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On a theorem of A. V. Sokolovskiĭ

by

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1. Let K be an algebraic number field. Denote by n and Δ the degree and the discriminant of the field K respectively and by $\zeta_K(s)$, $s = \sigma+it$, the Dedekind zeta-function (see [5]).

The function $\zeta_K(s)$ is defined for $\sigma > 1$ by the absolute convergent series

$$\sum_{m=1}^{\infty} F(m)m^{-s}, \tag{1.1}$$

where $F(m)$ is the number of ideals of the ring of algebraic integers R_K, with the norm equal to m. The function $\zeta_K(s)$ can be continued analytically to a meromorphic function with a simple pole at $s = 1$.

In 1968 A. V. Sokolovskiĭ (see [4]) using Vinogradov's methods proved that $\zeta_K(s)$ has no zeros in the region

$$\sigma \geqslant 1-\frac{A}{\log^{2/3}t(\log\log t)^{1/3}}, \quad t \geqslant t_0, \tag{1.2}$$

where A and t_0 are constants depending on K (compare [8]).

The aim of this note is to express the constants A and t_0 in Sokolovskiĭ's theorem explicitly in terms of degree n and discriminant Δ of the field K and to extend the region (1.2) closer to the real axis of the complex plane.

We shall prove the following

Theorem. *$\zeta_K(s)$ has no zeros in the region*

$$\sigma \geqslant 1-\frac{1}{c_1 n^{11}|\Delta|^3\log^{2/3}t(\log\log t)^{1/3}}, \quad t \geqslant 4, \tag{1.3}$$

where c_1 is a pure numerical constant, $c_1 > 1$.

It has to be noted that for the refinement of the zero-free domain (1.2) of $\zeta_K(s)$ with respect to n and Δ it was essential to use some results of K. Mahler and C. Siegel.

2. It is well-known that for $s = \sigma + it$, $\sigma > 1$,

$$\zeta_K(s) = \sum_C \Big(\sum_{\mathfrak{a}\in C} (N\mathfrak{a})^{-s}\Big), \tag{2.1}$$

where the inner sum is taken over all ideals $\mathfrak{a}$ of K belonging to an ideal class C (see [5], p. 57) and the outer sum is taken over all h ideal classes C.

It is also well-known that

$$f_C(s) = \sum_{\mathfrak{a}\in C} (N\mathfrak{a})^{-s} = (N\mathfrak{a}')^s \sum_{\alpha\in\mathfrak{a}'} |N(\alpha)|^{-s}, \tag{2.2}$$

where the last sum is taken over a full system of in pairs not associated algebraic integers belonging to $\mathfrak{a}'$ from the inverse class C^{-1} (see [5], p. 58).

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a basis for $\mathfrak{a}'$. Then each α of $\mathfrak{a}'$ can be written in a unique way as a sum

$$\alpha = a_1\alpha_1 + \ldots + a_n\alpha_n$$

with rational integral coefficients $a_1, \ldots, a_n$.

Every element $\alpha \in K$ can be considered as an element of the n-dimensional real space $\boldsymbol{R}^n$:

$$x(\alpha) = (x_1, \ldots, x_{r_1};\ y_1, \ldots, y_{r_2},\ z_1, \ldots, z_{r_2})$$

where $n = r_1 + 2r_2$ (see [1], II, § 3).

Denote by $\mathfrak{M}$ the n-dimensional lattice in $\boldsymbol{R}^n$ formed of images of algebraic integers $\alpha \in K$, divided by $\mathfrak{a}'$, and denote by V the fundamental domain of K (see [1], p. 352). Then the summation in (2.2) reduces to the summation over rational integers $a_1, \ldots, a_n$, such that $x(\alpha) \in \mathfrak{M} \cap V$ where $\alpha = a_1\alpha_1 + \ldots + a_n\alpha_n \in K$. We have

$$f_C(s) = (N\mathfrak{a}')^s \sum_{\substack{a_1 \\ x(\alpha)\in\mathfrak{M}\cap V}} \ldots \sum_{a_n} |Nx(\alpha)|^{-s}. \tag{2.3}$$

Denote further by

$$\alpha^{(i)} = a_1\alpha_1^{(i)} + \ldots + a_n\alpha_n^{(i)}, \quad i = 1, 2, \ldots, n,$$

the conjugates of α and namely $\alpha^{(i)}$ are real if $1 \leqslant i \leqslant r_1$ and $\alpha^{(i)}$ are complex conjugates of $\alpha^{(i+r_2)}$ if $r_1+1 \leqslant i \leqslant r_2$, so that $N\alpha = \alpha^{(1)}\cdot \ldots \cdot \alpha^{(n)}$.

Denote by $\overline{V}$ the set which we get multiplying the elements of V by images of all roots of unity belonging to K.

Then we can write the series (2.3) as follows

$$f_C(s) = \frac{1}{m}(N\mathfrak{a}')^s \sum_{\substack{a_1 \\ x(\alpha)\in\mathfrak{M}\cap\overline{V}}} \ldots \sum_{a_n} \frac{e^{-it\log|Nx(\alpha)|}}{|Nx(\alpha)|^{\sigma}}, \tag{2.4}$$

where m denotes the number of roots of unity belonging to K (see [4], p. 323).

In the n-dimensional real space $\boldsymbol{R}^n$ we shall define for any ideal $\mathfrak{a}$, the sets $K_{\mathfrak{a}}^X$, $X > 0$, as follows:

$$K_{\mathfrak{a}}^X = \{(u_1, \ldots, u_n)\colon u_i \in \boldsymbol{R},\ \max_{1\leqslant i\leqslant n} |u_i| \leqslant X,\ x(u) \in \overline{V}\} \tag{2.5}$$

and $x(u) = u_1x(\alpha_1) + \ldots + u_nx(\alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ form a basis for $\mathfrak{a}$ (see [4], p. 324).

In the following we shall always assume that $N\mathfrak{a} \leqslant |\Delta|^{1/2}$ because in each ideal class C there exists at least one such ideal (see [5], p. 42).

Following Mahler (see [7], p. 429), a *ceiling* is a positive valued function $\lambda(\mathfrak{p})$ of the variable prime divisor $\mathfrak{p}$ with the following properties:

(A) At all infinite prime divisors $\mathfrak{q}$, $\lambda(\mathfrak{q})$ may assume arbitrary positive values.

(B) At every finite prime divisor τ, $\lambda(\tau)$ is of the form

$$\lambda(\tau) = p_\tau^{-l_\tau/e_\tau}$$

where e_τ is the order of τ, p_τ is the corresponding rational prime, and l_τ is any rational integer.

(C) $\lambda(\mathfrak{p})$ is equal to 1 except at finitely many prime divisors.

(D) $$\prod_{\mathfrak{p}} \lambda(\mathfrak{p})^{n_\mathfrak{p}} = 1.$$

There is a one-to-one correspondence, $\mathfrak{a} \sim [\mathfrak{a}]$, between the finite divisors $\mathfrak{a}$ and the fractional ideals $[\mathfrak{a}]$ in K which consist of all field elements α that are divisible by $\mathfrak{a}$ (see [1], p. 244).

3. The proof of (1.3) rests on the following lemmas.

LEMMA 1 (theorem of Mahler, see [7], p. 436 and [6], p. 400). *Let $\lambda(\mathfrak{p})$ be an arbitrary ceiling of K, and let $\mathfrak{a}_\lambda$ be the corresponding divisor. Then there exists a basis $\alpha_1, \ldots, \alpha_n$ of the ideal $[\mathfrak{a}_\lambda]$ such that*

$$A_1^{-(n-1)}\lambda(\mathfrak{q}) \leqslant |\alpha_k|_{\mathfrak{q}} \leqslant A_1\lambda(\mathfrak{q}) \tag{3.1}$$

for all infinite prime divisors $\mathfrak{q}$, and $k = 1, \ldots, n$, where $A_1 = n^n|\Delta|^{1/2}$, $n \geqslant 2$.

LEMMA 2. *For each ideal $\mathfrak{a}$ of K there exists a basis $\alpha_1, \ldots, \alpha_n$ such hat*

$$A_1^{-(n-1)}(N\mathfrak{a})^{1/2n} \leqslant |\alpha_k^{(i)}| \leqslant A_1(N\mathfrak{a})^{1/n}, \tag{3.2}$$

where $A_1 = n^n|\Delta|^{1/2}$ and $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, n$.

Proof. Writing

$$\mathfrak{a} = \prod_\tau \tau^{l_\tau}$$

with prime ideals τ and rational integers $l_\tau > 0$, we define the ceiling of K as follows

$$\lambda(\mathfrak{p}) = \begin{cases} (N\tau)^{-l_\tau/n_\tau} & \text{for } \mathfrak{p} \sim \tau, \\ 1 & \text{for the other finite prime divisors,} \\ (N\mathfrak{a})^{1/n} & \text{for the infinite prime divisors,} \end{cases}$$

where $n_\tau = e_\tau \cdot f_\tau$ and e_τ is order of prime ideal τ, f_τ its degree. Then

$$\prod_{\mathfrak{p}} \lambda(\mathfrak{p})^{n_\mathfrak{p}} = 1.$$

From Lemma 1 it follows that there exists a basis $a_1, \ldots, a_n$ of the ideal $\mathfrak{a}$ ($\mathfrak{a}$ corresponds to the above ceiling), satisfying (3.2).

LEMMA 3. *In any field K there exists a fundamental system of $r = r_1 + r_2 - 1$ units $\varepsilon_1, \ldots, \varepsilon_r$ (in the case of rational field and of imaginary quadratic fields $r = 0$) such that*

$$\log|\varepsilon_k^{(j)}| < n|\Delta|^2, \tag{3.3}$$

where $k = 1, \ldots, r$ and $j = 1, \ldots, n$.

This statement can easily be deduced from Siegel's paper [12], p. 85.

LEMMA 4. *For each ideal $\mathfrak{a}$ of K with $N\mathfrak{a} \leqslant |\Delta|^{1/2}$ there exists an integral basis $a_1, \ldots, a_n$ such that for any point $(u_1, \ldots, u_n)$ of $K_\mathfrak{a}^{2X} \setminus K_\mathfrak{a}^X$ (see (2.5)) we have the inequality*

$$A_2 X < |u_1 a_1^{(i)} + \ldots + u_n a_n^{(i)}| < A_3 X, \tag{3.4}$$

where $i = 1, \ldots, n$ and $A_2 = \exp(-4n^3|\Delta|^2)$, $A_3 = 2|\Delta| n^{n+1}$.

Proof (compare [4], Lemma 1). From (3.2) it follows that

$$|u_1 a_1^{(l)} + \ldots + u_n a_n^{(l)}| \leqslant 2XnA_1(N\mathfrak{a})^{1/n} \leqslant A_3 X. \tag{3.5}$$

Now we have to estimate $|u^{(l)}|$ from below.

Consider the system of n linear equations

$$u^{(l)} = u_1 a_1^{(l)} + \ldots + u_n a_n^{(l)}$$

where $l = 1, \ldots, n$. By Cramer's rule

$$u_i = c_{1i} u^{(1)} + \ldots + c_{ni} u^{(n)}$$

and

$$c_{ki} = \frac{D_{ki}}{D_0}, \quad \text{where} \quad |D_0| = |\det[a_i^{(k)}]_{1\leqslant i,k\leqslant n}| = |\Delta|^{1/2} N\mathfrak{a}.$$

Owing to (3.2) and by the use of Hadamard inequality, we have

$$|D_{ki}| \leqslant n^{(n-1)/2} \big(\max_{1\leqslant s,t\leqslant n} |a_s^{(t)}|\big)^{n-1} \leqslant n^{n^2-1} |\Delta|^{n/2}.$$

Hence

$$|c_{ki}| \leqslant n^{n^2-1} |\Delta|^{\frac{1}{2}(n-1)}. \tag{3.6}$$

Putting $|u^{(i)}| < A_4 X$, $A_4 = n^{-n^2} |\Delta|^{-\frac{1}{2}(n-1)}$ and using (3.6), we get

$$|u_i| = |c_{1i} u^{(1)} + \ldots + c_{ni} u^{(n)}| \leqslant X.$$

From this it follows that all solutions $(u_1, \ldots, u_n)$ of the system of inequalities $|u_1 a_1^{(l)} + \ldots + u_n a_n^{(l)}| < A_4 X$, $l = 1, \ldots, n$, belong to $K_\mathfrak{a}^X$. Hence for any $(u_1, \ldots, u_n) \in K_\mathfrak{a}^{2X} \setminus K_\mathfrak{a}^X$ there exists such j $(1 \leqslant j \leqslant n)$ that

$$|u_1 a_1^{(j)} + \ldots + u_n a_n^{(j)}| \geqslant A_4 X. \tag{3.7}$$

Furthermore, for any $u = (u_1, \ldots, u_n)$ belonging to V we have (see [1], p. 359)

$$\log|u^{(i)}| = \frac{1}{n} \log|N(u)| + \sum_{k=1} \xi_k \log|\varepsilon_k^{(i)}|, \tag{3.8}$$

where $i = 1, \ldots, n$, $0 \leqslant \xi_k < 1$ and $\varepsilon_1, \ldots, \varepsilon_r$ are fundamental units of K, satisfying (3.3).

Putting $i = j$, we get from (3.3)

$$|u^{(j)}| = |N(u)|^{1/n} \prod_{k=1}^{r} |\varepsilon_k^{(j)}|^{\xi_k} \leqslant |N(u)|^{1/n} \exp(n^2|\Delta|^2).$$

Hence, owing to (3.7), we have

$$|N(u)| \geqslant A_4^n \exp(-n^2|\Delta|^2) X^n$$

and finally from (3.5)

$$|u^{(i)}| = \frac{|N(u)|}{\prod_{\substack{k=1\\k\neq i}}^{n} |u^{(k)}|} \geqslant \exp(-4n^3|\Delta|^2) X,$$

and this completes the proof of Lemma 4.

LEMMA 5 (compare [4], Lemma 2). *Putting*

$$F(u_1, \ldots, u_n) = -\frac{t}{2\pi} \log|N x(u)|$$

we have for any $(u_1, \ldots, u_n) \in K_\mathfrak{a}^{2X} \setminus K_\mathfrak{a}^X$ and $m \geqslant 1$ the estimates

$$\left|\frac{\partial^m F}{\partial u_i^m}\right| \leqslant A_5^m (m-1)!\,|t|\, X^{-m}, \tag{3.9}$$

where $A_5 = \exp(5n^3|\Delta|^2)$.

Proof. Applying (3.4) and (3.2) we obtain easily (3.9).

LEMMA 6 (Turán's second main theorem (see [15] and [13])). *Let $k \geqslant 2$, $m > 0$ are rational integers, b_j arbitrary complex numbers, $1 \leqslant j \leqslant k$. If z_j, $1 \leqslant j \leqslant k$, are complex numbers such that*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_k|$$

then

$$\max_{m+1 \leqslant \nu \leqslant m+k} |b_1 z_1^\nu + \ldots + b_k z_k^\nu| \geqslant \left(\frac{k}{8e(m+k)}\right)^k \min_{1 \leqslant j \leqslant k} |b_1 + \ldots + b_j|. \tag{3.10}$$

LEMMA 7 (compare [4], Lemma 3). *For arbitrary fixed u_j $(j \neq i)$ such that $(u_1, \ldots, u_n) \in K_\mathfrak{a}^{2X} \setminus K_\mathfrak{a}^X$ and for any rational integer $m_1 \geqslant 1$ we can divide the interval in which u_i is determined into at most*

$$A_6^{m_1+n} = \exp\left(7n^3|\Delta|^2(m_1+n)\right)$$

sub-intervals in such a way that for each sub-interval there exists m $(m_1+1 \leqslant m \leqslant m_1+n)$ such that the following inequality holds

$$\left|\frac{\partial^m F}{\partial u_i^m}\right| \geqslant A_7^m (m-1)!\, |t|\, X^{-m} \tag{3.11}$$

with $A_7 = n^{-2n}|\Delta|^{-7}$, for every point of the sub-interval.

Proof. This lemma can be deduced in the same way as Lemma 3 in [4] by the use of (3.10) and (3.4).

LEMMA 8 (Vinogradov's theorem (see [2], p. 210)). *Let $k \geqslant 11$, $m > 0$, $p > 0$ be rational integers. Let*

$$S = \sum_{x=1}^{p} e^{2\pi i m(a_{k+1}x^{k+1} + \ldots + a_1 x + a_0)},$$

where $a_{k+1}, \ldots, a_1, a_0$ are real.

Let r be one of the numbers $k+1, \ldots, 3, 2$ and suppose that

$$a_r = \frac{a}{q} + \frac{\theta}{q^2}, \quad \text{where} \quad (a, q) = 1,\ 1 < q < p^r,\ |\theta| \leqslant 1.$$

Then

$$|S| < (8k)^{\frac{1}{2}kl} m^{\frac{2\varrho}{\tau}} p^{1-\varrho}$$

where

$$l = \log\frac{12k(k+1)}{\tau}, \quad \varrho = \frac{\tau}{3k^2 l}.$$

Here τ is defined in terms of q and p by

$$q = p^\tau \quad \text{for} \quad 1 < q \leqslant p,$$
$$\tau = 1 \quad \text{for} \quad p < q \leqslant p^{r-1}$$
$$q = p^{r-\tau} \quad \text{for} \quad p^{r-1} < q < p^r.$$

LEMMA 9. *If*

$$m_1 = \left[11\,\frac{n+2}{n}\,\frac{\log t}{\log X}\right], \tag{3.12}$$

$$t^{1/(n+2)} \leqslant X < A_2^{-1} t^{(n+1)/n}, \tag{3.13}$$

$A_2 = \exp(-4n^3|\Delta|^2)$ and $t > \exp(n^5|\Delta|^3 c_0)$, then

$$|S_i| = \Bigg|\sum_{\substack{a < a_i \leqslant a' \\ (a_1, \ldots, a_n) \in K_\mathfrak{a}^{2X} \setminus K_\mathfrak{a}^X}} e^{2\pi i F(a_1, \ldots, a_n)}\Bigg| \leqslant A_3 X^{1 - \frac{1}{c_2 n^3 m_1^2}}, \tag{3.14}$$

where $A_3 = \exp(c_3 n \log^2 n)$; c_0, c_2, c_3 are numerical constants and

$$F(a_1, \ldots, a_n) = -\frac{t}{2\pi}\log|N\alpha(\alpha)|.$$

Lemma 9 follows from Lemmas 8 and 7. The method of the proof is essentially the same as that of Lemma 5 in [4]. We mention only that we have to use Lemma 8 with $1/cn \leqslant \tau \leqslant 1$.

LEMMA 10 (Vinogradov's theorem (see [3], p. 55)). *Let $k \geqslant 11$ and*

$$S = \sum_{x=1}^{p} e^{2\pi i(a_k x^k + \ldots + a_1 x)}$$

where $a_k, \ldots, a_1$ are real.

If

$$D_l = (20k)^{\frac{k(k+1)}{2}l}, \quad b_l = kl + \left[\frac{k(k+1)}{4} + 1\right],$$

where $l > 0$ rational integer, then for $b \geqslant b_l$

$$\int_0^1 \ldots \int_0^1 |S|^{2b}\, da_n \ldots da_1 < D_l p^{2b - \frac{k(k+1)}{2} + \frac{k(k+1)}{2}\left(1-\frac{1}{k}\right)^l}.$$

COROLLARY. *Let p, l, k be positive rational integers, $k \geqslant 11$, $b = \left[k^2 l + \frac{k(k+1)}{4} + 1\right]$.*

Let x_j run through integers $1, 2, \ldots, p$ and let

$$Z_s = x_1^s + \ldots + x_b^s - x_{b+1}^s - \ldots - x_{2b}^s.$$

Then the number of solutions of the system of equations

$$Z_1 = \xi_1, \quad Z_2 = \xi_2, \quad \ldots, \quad Z_k = \xi_k$$

is not greater than U where

$$U = (20k)^{\frac{k^2(k+1)}{2}l}\, p^{2b - \frac{k(k+1)}{2} + \frac{k(k+1)}{2}e^{-l}}. \tag{3.15}$$

This corollary can easily be deduced from Lemma 10.

LEMMA 11 (compare [4], Lemma 8). *If*

$$m = \left[\frac{\log t}{\log X}\right] + 1 \tag{3.16}$$

and

$$1 < X < t^{1/(n+2)} \tag{3.17}$$

then

$$|S_i| = \Bigg|\sum_{\substack{a < a_i \leqslant a' \\ (a_1, \ldots, a_n) \in K_\alpha^{2X} \setminus K_\alpha^X}} e^{2\pi i F(a_1, \ldots, a_n)}\Bigg| < A_9 X^{1 - \frac{1}{A_{10} m^2}} \tag{3.18}$$

where $A_9 = \exp(c_4 n^2 |\Delta|^2)$, $A_{10} = c_5 n^4$ and c_4, c_5 are numerical constants.

Proof. First of all, it is easy to verify that (3.18) holds for $X < \exp\big(A_{10}(\log A_9) m^2\big)$. So, we may assume that

$$X \geqslant \exp\big(A_{10}(\log A_9) m^2\big).$$

Notice that $m \geqslant n+3$ and $X^{m-1} \leqslant t < X^m$. We put

$$Y = [X^{1/3}], \qquad m_0 = 3m,$$

$$r = 2b = 2\left[5nm_0^2 + \frac{m_0(m_0+1)}{4} + 1\right].$$

Then

$$|S_i| \leqslant \frac{1}{Y^2}\Bigg|\sum_{\substack{a \leqslant a_i < a' \\ (a_1, \ldots, a_n) \in K_\alpha^{2X} \setminus K_\alpha^X}} \sum_{x=1}^{Y} \sum_{y=1}^{Y} e^{2\pi i F(a_1, \ldots, a_n)}\Bigg| + 2Y^2.$$

Now, since by Lemma 4 $|Nx(\alpha)| > A_2^n$ for $X > 1$, by Taylor's expansion with Lagrange's remainder, we get

$$|S_i| \leqslant \frac{1}{Y^2} \sum_{\substack{a \leqslant a_i < a' \\ (a_1, \ldots, a_n) \in K_\alpha^{2X} \setminus K_\alpha^X}} |S_{a_i}| + 3X^{2/3}, \tag{3.19}$$

where

$$S_{a_i} = \sum_{x=1}^{Y} \sum_{y=1}^{Y} e^{2\pi i(A_1 xy + \ldots + A_{m_0} x^{m_0} y^{m_0})}, \qquad A_k = \frac{1}{k!}\frac{\partial^k F}{\partial a_i^k}.$$

We divide the sum in (3.19) into at most $A_6^{\frac{m_0(m_0+1)}{2} + nm_0}$ sums

$$S' = \sum_{\substack{A < a_i \leqslant A' \\ (a_1, \ldots, a_n) \in K_\alpha^{2X} \setminus K_\alpha^X}} |S_{a_i}|$$

according to Lemma 7 in such a way that for each of them for $m_1 = 1, 2, \ldots$ $\ldots, m_0$ there exists M: $m_1 + 1 \leqslant M \leqslant m_1 + n$ such that $\left|\frac{\partial^M F}{\partial a_i^M}\right|$ satisfies (3.9) and (3.11) in every point of the interval $\langle A, A' \rangle$.

By Hölder's inequality we have

$$|S_{a_i}|^r \leqslant Y^{r-1} \sum_{x=1}^{Y} \sum_{y_1, \ldots, y_{2b}} e^{2\pi i(A_1 V_1 x + \ldots + A_{m_0} V_{m_0} x^{m_0})},$$

where

$$V_k = y_1^k + \ldots + y_b^k - y_{b+1}^k - \ldots - y_{2b}^k$$

and $y_1, \ldots, y_{2b}$ run through rational integers $1, \ldots, Y$.

Applying Hölder's inequality again, we get

$$|S_{a_i}|^{r^2} \leqslant Y^{2r(r-1)} \sum_{x_1, \ldots, x_{2b}} \sum_{y_1, \ldots, y_{2b}} e^{2\pi i(A_1 V_1 W_1 + \ldots + A_{m_0} V_{m_0} W_{m_0})}$$

where

$$W_k = x_1^k + \ldots + x_b^k - x_{b+1}^k - \ldots - x_{2b}^k$$

and $x_1, \ldots, x_{2b}$ run through rational integers $1, \ldots, Y$. So V_s and W_s run through rational integers ξ_s such that

$$|\xi_s| < bY^s, \qquad s = 1, \ldots, m_0.$$

By the corollary of Lemma 10 the number of solutions of the system $W_1 = \xi_1, \ldots, W_{m_0} = \xi_{m_0}$ is not greater than U (see (3.15)). Hence

$$|S_{a_i}|^{r^2} \leqslant Y^{2r(r-1)} U^2 \prod_{s=1}^{m_0} H_s, \tag{3.20}$$

where

$$H_s = \sum_{|\eta_s| < bY^s} \Bigg|\sum_{|\xi_s| < bY^s} e^{2\pi i A_s \eta_s \xi_s}\Bigg|.$$

It is obvious that

$$\Bigg|\sum_{|\xi_s| < bY^s} e^{2\pi i A_s \eta_s \xi_s}\Bigg| \leqslant \min\left\{rY^s, \frac{1}{2(A_s \eta_s)}\right\},$$

where (x) indicates the distance between x and the nearest integer.

For $s \geqslant [\frac{3}{2}m+1]$ since $|A_s\eta_s| < \frac{1}{2}$ we have evidently $(A_s\eta_s) = |A_s\eta_s|$ and

$$|H_s| < Y^{2s}X^{\frac{1}{3}s-m+\frac{4}{3}}.$$

Hence

$$\Big|\prod_{s=1}^{m_0} H_s\Big| \leqslant r^{2m}\, Y^{m_0(m_0+1)} X^{-\frac{1}{100n^2}m^2}.$$

Now, from (3.20) and (3.15), we get

$$|S_{a_i}|^{r^2} \leqslant Y^{2r^2}(60m)^{45m^2(3m+1)n} X^{-\frac{1}{100n^2}m^2}$$

and finally by (3.19)

$$|S_i| \leqslant 4X^{1-\frac{1}{c_5 n^4 m^2}}.$$

LEMMA 12. *If*

$$m_1 = \left[11\,\frac{n+2}{n}\,\frac{\log t}{\log X}\right],$$

(3.21) $\quad 1 < X < A_2^{-1}t^{(n+1)/n}, \quad A_2 = \exp(-4n^3|\Delta|^2), \quad t > \exp(n^5|\Delta|^3 c_0)$

then

$$(3.22)\qquad |S_i| = \Big|\sum_{\substack{a<a_i\leqslant a' \\ (a_1,\ldots,a_n)\in K_\alpha^{2X}\setminus K_\alpha^{X}}} e^{2\pi i F(a_1,\ldots,a_n)}\Big| \leqslant A_{11}X^{1-\frac{1}{A_{12}m_1^2}},$$

where $A_{11} = \exp(c_6 n^2|\Delta|^2)$, $A_{12} = c_7 n^4$.

This lemma is a simple corollary of Lemmas 9 and 11.

LEMMA 13 (compare [4], Lemma 9). *In the region* $\sigma \geqslant 1-\dfrac{1}{n+1}$, $t \geqslant 1$, $s = \sigma+it$ *of the complex plane, we have the estimate*

$$(3.23)\qquad \Big|\zeta_K(s) - \sum_{1\leqslant m<t^{n+1}} F(m)m^{-s}\Big| \leqslant \exp(c_8 n^4|\Delta|^2),$$

where c_8 *is purely numerical constant.*

The method of the proof of (3.23) is standard and therefore the proof of Lemma 13 will be dropped. We mention only that it needs refinements with respect to n and Δ of a series of Landau's classical results ([5], th. 203, 210). These refinements can simply be obtained by the use of Lemmas 2 and 3.

LEMMA 14 (see [14], p. 185). *In the region* $-1 \leqslant \sigma \leqslant 2$, $-\infty < t < \infty$ *of the complex plane, we have the estimate*

$$(3.24)\qquad |(s-1)\zeta_K(s)| \leqslant A_{13}(|t|+1)^{A_{14}}, \quad s = \sigma+it,$$

where

$$A_{13} = c_9^n|\Delta|^{3/2}, \quad A_{14} = \tfrac{3}{2}n+2$$

and c_9 *is a pure numerical constant.*

LEMMA 15 (compare [4], Lemma 10). *In the region*

$$(3.25)\qquad \sigma \geqslant \sigma_0 = 1-\frac{1}{c_{10}n^5}\left(\frac{\log\log t}{\log t}\right)^{2/3}, \quad t \geqslant 1.1,$$

of the complex plane, we have the estimate

$$(3.26)\qquad |\zeta_K(s)| \leqslant A_{15}\log t$$

where $A_{15} = \exp(c_{11}n^6|\Delta|^3)$ *and* c_{10}, c_{11} *are numerical constants.*

Proof. Denote

$$K_i = K_\alpha^{2^i t_0} \quad \text{where} \quad t_0 = \exp\log^{2/3}t(\log\log t)^{1/3}) \quad (\text{see (2.5)}).$$

Owing to (2.1), (2.4) and (3.23) we get in the region $\sigma \geqslant 1-\dfrac{1}{n+1}$, $t>1$ the estimate

$$(3.27)\qquad |\zeta_K(s)| \leqslant \exp(c_8 n^4|\Delta|^2) + |\Delta|^{1/2}\sum_{j=1}^{h}\Big|\sum_{\substack{(a_1,\ldots,a_n)\in K_0 \\ 0<|Nx(\alpha)|<N\mathfrak{a}_j t^{n+1}}} |Nx(\alpha)|^{-s}\Big| + $$
$$+|\Delta|^{1/2}\sum_{j=1}^{h}\sum_{i=1}^{i_0}\Big|\sum_{\substack{(a_1,\ldots,a_n)\in K_i\setminus K_{i-1} \\ 0<|Nx(\alpha)|<N\mathfrak{a}_j t^{n+1}}} |Nx(\alpha)|^{-s}\Big|$$

where $\mathfrak{a}_j$ is an ideal belonging to the inverse class C_j^{-1}, $N\mathfrak{a}_j \leqslant |\Delta|^{1/2}$ and h is the class-number. For h we have the estimate

$$h \leqslant |\Delta|^{(n^2+1)/2} \quad (\text{see [11], p. 160}).$$

By Lemma 4, $i_0 \leqslant 3\log t$.

We estimate the first sum in (3.27) using Landau's theorem (see [5], th. 203). We get simply

$$\Big|\sum_{\substack{(a_1,\ldots,a_n)\in K_0 \\ 0<|Nx(\alpha)|<N\mathfrak{a}_j t^{n+1}}} |Nx(\alpha)|^{-s}\Big| \leqslant \sum_{k\leqslant A_3^n t_0^n} F(k)k^{-\sigma_0} \leqslant \exp(c_{12}n^4|\Delta|^2)\log t.$$

We estimate the remaining sums of (3.27) applying Lemma 12 with $X = 2^{i-1}t_0$. We obtain

$$\Big|\sum_{\substack{(a_1,\ldots,a_n)\in K_i\setminus K_{i-1} \\ 0<|Nx(\alpha)|<N\mathfrak{a}_j t^{n+1}}} |Nx(\alpha)|^{-s}\Big| \leqslant \exp(c_{13}n^4|\Delta|^2)$$

and finally owing to (3.27) we get (3.26) for $\sigma \geqslant \sigma_0$ and $t \geqslant \exp(c_0 n^5 |\Delta|^3)$.

For $1.1 \leqslant t < \exp(c_0 n^5 |\Delta|^3)$, $\sigma \geqslant \sigma_0$ we obtain by Lemma 14

$$|\zeta_K(s)| \leqslant \exp(c_{14} n^6 |\Delta|^3).$$

LEMMA 16 (see [10], p. 435, th.4.4). *If $F(s)$ is a function regular in the circle $|s-s_0| \leqslant r$ and satisfying the inequality $\left|\frac{F(s)}{F(s_0)}\right| \leqslant M$ in this circle, then*

$$-\operatorname{Re}\frac{F'}{F}(s_0) \leqslant \frac{4}{r}\log M - \operatorname{Re}\sum_{\varrho}\frac{1}{s_0-\varrho}, \tag{3.28}$$

where ϱ runs through the zeros of $F(s)$ such that $|\varrho - s_0| \leqslant \frac{1}{2}r$ (a zero of order m being counted m times).

4. Proof of the theorem. Let $\beta + i\tau$ be a zero of $\zeta_K(s)$ such that $\tau \geqslant 1.5$ and $\beta > 0$. Denote

$$r(\tau) = \frac{\{\log\log(2\tau+1)\}^{2/3}}{c_{10} n^5 \log^{2/3}(2\tau+1)}, \qquad \alpha_0 = 1 + \frac{r(\tau)}{c^* n^6 |\Delta|^3 \log\log(2\tau+1)} \tag{4.1}$$

and let $s_0 = \alpha_0 + i\tau$, $s_0' = \alpha_0 + i2\tau$.

Consider the circles $|s-s_0| \leqslant r(\tau)$ and $|s-s_0'| \leqslant r(\tau)$. Both circles lie in the region (3.25).

For $\sigma > 1$ we have by (3.24)

$$\left|\frac{1}{\zeta_K(s)}\right| \leqslant \zeta_K(\sigma) \leqslant \frac{1}{\sigma-1} c_9^n |\Delta|^{3/2}.$$

Hence, by Lemma 15 for $|s-s_0| \leqslant r(\tau)$ and $|s-s_0'| \leqslant r(\tau)$, we get

$$\begin{gathered} \left|\frac{\zeta_K(s)}{\zeta_K(s_0)}\right| < c^* \exp(c_{15} n^6 |\Delta|^3)\log^2(2\tau+1), \\ \left|\frac{\zeta_K(s)}{\zeta_K(s_0')}\right| < c^* \exp(c_{15} n^6 |\Delta|^3)\log^2(2\tau+1), \end{gathered} \tag{4.2}$$

where c_{15} is a numerical constant independent on c^*.

Consider the third circle $|s-\alpha_0| \leqslant \frac{1}{2}$. In this circle by (3.24) we have

$$\left|\frac{(s-1)\zeta_K(s)}{(\alpha_0-1)\zeta_K(\alpha_0)}\right| < c^{*2}\exp(c_{16} n^2 |\Delta|)\log^2(2\tau+1).$$

Applying Lemma 16 we have for $\beta > \alpha_0 - \frac{1}{2}r(\tau)$ the estimates

$$-\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(s_0') \leqslant (c_{17} + 0.1 c_{10} c^*) n^{11} |\Delta|^3 \log^{2/3}(2\tau+1)\{\log\log(2\tau+1)\}^{1/3},$$

$$\begin{aligned} -\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(s_0) &\leqslant (c_{17} + 0.1 c_{10} c^*) n^{11} |\Delta|^3 \log^{2/3}(2\tau+1) \times \\ &\quad \times \{\log\log(2\tau+1)\}^{1/3} - \frac{1}{\alpha_0-\beta}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} -\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(\alpha_0) &\leqslant \frac{1}{\alpha_0-1} + c_{18} n^7 |\Delta| \log c^* \log^{2/3}(2\tau+1) \times \\ &\quad \times \{\log\log(2\tau+1)\}^{1/3}, \end{aligned}$$

c_{17}, c_{18} are numerical constants independent on c^*.

Now from the well-known inequality for $\sigma > 1$

$$-3\frac{\zeta_K'}{\zeta_K}(\sigma) - 4\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(\sigma+it) - \operatorname{Re}\frac{\zeta_K'}{\zeta_K}(\sigma+2it) \geqslant 0,$$

putting $t = \tau$ and $\sigma = \alpha_0$, we have by (4.3) for $c^* > \left(\frac{10 c_{18}}{c_{10}}\right)^2$

(4.4)
$$\frac{3}{\alpha_0-1} + (c_{19} + 0.8 c_{10} c^*) n^{11} |\Delta|^3 \log^{2/3}(2\tau+1)\{\log\log(2\tau+1)\}^{1/3} - \frac{4}{\alpha_0-\beta} \geqslant 0$$

where c_{19} does not depend on c^*.

Hence

$$\begin{aligned} \beta \leqslant 1 &+ \frac{1}{c_{10} c^* n^{11} |\Delta|^3 \log^{2/3}(2\tau+1)\{\log\log(2\tau+1)\}^{1/3}} - \\ &- \frac{1}{(c_{19} + 0.95 c_{10} c^*) n^{11} |\Delta|^3 \log^{2/3}(2\tau+1)\{\log\log(2\tau+1)\}^{1/3}}. \end{aligned}$$

Since we can put $c^* > \frac{20 c_{19}}{c_{10}}$, we get

$$\beta \leqslant 1 - \frac{1}{c_{20} n^{11} |\Delta|^3 \log^{2/3}(2\tau+1)\{\log\log(2\tau+1)\}^{1/3}}.$$

If $\beta < \alpha_0 - \frac{1}{2}r(\tau)$ we get a similar result. It means that the proof of (1.3) is complete.

References

[1] З. И. Боревич, И. Р. Шафаревич, *Теория чисел*, Москва 1972.
[2] И. М. Виноградов, *Верхняя граница модуля тригонометрической суммы*, И. А. Н. СССР, серия матем., 14 (1950), pp. 199–214.
[3] — *Метод тригонометрической суммы в теории чисел*, Москва 1971.
[4] А. В. Соколовский, *Теорема о нулях дзета функции Дедекинда и расстояние между „соседними" простыми идеалами*, Acta Arith. 13 (1968), pp. 321–334.
[5] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Leipzig und Berlin 1927.
[6] J. S. Luthar, *A note on a result of Mahler*, J. Austral. Math. Soc. 6 (1966), pp. 399–401.
[7] K. Mahler, *Inequalities for ideal bases in algebraic number fields*. ibid. 4 (1964), pp. 425–448.
[8] T. Mitsui, *On the prime ideal theorem*, J. Math. Soc. Japan 20 (1–2) (1968), pp. 233–247.
[9] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Warszawa 1974.
[10] К. Прахар, *Распределение простых чисел*, Москва 1967.
[11] R. Remak, *Elementare Abschätzungen von Fundamentaleinheiten und des Regulator's eines algebraischen Zahlkörpres*, J. Reine Angew. Math. 165 (1931), pp. 159–180.
[12] C. L. Siegel, *Abschätzung von Einheiten*, Nachr. Akad. Wiss. Göttingen (1969), pp. 71–86.
[13] Vera T. Soś and P. Turán, *On some new theorems in the theory of diophantine approximations*, Acta Math. Hung. 6 (1965), pp. 241–257.
[14] W. Staś, *Über eine Anwendung der Methode von Turán auf die Theorie des Restgliedes im Primidealsatz*, Acta Arith. 5 (1959), pp. 179–195.
[15] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953.

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Sur la dimension diophantienne des corps p-adiques

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1. La notion de dimension diophantienne. Soient A un anneau intègre et p un élément de $A[X_i]_{i\in I}$, nous dirons que p est anisotrope, ou anisotrope sur A, si I est fini, si p est sans terme constant et si le seul zéro de p dans A est le zéro banal; sous ces hypothèses, si p est de degré $d > 1$ on appelle ordre de p le nombre $\log_d n$, où n est le nombre des indéterminées de p. Soit K un corps commutatif, on appelle dimension diophantienne de K et on note $\mathrm{dd}(K)$ la borne supérieure, finie ou non, des ordres des polynômes à coefficients dans K, de degrés strictement supérieurs à un, homogènes et anisotropes.

Nous nous proposons de montrer que, si K est une extension finie du corps des nombres p-adiques, on a $\mathrm{dd}(K) \geqslant 3$. Nous généraliserons ainsi le résultat que J. Browkin avait obtenu dans le cas du corps des nombres p-adiques; pour cela, nous nous servons, outre les travaux déjà cités de Browkin [1], d'une idée que nous avons remarquée dans des travaux non publiés de S. Schanuel.

2. Le polynôme de Browkin–Schanuel. Rappelons et précisons quelques définitions de Browkin [1]. On désigne par p un nombre premier et par r un entier $\geqslant 0$. On suppose $r \geqslant 3$ lorsque p vaut 2. On a $p^r \geqslant 2r+1$ et on note n la partie entière de $p^r/(2r+1)$. Pour i entier compris entre 1 et $r+1$, on note a_i le nombre

$$(-1)^{i+1}\binom{r+i-1}{i-1}\binom{2r+1}{r+i}.$$

Si $s_1, \ldots, s_j$ sont des entiers $\geqslant 0$ et $X_1, \ldots, X_j$ des indéterminées, on note $d(s_1, \ldots, s_j)$ l'opérateur $\dfrac{\partial^{s_1+\ldots+s_j}}{\partial X_1^{s_1}\ldots\partial X_j^{s_j}}$. Pour k entier tel que $1 \leqslant k \leqslant n$, on définit les éléments ψ_k et φ_k de $\boldsymbol{Z}[X_1, \ldots, X_n]$ par:

$$\psi_k = \sum_{i=1}^{r+1} a_i X_1^{p^r-(r+i)(k-1)} \prod_{2\leqslant j\leqslant k} X_j^{r+i},$$

$$\varphi_k = \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} \psi_k(X_{i_1}, \ldots, X_{i_k}).$$