \lambda_n^{-1} < \infty.$$

Since $\alpha+1$ is a totally disconnected space, we may choose clopen subsets E_n of $\alpha+1$ so that

$$|f_n|^{-1}\big([\lambda_{n-1}^{-1},\infty)\big) \subset E_n \subset |f_n|^{-1}\big((\lambda_n^{-1},\infty)\big)$$

for all $n > 1$. Then there is no x in $\alpha+1$ with $x \in E_n$ for infinitely many n's, since $\lambda_n f_n(x) \nrightarrow 0$ for such an x. Hence

(2) $$\chi_{E_n} \to 0 \text{ weakly in } C(\alpha+1).$$

Now set $g_n = f_n\chi_{E_n}$ for all n. Then since $x \notin E_n$ implies $|f_n(x)| < \lambda_{n-1}^{-1}$, $\|f_n - g_n\| < \lambda_{n-1}^{-1}$ for all $n > 1$, so by (1) we have

(3) $$\sum\|g_n - f_n\| < \infty.$$

Now assume (a) and choose $M < \infty$ so that $a \leqslant \omega^M$. Since $a+1$ is a clopen subset of ω^M, we may apply (b) of Lemma 3.1 to choose a subsequence (g_n') of the g_n's so that no $M+1$ of the corresponding E_n''s have a common point. Now let $\varepsilon > 0$. We may choose continuous functions h_n on $a+1$ and a finite subset G of the real numbers so that for all n,

$$h_n(a+1) \subset G, \qquad |h_n|^{-1}(0, \infty) \subset E_n' \tag{4}$$

and $\|h_n - g_n'\| < \varepsilon/M$.

It follows that for all sequences of scalars (c_j) with only finite many non-zero terms and any x in $a+1$, that

$$\Big|\sum_j c_j(h_j - g_j')(x)\Big| = \Big|\sum_{j\in N_x} c_j(h_j - g_j')(x)\Big|$$

$$\leqslant \sup |c_j|\, |N_x| \frac{\varepsilon}{M} \leqslant \sup |c_j|\,\varepsilon,$$

where $N_x = \{n \in N\colon x \in E_n'\}$. Thus we have

$$\Big\|\sum c_j(h_j - g_j')\Big\| \leqslant \sup_j |c_j|\,\varepsilon. \tag{5}$$

We now appeal to the results of [9], which yield that (h_j) has a $(1+\varepsilon)$-unconditional subsequence (h_j'). It then follows from (5), (3) and the standard perturbation result that (f_n) has a $(1+\varepsilon)^3$-unconditional subsequence. Since (h_j) is a semi-normalized and weakly null, it follows from the results of [9] that there exists a subsequence (h_j') of (h_j) so that for any n, scalars $c_1, \ldots, c_n$, and finite set F with $|F| \leqslant M$,

$$\Big(\sup_x \sum_{j\in F} c_i h_j'(x)\Big)^+ \leqslant (1+\varepsilon) \sup_x \Big(\sum_{j=1}^n c_j h_j'(x)\Big)^+ \tag{6}$$

(f^+ denotes the maximum of f and 0). But then (h_j') is itself $(1+\varepsilon)$-unconditional. Indeed, fix n, scalars $c_1, \ldots, c_n$, F a subset of $\{1, \ldots, n\}$ and $x \in a+1$. Then since $h_n^{-1}(0, \infty) \subset E_n'$ for all n, letting $F' = \{j \leqslant n\colon h_j'(x) \neq 0\}$, $|F'| < M$, hence

$$\Big|\sum_{j\in F} c_j h_j'(x)\Big| \leqslant \Big\|\sum_{j\in F'} c_j h_j'\Big\| \leqslant (1+\varepsilon)\Big\|\sum_{j=1}^n c_j h_j'\Big\|;$$

the arbitrariness of x thus yields that

$$\Big\|\sum_{j\in F} c_j h_j'\Big\| \leqslant (1+\varepsilon)\Big\|\sum c_j h_j'\Big\|.$$

We pass now to the proof of Theorem 3.4 (b). The E_n's chosen in the first part of our argument have the property that $\omega^\omega + 1$ belongs to at most finitely many of them. We may thus assume that E_n is a compact subset of ω^ω for all n. We then choose simple continuous $\bar{g}_n$'s supported on the E_n's so that $\sum \|\bar{g}_n - f_n\| < \infty$ with $\|\bar{g}_n\| \leqslant 1$ for all n. By the standard perturbation result, it suffices to show that the $\bar{g}_n$'s have a $(1+\varepsilon)$-unconditional subsequence for any $\varepsilon > 0$. Thanks to the statement containing (6), our proof of Theorem 3.4 (a) yields that for any compact subset A of ω^ω, any semi-normalized weakly null sequence (h_n) in $C(A)$ and any $\varepsilon > 0$, there exists a subsequence (h_n') so that for all n, $c_1, \ldots, c_n$ and $F \subset \{1, \ldots, n\}$,

$$\Big(\sup_{x\in A} \sum_{j\in F} c_j h_j(x)\Big)^+ \leqslant (1+\varepsilon)\Big(\sup_{x\in A} \sum_{j=1}^n c_j h_j(x)\Big)^+. \tag{7}$$

Now fix $\varepsilon > 0$. By a standard result of Bessaga and Pełczyński [3], we may assume without loss of generality that $(\bar{g}_n)$ is already a $(1+\varepsilon)$-monotone basic sequence.

We now choose infinite subsets $N = M_1, M_2, \ldots$ of N so that setting $m_i = \min M_i$ then for all i, $m_i < m_{i+1}$, $M_i \supset M_{i+1}$ and for any set A in the Boolean ring generated by $\{\bar{g}_j^{-1}(c)\colon c \neq 0 \text{ and } j \leqslant i\}$, either $\{\bar{g}_j | A \cdot j \in M_{i+1}\}$ satisfies the statement containing (7) or $\sum_{j\in M_{i+1}} \|\bar{g}_j | A\| < \varepsilon$.

Finally, set $h_j = \bar{g}_{m_j}$ for all j. We shall show that the h_j's are almost two-unconditional. Fix n, scalars $c_1, \ldots, c_n$ and F a subset of $\{1, \ldots, n\}$. We may assume without loss of generality that there is an x so that

$$1 = \Big\|\sum_{j\in F} c_j h_j\Big\| = \Big|\sum_{j\in F} c_j h_j(x)\Big|.$$

We shall show that

$$(2 + 4\varepsilon + 2\varepsilon^2)^{-1} \leqslant \Big\|\sum c_j h_j\Big\|, \tag{8}$$

which is certainly enough to complete the proof.

Let k be the least integer so that $h_j(x) \neq 0$ and set

$$A = \{y \in \omega^\omega\colon h_j(y) = 0 \text{ for all } j < k \text{ and } h_k(y) = h_k(x)\}.$$

A is thus a compact subset of ω^ω containing x. Now if

$$\sum_{j>k} \|h_j | A\|_\infty < \varepsilon, \tag{9}$$

then

$$1 = \Big|\sum_{j\in F} c_j h_j(x)\Big| \leqslant |c_k h_k(x)| + \sup_j |c_j|\,\varepsilon \leqslant (2 + 4\varepsilon + 2\varepsilon^2)\Big\|\sum_{j=1}^n c_j h_j\Big\|.$$

Now suppose (9) does not hold; let $f = \sum_{\substack{j\in F\\ j>k}} c_j h_j$ and $g = \sum_{j=k+1}^n c_j h_j$. If $k \notin F$ then

$$\Big|\sum_{j\in F} c_j h_j(x)\Big| = |f(x)| \leqslant (1+\varepsilon)\,\|g \mid A\|_\infty$$

(by the statement containing (7) and the construction of the (h_j's))

$$\leqslant (1+\varepsilon)\,\|g\| \leqslant 2(1+\varepsilon)\Big\|\sum_{j=1}^{n} c_j h_j\Big\|$$

by the $(1+\varepsilon)$-monotonicity of the h_j's. (In reality, it is only in the very last inequality that the constant "2" enters in a crucial way.)

Finally, suppose $k\in F$. Then if $c_k h_k(x)$ and $f(x)$ are of opposite sign, either $|c_k h_k(x)|$ or $|f(x)|$ is larger than $\big|\sum_{j\in F} c_j h_j(x)\big|$. But

$$\max\{|c_k h_k(x)|,\ |f(x)|\} \leqslant 2(1+\varepsilon)\Big\|\sum_{j=1}^{n} c_j h_j\Big\|$$

by the above argument. Now suppose $c_k h_k(x)$ and $f(x)$ are of the same sign; we may suppose both are positive. Then by (7) there exists a $y\in A$ with $f(x)\leqslant (1+\varepsilon)\,g(y)$. By the definition of A,

$$\begin{aligned}\Big|\sum_{j\in F} c_j h_j(x)\Big| &= c_k h_k(x)+f(x) \leqslant c_k h_k(x)+(1+\varepsilon)\,g(y)\\ &= c_k h_k(y)+(1+\varepsilon)\,g(y) \leqslant (1+\varepsilon)\big(c_k h_k(y)+g(y)\big)\\ &= (1+\varepsilon)\sum_{j=1}^{n} c_j h_j(y) \leqslant (1+\varepsilon)\Big\|\sum_{j=1}^{n} c_j h_j\Big\|.\end{aligned}$$

This completes the proof.

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Received March 9, 1976 (1128)