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Existence, uniqueness and continuous dependence [on the parameter of solutions of a system of differential equations with deviating argument

by S. Czerwik (Katowice)

Abstract. This paper is concerned with the problem of the existence, uniqueness and continuous dependence on parameter of solutions of the initial-value problem

$$\varphi_i'(x) = h_i(x, \varphi_1[f_{i,1}(x)], \ldots, \varphi_n[f_{i,n}(x)], \varphi_1'[g_{i,1}(x)], \ldots, \varphi_n'[g_{i,n}(x)], u),$$
$$\varphi_i(0) = \theta,\ i = 1, \ldots, n$$

(θ denotes the zero of a Banach space). The functions φ_i, $i = 1, \ldots, n$, are unknown functions belonging to a special function class, $h_i, f_{i,k}, g_{i,k}$, $i, k = 1, \ldots, n$, are known functions and u is a real parameter. The proof is based on the contraction principle for Lipschitz transformations due to J. Matkowski.

1. Introduction. In the present paper we consider the problem of the existence and uniqueness of solutions (assuming values in a Banach space) of the system of functional-differential equations with a parameter u

$$\varphi_1'(x) = h_i\big(x, \varphi_1[f_{i,1}(x)], \ldots, \varphi_n[f_{i,n}(x)], \varphi_1'[g_{i,1}(x)], \ldots, \varphi_n'[g_{i,n}(x)], u\big),\quad i = 1, \ldots, n, \tag{1}$$

with the initial condition

$$\varphi_i(0) = \theta, \quad i = 1, \ldots, n, \tag{2}$$

in the interval $I = \langle 0, \infty)$, where θ is the zero of the Banach space (a general initial condition $\varphi_i(x_0) = e_i$ can be reduced to the above one).

The functions φ_i, $i = 1, \ldots, n$, are unknown functions belonging to a certain functions class G, which is defined below and h_i, f_{ik}, g_{ik}, $i, k = 1, \ldots, n$, are known functions; u is a real parameter.

For $i = 1$, the corresponding problem has been investigated in papers [1]–[6].

In this paper we shall give certain conditions ensuring the existence, uniqueness and continuous dependence on the parameter of solutions of the initial-value problem (1)–(2).

2. Preliminaries. We define the numbers a^r_{ik}, $i, k = 1, \ldots, n$, $r = 1, \ldots, n$ as follows:

(3) $$a^1_{ik} = \begin{cases} a_{ik}, & i \neq k, \\ 1 - a_{ik}, & i = k, \end{cases} \quad i, k = 1, \ldots, n,$$

(4) $$a^{r+1}_{ik} = \begin{cases} a^r_{1,1} a^r_{i+1,k+1} + a^r_{i+1,1} a^r_{1,k+1}, & i \neq k, \\ a^r_{1,1} a^r_{i+1,k+1} - a^r_{i+1,1} a^r_{1,k+1}, & i = k, \end{cases}$$

$$r = 1, \ldots, n-1, \ i, k = 1, \ldots, n-r.$$

Let

(5) $$a^r_{ii} > 0, \quad r = 1, \ldots, n, \ i = 1, \ldots, n+1-r.$$

The following fixed-point theorem is known (cf. [8]).

THEOREM 1. *Let (X_i, d_i), $i = 1, \ldots, n$ be complete metric spaces. Suppose that the transformations T_i: $X_1 \times \ldots \times X_n \to X_i$, $i = 1, \ldots, n$, fulfil the conditions*

(6) $$d_i[T_i(x_1, \ldots, x_n), T_i(y_1, \ldots, y_n)] \leqslant \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k),$$

$$x_k, y_k \in X_k, \ i, k = 1, \ldots, n,$$

where $a_{ik} > 0$. If the numbers a^r_{ik}, $r = 1, \ldots, n$, $i, k = 1, \ldots, n+1-r$ defined by (3) *and* (4) *fulfil inequalities* (5), *then the system of equations*

(7) $$x_i = T_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n,$$

has exactly one solution $x_i \in X_i$, $i = 1, \ldots, n$. Moreover,

$$x_i^{\mathsf{1}} = \lim_{k\to\infty} x_{ik}, \quad i = 1, \ldots, n,$$

where $x_{i,0} \in X_i$, $i = 1, \ldots, n$, are arbitrarily chosen and

$$x_{i,k+1} = T_i(x_{1,k}, \ldots, x_{n,k}), \quad i = 1, \ldots, n, \ k = 0, 1, 2, \ldots$$

3. Existence and uniqueness. In this section we establish a theorem on the existence of a unique solution of the initial-value problem (1)–(2). We assume the following:

HYPOTHESIS 1.

(i) Let $(B, \|\cdot\|)$ be a Banach space. The functions h_i: $I \times B^{2n} \times R \to B$, $I = \langle 0, \infty)$, $R = (-\infty, +\infty)$, $i = 1, \ldots, n$, are continuous in $I \times B^{2n} \times R$.

(ii) There exist functions L_{ik}: $I \to (0, \infty)$, P_{ik}: $I \to (0, \infty)$, $i, k = 1, \ldots, n$ such that for every $(x_1, \ldots, x_n, u_1, \ldots, u_n)$, $(z_1, \ldots, z_n, v_1, \ldots, v_n) \in B^{2n}$, $x \in I$, and $u \in R$, we have

$$\|h_i(x, x_1, \ldots, x_n, u_1, \ldots, u_n, u) - h_i(x, z_1, \ldots, z_n, v_1, \ldots, v_n, u)\|$$
$$\leqslant \sum_{k=1}^{n} L_{ik}(x)\|x_k - z_k\| + \sum_{k=1}^{n} P_{ik}(x)\|u_k - v_k\|.$$

(iii) Functions f_{ik}, g_{ik}: $I \to I$ are continuous in I.

(iv) Let L be a positive number. For every $u \in R$ there exist constants $N_i > 0$ and $(r_{1,i}, \ldots, r_{n,i}, p_{1,i}, \ldots, p_{n,i}) \in B^{2n}$, $i = 1, \ldots, n$ such that the following inequalities hold:

$$J_i = \|h_i(x, r_{1,i}, \ldots, r_{n,i}, p_{1,i}, \ldots, p_{n,i}, u)\| \leqslant N_i \exp(Lx),$$
$$x \in I, \ i = 1, \ldots, n.$$

(v) There exist constants b_{ik}, c_{ik}, i, $k = 1, \ldots, n$, such that

$$L^{-1} L_{ik}(x) \exp\big(Lf_{ik}(x)\big) \leqslant b_{ik} \exp(Lx),$$
$$P_{ik}(x) \exp\big(Lg_{ik}(x)\big) \leqslant c_{ik} \exp(Lx),$$
$$x \in I, \ i, \ k = 1, \ldots, n.$$

Next, we define G as the space of those functions φ: $I \to B$, which are continuous in I and satisfy

$$\|\varphi(x)\| = 0\big(\exp(Lx)\big) \ \text{(1)}. \tag{8}$$

In the space G we define the norm (cf. [1])

$$|\varphi| = \sup_{x \in I} \big(\|\varphi(x)\| \exp(-Lx)\big). \tag{9}$$

Now we shall verify that G with the norm (9) is a Banach space.

Let $\{\varphi_k\}$ be a Cauchy sequence of elements of G and take an $\varepsilon > 0$. There exists a positive integer N such that for k, $m \geqslant N$

$$|\varphi_k - \varphi_m| < \varepsilon. \tag{10}$$

For $x \in \langle 0, d\rangle$, where $d > 0$, we have

$$\|\varphi_k(x) - \varphi_m(x)\| \leqslant \sup_{x \in \langle 0, d\rangle} [\exp(Lx)\|\varphi_k(x) - \varphi_m(x)\| \exp(-Lx)]$$
$$\leqslant K|\varphi_k - \varphi_m| \leqslant K\varepsilon, \quad k, m \geqslant N, \ K = \exp(Ld),$$

and consequently $\{\varphi_k\}$ converges uniformly in $\langle 0, d\rangle$ to a function φ. The number $d > 0$ being arbitrary, it follows that φ is continuous in I. Also, $\varphi(x) \in B$ for $x \in I$. Letting $k \to \infty$ in (10), we see that $|\varphi - \varphi_m| \leqslant \varepsilon$ and next from the inequality

$$\big||\varphi| - |\varphi_m|\big| \leqslant |\varphi - \varphi_m| \leqslant \varepsilon$$

(1) I.e., there exists a constant $M > 0$ such that $\|\varphi(x)\| < M\exp(Lx)$, $x \in I$.

we have

$$|\varphi| \leqslant |\varphi_m| + \varepsilon < \infty.$$

Therefore $\varphi \in G$ and so G is a Banach space.

We shall prove:

THEOREM 2. *Let Hypothesis* 1 *be fulfilled and let the numbers* a^r_{ik}, $i, k = 1, \ldots, n$, $r = 1, \ldots, n$, *defined by* (3) *and* (4), *where* $a_{ik} = b_{ik} + c_{ik}$, *fulfil inequalities* (5). *Then, for every* $u \in R$, *the initial-value problem* (1)–(2), *where* θ *is the zero of* B, *has exactly one solution* $\varphi_i \in G$, $i = 1, \ldots, n$, *given as the limit of successive approximations.*

Proof. Let $u \in R$ be fixed. We assume $\varphi_i'(x) = v_i(x)$. Then the system of equations (1) with the initial condition (2) is equivalent to the system of equations

$$v_i(x) = h_i\Big(x, \int\limits_0^{f_{i1}(x)} v_1(s)\,ds, \ldots, \int\limits_0^{f_{in}(x)} v_n(s)\,ds, v_1[g_{i1}(x)], \ldots, v_n[g_{in}(x)], u\Big),$$
$$i = 1, \ldots, n.$$

We define transformations $T_i\colon X_1 \times \ldots \times X_n \to X_i$, where $X_i = G$, $i = 1, \ldots, n$, by $T_i(\varphi_1, \ldots, \varphi_n) = \Phi_i$ for $\varphi_i \in G$, $i = 1, \ldots, n$, where

$$(11)\quad \Phi_i(x)$$
$$= h_i\Big(x, \int\limits_0^{f_{i1}(x)} \varphi_1(s)\,ds, \ldots, \int\limits_0^{f_{in}(x)} \varphi_n(s)\,ds, \varphi_1[g_{i1}(x)], \ldots, \varphi_n[g_{in}(x)], u\Big),$$
$$i = 1, \ldots, n.$$

We shall prove that if $\varphi_i \in G$, then also $\Phi_i \in G$, $i = 1, \ldots, n$. By Hypothesis 1, Φ_i are continuous in I and $\Phi_i(x) \in B$ for $x \in I$, $i = 1, \ldots, n$. Next, from (ii)–(v) we obtain:

$$\|\Phi_i(x)\|$$
$$\leqslant \Big\|h_i\Big(x, \int\limits_0^{f_{i1}(x)} \varphi_1(s)\,ds, \ldots, \int\limits_0^{f_{in}(x)} \varphi_n(s)\,ds, \varphi_1[g_{i1}(x)], \ldots, \varphi_n[g_{in}(x)], u\Big) -$$
$$- h_i(x, r_{1,i}, \ldots, r_{n,i}, p_{1,i}, \ldots, p_{n,i}, u)\Big\| + J_i$$
$$\leqslant \sum_{k=1}^{n} L_{ik}(x)\Big\|\int\limits_0^{f_{ik}(x)} \varphi_k(s)\,ds - r_{k,i}\Big\| + \sum_{k=1}^{n} P_{ik}(x)\|\varphi_k[g_{ik}(x)] - p_{k,i}\| + J_i.$$

If $\varphi_i \in G$, $i = 1, \ldots, n$, then also $\int\limits_0^{f_{ik}(x)} \varphi_k(s)\,ds$, $i, k = 1, \ldots, n$, are functions belonging to G. Next,

$$\|\Phi_i(x)\| \leqslant \sum_{k=1}^{n} L_{ik}(x)\left|\int\limits_0^{f_{ik}(x)} \varphi_k(s)\,ds - r_{k,i}\right| \exp\big(Lf_{ik}(x)\big) +$$
$$+\sum_{k=1}^{n} P_{ik}(x)\,|\varphi_k - p_{k,i}|\exp\big(Lg_{ik}(x)\big) + J_i$$
$$\leqslant \exp(Lx)\sum_{k=1}^{n}(Lb_{ik}+c_{ik})\,d_{ik} + J_i,$$

where

$$d_{ik} = \max\left(\left|\int\limits_0^{f_{ik}(x)} \varphi_k(s)\,ds - r_{k,i}\right|,\ |\varphi_k - p_{k,i}|\right), \qquad i,\ k = 1, \ldots, n.$$

Hence from (iv) and (v) we get

$$\|\Phi_i(x)\| \leqslant \exp(Lx)\sum_{k=1}^{n} ca_{ik}d_{ik} + J_i$$
$$\leqslant \exp(Lx)\Big(\sum_{k=1}^{n} ca_{ik}d_{ik} + N_i\Big),$$

where

$$c = \max(L, L^{-1}).$$

It follows that $\Phi_i \in G$, $i = 1, \ldots, n$.

Now assume that $\varphi_i, \psi_i \in G$, $i = 1, \ldots, n$, and let $\Phi_i = T_i(\varphi_1, \ldots, \varphi_n)$, $\Psi_i = T_i(\psi_1, \ldots, \psi_n)$. From (ii), (iii) and (v) we have the following inequalities:

$$\|\Phi_i(x) - \Psi_i(x)\|$$
$$\leqslant \Big\|h_i\Big(x, \int\limits_0^{f_{i1}(x)} \varphi_1(s)\,ds, \ldots, \int\limits_0^{f_{in}(x)} \varphi_n(s)\,ds, \varphi_1[g_{i1}(x)], \ldots, \varphi_n[g_{in}(x)], u\Big) -$$
$$- h_i\Big(x, \int\limits_0^{f_{i1}(x)} \psi_1(s)\,ds, \ldots, \int\limits_0^{f_{in}(x)} \psi_n(s)\,ds, \psi_1[g_{i1}(x)], \ldots, \psi_n[g_{in}(x)], u\Big)\Big\|$$
$$\leqslant \sum_{k=1}^{n} L_{ik}(x)\int\limits_0^{f_{ik}(x)} \|\varphi_k(s) - \psi_k(s)\|\,ds + \sum_{k=1}^{n} P_{ik}(x)\|\varphi_k[g_{ik}(x)] - \psi_k[g_{ik}(x)]\|$$
$$\leqslant \sum_{k=1}^{n} L_{ik}(x)\,|\varphi_k - \psi_k|\int\limits_0^{f_{ik}(x)} \exp(Ls)\,ds + \sum_{k=1}^{n} P_{ik}(x)\,|\varphi_k - \psi_k|\exp\big(Lg_{ik}(x)\big)$$
$$\leqslant \sum_{k=1}^{n} L_{ik}(x)\,|\varphi_k - \psi_k|\,L^{-1}\exp\big(Lf_{ik}(x)\big) + \sum_{k=1}^{n} c_{ik}\,|\varphi_k - \psi_k|\exp(Lx)$$
$$\leqslant \exp(Lx)\sum_{k=1}^{n}(b_{ik}+c_{ik})\,|\varphi_k - \psi_k| \leqslant \exp(Lx)\sum_{k=1}^{n} a_{ik}\,|\varphi_k - \psi_k|.$$

Consequently

$$|\Phi_i - \Psi_i| \leqslant \sum_{k=1}^{n} a_{ik} |\varphi_k - \psi_k|. \tag{12}$$

Thus relation (6) is fulfilled and so from (5) and Theorem 1 we obtain the assertion of Theorem 2.

4. Continuous dependence on the parameter. In this section we prove the continuous dependence of the solution of problem (1)–(2) on the parameter u.

We assume the following

HYPOTHESIS 2. There exist constants M_i and functions A_i, ω_i: $I \to I$, $i = 1, \ldots, n$, $\omega_i(u) \to 0$ as $u \to 0+$, $i = 1, \ldots, n$, such that for every $x \in I$, $u_1, u_2 \in R$, $(y_1, \ldots, y_{2n}) \in B^{2n}$

$$\|h_i(x, y_1, \ldots, y_{2n}, u_1) - h_i(x, y_1, \ldots, y_{2n}, u_2)\| \leqslant A_i(x)\,\omega_i(\|u_1 - u_2\|)$$

and

$$\sup_{x \in I}[\exp(-Lx) A_i(x)] \leqslant M_i, \quad i = 1, \ldots, n,$$

where $\|x\|$ denotes the absolute value of x.

THEOREM 3. *Suppose that hypotheses of Theorem* 1 *and Hypothesis* 2 *are fulfilled. If, moreover,*

$$\max_i \Big(\sum_{k=1}^{n} a_{ik} \Big) = s < 1, \tag{13}$$

then the solution of the system of equations (1) *with condition* (2) *depends continuously on* u.

Proof. For $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\varphi_i \in G$, $i = 1, \ldots, n$, we define $|\varphi| = \max_i |\varphi_i|$. Let $F = (F_1, \ldots, F_n)$, where

$$\begin{aligned} F_i &= F_i(x, \varphi, u) \\ &= h_i\Big(x, \int_0^{f_{i1}(x)} \varphi_1(s)\,ds, \ldots, \int_0^{f_{in}(x)} \varphi_n(s)\,ds, \varphi_1[g_{i1}(x)], \ldots, \varphi_n[g_{in}(x)], u\Big). \end{aligned}$$

Let $\psi = (\psi_1, \ldots, \psi_n)$, $\psi_i \in G$, $i = 1, \ldots, n$. Just as in the proof of Theorem 2, we obtain

$$|F_i(x, \varphi, u) - F_i(x, \psi, u)| \leqslant \sum_{k=1}^{n} a_{ik} |\varphi_k - \psi_k|, \quad i = 1, \ldots, n;$$

whence by (13)

$$|F(x, \varphi, u) - F(x, \psi, u)| \leqslant s |\varphi - \psi|. \tag{14}$$

In view of Hypothesis 2 we have

$$(15) \qquad |F_i(x, \varphi, u_1) - F_i(x, \varphi, u_2)| \leqslant M_i \omega_i(\|u_1 - u_2\|), \qquad i = 1, \ldots, n.$$

From (14), (15), applying Banach's fixed-point principle [7], we obtain our assertion.

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INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY
KATOWICE, POLAND

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