Hyperspaces of Peano continua are Hilbert cubes*

by

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Abstract. The main results of this paper are that the hyperspace 2^X of nonempty closed subsets of a nondegenerate Peano continuum X is homeomorphic to the Hilbert Cube Q, the hyperspace $C(X)$ of nonempty subcontinua is a Q-factor, and $C(X)$ is homeomorphic to Q if X contains no free arcs. Proofs are based on previous results of Schori and West for hyperspaces of graphs, and on partition techniques for Peano continua.

§ 1. **Introduction.** In this paper, we prove the results announced in [3], that the hyperspace of nonempty closed subsets of a space X is homeomorphic to the Hilbert cube if and only if X is a nondegenerate Peano space. This result was conjectured by Wojdyslawski [10] in 1938 and this paper is the last of a series of papers [8], [9] and [4] which prove respectively that the hyperspace of the closed unit interval, graphs, and polyhedra are Hilbert cubes. These papers, variously by the authors and J. E. West, use many techniques from infinite-dimensional topology where each paper uses different techniques and assumes the results of the previous papers. See [3], [6] and [7] for further historical background.

The proof in this paper that if X is a non-degenerate Peano continuum, then 2^X is a Hilbert cube proceeds from the following basic results:

(i) Hyperspaces of graphs are Hilbert cubes [9];

(ii) The Inverse Sequence Approximation Lemma 2.1 of [4] and stated as 3.1 of this paper; and

(iii) Maps between hyperspaces of graphs which are induced by C-monotone piecewise-linear maps are near-homeomorphisms [4].

The special case of the hyperspace of a polyhedron was treated in detail in [4], where the construction of the appropriate inverse sequences was accomplished via the Polyhedral Subdivision Theorem 4.1. The analogue of this theorem in the general case of a Peano continuum is the Partition Refinement Theorem 2.3 of the present paper. In §§ 3–5 we use the Refinement Theorem to obtain proofs of the general hyperspace results. The remainder (and major part) of this paper is devoted to a proof of the Refinement Theorem.

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* Research supported in part by NSF Grant GP 44349.

§ 2. **The Partition Refinement Theorem.** In what follows, the space X will always be compact metric.

DEFINITIONS 2.1 ([1], [5]). A *partition* of X is a finite collection G of mutually disjoint connected open subsets whose closures cover X. Two distinct elements of a partition are *adjacent* if their closures meet each other.

A metric space has *Property* S if, for each $\varepsilon>0$, the space is the union of a finite collection of connected subsets each of diameter less then ε. Of course, for compact metric spaces, Property S is equivalent to local connectedness.

If each element of a partition G has Property S, we say that G is an S-*partition*. If, furthermore, each element of G has diameter less than ε, then G is an ε-S-*partition*. Note that if X admits an S-partition, then it must be locally connected.

If H and G are two partitions of the same space such that each element of H is contained in an element of G, then H is a *refinement* of G. We shall be interested in partitions which admit arbitrarily fine refinements. Clearly, if G is such a partition, then G must be an S-partition. Conversely, every S-partition does admit arbitrarily fine refinements (see Theorem 6.1).

DEFINITION 2.2. A compact connected graph $\Gamma\subset X$ is a *nerve* of a partition G of X if the following conditions are satisfied:

(i) for each $g\in G$, $\Gamma\cap\bar{g}$ is a connected subgraph for which each point of $\Gamma\cap\operatorname{Bd}g$ is an endpoint;

(ii) the set $\bigcup\{\Gamma\cap\operatorname{Bd}g\colon g\in G\}$ of *boundary vertices* of the nerve is in 1-1 correspondence with the set of maximal nonempty intersections $\overline{g_1}\cap\ldots\cap\overline{g_k}$ of closures of adjacent elements of G, with each boundary vertex contained in the corresponding intersection.

For $g\in G$, define $\operatorname{St}^2(g;G)=\bigcup\{\overline{g'}\colon \overline{g'}\cap\overline{g''}\neq\emptyset\neq\overline{g''}\cap\bar{g}$ for some $g''\in G\}$. We shall use the Hausdorff metric d^* on 2^X (and all of its subspaces), induced by an arbitrary metric d on X.

PARTITION REFINEMENT THEOREM 2.3. *Let G be an S-partition of X, with a nerve Γ, and $\varepsilon>0$. Then there exists an ε-S-refinement H of G, with a nerve Λ and a C-monotone piecewise-linear map $\varphi\colon\Lambda\to C(\Gamma)$ such that:*

(i) *for each $x\in\Lambda$, $\varphi(x)\subset\operatorname{St}^2(g;G)$ for some $g\in G$ with $x\in\bar{g}$;*

(ii) *$\operatorname{diam}\varphi(\Lambda\cap\bar{h})<\varepsilon$ for each $h\in H$.*

§ 3. **The hyperspace** 2^X. In the proof of our main result we will refer to the different parts of the following result.

INVERSE SEQUENCE APPROXIMATION LEMMA 3.1. (2.1 of [4]). *Let Y be a compact metric space, and let*

$$Q_1\xleftarrow{f_1}Q_2\xleftarrow{f_2}\ldots$$

be an inverse sequence of maps and copies of the Hilbert cube in Y such that

(i) *$Q_i\to Y$ (in 2^Y);*

(ii) $\sum_{i=1}^{\infty}d(f_i,\mathrm{id})<\infty$;

(iii) *$\{f_i\circ\ldots\circ f_j\colon j\geqslant i\}$ is an equi-uniformly continuous family for each i; and*

(iv) *each f_i is a near-homeomorphism. Then $Y\approx Q$.*

THEOREM 3.2. *$2^X\approx Q$ for every nondegenerate Peano continuum X.*

Proof. The structure of the proof is the same as for the polyhedral special case [4], with refining partitions and their nerves taking the place of polyhedral subdivisions and their one-dimensional skeletons. Suppose that S-partitions $G_1,\ldots,G_i$ of X have been constructed, with corresponding nerves $\Gamma_1,\ldots,\Gamma_i$ and piecewise-linear C-monotone maps $\varphi_n\colon\Gamma_{n+1}\to C(\Gamma_n)$, $1\leqslant n<i$, such that each G_{n+1} refines G_n and

$$\operatorname{mesh}G_{n+1}<\min\left\{\frac{1}{3\cdot2^{-(n+1)}},\ \varepsilon_n\right\},$$

where $\varepsilon_n>0$ is chosen such that if the $4\varepsilon_n$-neighborhoods of the elements of a subcollection of G_n have a common intersection, then the closures of the elements have a common intersection. (We may take $G_1=\{X\}$, with Γ_1 any compact connected graph in X.) Let $f_n\colon 2^{\Gamma_{n+1}}\to2^{\Gamma_n}$ be the hyperspace map induced by φ_n, $1\leqslant n<i$. For $1\leqslant m<n$ define $f_m^n=f_m\circ\ldots\circ f_{n-1}\colon 2^{\Gamma_n}\to2^{\Gamma_m}$. Choose $0<\delta_i<1/i$ such that for A and B in 2^{Γ_i} with $d^*(A,B)<\delta_i$, we have $d^*(f_j^i(A),f_j^i(B))<1/i$ for each $j<i$.

There exists by the Partition Refinement Theorem an S-refinement G_{i+1} of G_i, with

$$\operatorname{mesh}G_{i+1}<\min\left\{\frac{1}{3\cdot2^{-(i+1)}},\ \varepsilon_i\right\},$$

a nerve Γ_{i+1}, and a C-monotone piecewise-linear map $\varphi_i\colon\Gamma_{i+1}\to C(\Gamma_i)$ such that:

(i) for each $x\in\Gamma_{i+1}$, $\varphi_i(x)\subset\operatorname{St}^2(g;G_i)$ for some $g\in G_i$ with $x\in\bar{g}$;

(ii) $\operatorname{diam}\varphi_i(\Gamma_{i+1}\cap\bar{h})<\frac{1}{12}\delta_i$ for each $h\in G_{i+1}$.

As before, we apply the Approximation Lemma to the inverse sequence $2^{\Gamma_1}\xleftarrow{f_1}2^{\Gamma_2}\xleftarrow{f_2}\ldots$ thus constructed. Conditions (i) and (iv) of the lemma are clear. It follows from the property (i) above that $d^*(f_i,\mathrm{id})<2^{-i}$. Thus condition (ii) of the lemma is satisfied.

Before turning to the verification of condition (iii), we show that for $m<n$ and $x\in\Gamma_n$, $f_m^n(\{x\})\subset\operatorname{St}^4(g_x;G_m)$ for some $g_x\in G_m$ such that $x\in\overline{g_x}$. The proof is by induction on $n-m$. For $n-m=1$, the result is given by property (i) of φ_m. By the inductive hypothesis there exists $h_x\in G_{m+1}$ such that $x\in\overline{h_x}$ and $f_{m+1}^n(\{x\})\subset\operatorname{St}^4(h_x;G_{m+1})$. Let $G_m^1=\{g\in G_m\colon g\supset h$ for some $h\in\operatorname{St}^4(h_x;G_{m+1})\}$. Then, since $\operatorname{mesh}G_{m+1}<\varepsilon_m$, the $4\varepsilon_m$-neighborhoods of elements of G_m^1 have a common intersection, and therefore the closures of these elements have a common intersection. We have

$$f_{m+1}^n(\{x\})\subset\bigcup\{\bar{g}\colon g\in G_m^1\}.$$

Let g_x be the element of G_m containing h_x; then $g_x \in G_m^1$. Property (i) of φ_m implies that for each $g_0 \in G_m$, $f_m^{m+1}(\overline{g_0} \cap \Gamma_{m+1}) \subset \mathrm{St}^3(g_0; G_m)$. And since $\mathrm{St}^3(g; G_m) \subset \mathrm{St}^4(g_x; G_m)$ for each $g \in G_m^1$, it follows that $f_m^n(\{x\}) \subset \mathrm{St}^4(g_x; G_m)$.

We now easily verify condition (iii) of the Approximation Lemma. Let $\varepsilon > 0$ and $k \geqslant 1$ be given. Choose $j \geqslant k$ such that $1/j < \varepsilon$. Choose $\mu > 0$ such that for $x, y \in X$ with $d(x, y) < \mu$, there exist elements h_x and h_y of G_{j+1} with $x \in \overline{h_x}$, $y \in \overline{h_y}$, and $\overline{h_x} \cap \overline{h_y} \neq \emptyset$. Now consider points $x, y \in \Gamma_i$, $i \geqslant j+1$, with $d(x, y) < \mu$. With h_x and h_y as above, we have $f_{j+1}^i(\{x\}) \subset \mathrm{St}^5(h_x; G_{j+1})$ and $f_{j+1}^i(\{y\}) \subset \mathrm{St}^5(h_y; G_{j+1})$. Thus by property (ii) of φ_j,

$$d^*\big(f_j^i(\{x\}), f_j^i(\{y\})\big) < 12(\delta_j/12) = \delta_j .$$

Hence for $A, B \in 2^{\Gamma_i}$, with $i \geqslant j+1$ and $d^*(A, B) < \mu$, we have $d^*\big(f_j^i(A), f_j^i(B)\big) < \delta_j$, and therefore $d^*\big(f_k^i(A), f_k^i(B)\big) < 1/j < \varepsilon$. Clearly, this implies condition (iii).

§ 4. The hyperspace $C(X)$.

THEOREM 4.1. *$C(X) \times Q \approx Q$ for every Peano continuum X, and $C(X) \approx Q$ if and only if X is nondegenerate and contains no free arcs (i.e., contains no open copies of the line).*

Proof. The result $C(X) \times Q \approx Q$ is obtained by simply restricting the maps of the inverse sequence constructed above, and stabilizing with Q. We thereby obtain an inverse sequence

$$C(\Gamma_1) \times Q \xleftarrow{g_1 \times \mathrm{id}} C(\Gamma_2) \times Q \xleftarrow{g_2 \times \mathrm{id}} \ldots$$

satisfying the conditions of the Approximation Lemma (it is easily seen that $C(\Gamma_i) \times Q \to C(X) \times Q$).

To obtain the stronger result $C(X) \approx Q$ when X is nondegenerate and contains no free arcs, we modify the above construction to obtain an inverse sequence

$$C(\Gamma_1^*) \xleftarrow{g_1^*} C(\Gamma_2^*) \xleftarrow{g_2^*} \ldots ,$$

where each Γ_i^* is a connected local dendron in X with a dense set of branch points. This is accomplished by adding stickers to the sequence of nerves $\{\Gamma_i\}$, thereby constructing a doubly-indexed sequence $\{\Gamma_{ij}\colon i \leqslant j\}$ of connected graphs, such that for each i,

$$\Gamma_i = \Gamma_{ii} \subset \Gamma_{i,\,i+1} \subset \ldots \subset \Gamma_i^* = \overline{\bigcup \Gamma_{ij}} .$$

This parallels the procedure previously employed for $C(K)$, K a polyhedron.

For graphs Γ_0 and Γ, with $\Gamma_0 \subset \Gamma \subset X$, and $\eta > 0$, we say that Γ is an *η-sticker expansion* of Γ_0 if $\overline{\Gamma \setminus \Gamma_0} = \bigcup \{\alpha_e\colon e \text{ an edge of } \Gamma_0\}$, where each α_e is an arc meeting Γ_0 only at an interior point of the corresponding edge e, $\alpha_e \cap \alpha_f = \emptyset$ if $e \neq f$, and $\operatorname{diam} \alpha_e < \eta$ for each e.

We now describe the inductive hypotheses of the construction. $G_1, \ldots, G_n$ are S-partitions of X, with

$$\operatorname{mesh} G_i < \min\left\{\frac{1}{3 \cdot 2^i}, \varepsilon_{i-1}\right\} \quad \text{for each } i ,$$

where $\varepsilon_{i-1} > 0$ is chosen as before: if the $4\varepsilon_i$-neighborhoods of the elements of a subcollection of G_{i-1} have a common intersection, then the closures of these elements have a common intersection. There exist connected graphs $\{\Gamma_{ij}\colon 1 \leqslant i \leqslant j \leqslant n\}$ and a corresponding collection $\{\gamma_{ij}\colon \Gamma_{i+1,\,j} \to C(\Gamma_{ij})\colon 1 \leqslant i < j \leqslant n\}$ of C-monotone piecewise-linear maps. Each Γ_{ij} is a nerve of G_i, and with respect to a suitable triangulation of Γ_{ij}, $\Gamma_{i,\,j+1}$ is an η_j-sticker expansion of Γ_{ij} ($\eta_j > 0$ is specified below). Thus for each $g \in G_i$, the subsets of boundary vertices $\Gamma_{ij} \cap \mathrm{Bd}\, g$ and $\Gamma_{i,\,j+1} \cap \mathrm{Bd}\, g$ are the same, and each sticker added to Γ_{ij} lies in some element of the partition G_i. In obtaining these sticker expansions we use the well-known fact that if a subset A of a Peano continuum X has empty interior, then the points of A which are accessible from $X \setminus A$ are dense in A. In particular, the interior of each edge e of a nerve Γ_{ij} contains a countable dense subset A_e, each point of which is accessible from $X \setminus e$. The corresponding sticker α_e of $\Gamma_{i,\,j+1}$ will meet Γ_{ij} at a point of A_e.

For each $i < n$, the map $\gamma_{i,\,i+1}\colon \Gamma_{i+1,\,i+1} \to C(\Gamma_{i,\,i+1})$ will have the following properties:

(i) for each $x \in \Gamma_{i+1,i+1}$, $\gamma_{i,i+1}(x) \subset \mathrm{St}^2(g_x; G_i)$ for some $g_x \in G_i$ with $x \in \overline{g_x}$;

(ii) $\operatorname{diam} \gamma_{i,\,i+1}(\Gamma_{i+1,\,i+1} \cap \overline{h}) < \frac{1}{24}\delta_i$ for each $h \in G_{i+1}$ ($\delta_i > 0$ is specified below).

To set forth relationships between the maps $\{\gamma_{ij}\}$, we need to define two auxiliary maps. For $1 \leqslant i \leqslant j < n$, let $\sigma_{ij}\colon \Gamma_{i,\,j+1} \to \Gamma_{ij}$ be the unique monotone retraction. Let $\tilde{\Gamma}_{i,\,i+1} = \bigcup \{e\colon e \text{ an edge of } \Gamma_{i+1,\,i+1}$ such that $\gamma_{i,\,i+1}$ maps the points of e in a 1-1 fashion onto the points of an edge of $\Gamma_{i,i+1}\}$. Thus $\tilde{\Gamma}_{i,i+1}$ is the smallest subgraph (not necessarily connected) of $\Gamma_{i+1,i+1}$ for which $\gamma_{i,\,i+1}(\tilde{\Gamma}_{i,\,i+1}) = \{\{x\}\colon x \in \Gamma_{i,\,i+1}\}$. For $i+1 < j \leqslant n$, we inductively define $\tilde{\Gamma}_{ij} \subset \Gamma_{i+1,\,j}$ by $\tilde{\Gamma}_{ij} = \sigma_{i+1,\,j-1}^{-1}(\tilde{\Gamma}_{i,\,j-1})$. Thus $\tilde{\Gamma}_{ij}$ is a sticker expansion of $\tilde{\Gamma}_{i,\,j-1}$, and there is a natural 1-1 correspondence between the edges of $\tilde{\Gamma}_{ij}$ and those of Γ_{ij}. For $i < j < n$, let $\tau_{ij}\colon \Gamma_{i+1,\,j+1} \to \Gamma_{i+1,\,j} \cup \tilde{\Gamma}_{i,j+1}$ be the unique monotone retraction.

We now continue with the description of the maps $\gamma_{i,\,j+1}\colon \Gamma_{i+1,\,j+1} \to C(\Gamma_{i,\,j+1})$, for $i < j < n$. Each of these maps has the following defining properties:

(i) $\gamma_{i,\,j+1}(x) = \gamma_{ij}(x)$ if $x \in \Gamma_{i+1,\,j}$;

(ii) $\gamma_{i,\,j+1}$ maps the points of each sticker of $\tilde{\Gamma}_{i,\,j+1} \setminus \tilde{\Gamma}_{ij}$ in a 1-1 fashion onto the points of the corresponding sticker of $\Gamma_{i,\,j+1} \setminus \Gamma_{ij}$;

(iii) $\gamma_{i,\,j+1} = \gamma_{i,\,j+1} \circ \tau_{ij}$.

Of course, for properties (i) and (ii) to be consistent, it is necessary that the stickers of $\tilde{\Gamma}_{i,\,j+1} \setminus \tilde{\Gamma}_{ij}$ and $\Gamma_{i,\,j+1} \setminus \Gamma_{ij}$ be properly aligned with each other: the attaching points of stickers on corresponding edges of $\tilde{\Gamma}_{ij}$ and Γ_{ij} must be corresponding points under the map γ_{ij}. There is no difficulty in achieving this alignment of stickers if we make the easily justified assumption that for each edge $\tilde{e}$ of $\tilde{\Gamma}_{ij}$

and corresponding edge $e = \gamma_{ij}(\tilde{e})$ of Γ_{ij}, the map γ_{ij} takes the countable dense subset $A_{\tilde{e}}$ of accessible points onto the countable dense subset A_e.

The positive constants δ_j and η_j alluded to above are chosen as follows. For $i<j\leqslant n$, let $g_{ij}: C(\Gamma_{i+1,j})\to C(\Gamma_{ij})$ be the map induced by γ_{ij}. Let

$$g_i^j = g_{ij}\circ \ldots \circ g_{j-1,j}: C(\Gamma_{jj})\to C(\Gamma_{ij}).$$

Choose $0<\delta_j<1/j$ such that for $A, B\in C(\Gamma_{jj})$ with $d^*(A, B)<\delta_j$, we have $d^*(g_i^j(A), g_i^j(B))<1/j$ for each $1\leqslant i<j$. Choose $z_i>0$ less than the minimum distance between points of distinct stickers of Γ_{ki}, for each $k<i$. We now set

$$\eta_j = \min(\{2^{-j}\}\cup\{z_i 2^{-(j+1)}: 1\leqslant i\leqslant j\}\cup\{\delta_i 2^{-(j+2)}: 1<i\leqslant j\}).$$

This completes the description of the inductive hypothesis. The construction can be continued in the obvious manner, with the following results. For each i, the sequence $\{\Gamma_{ij}: i\leqslant j\}$ of graphs converges to a connected local dendron $\Gamma_i^*\subset X$ with a dense set of branch points, and $\Gamma_i^*\approx \operatorname{invlim}(\Gamma_{ij}, \sigma_{ij})$. And for each i, the sequence of induced maps $\{g_{ij}: C(\Gamma_{i+1,j})\to C(\Gamma_{ij}): j>i\}$ converges to a map $g_i^*: C(\Gamma_{i+1}^*)\to C(\Gamma_i^*)$ which is a near-homeomorphism (by the same argument as given in the proof of Theorem 6.2 of [4]). Finally, using the form of argument of the previous proof for $2^X\approx Q$, it is not difficult to verify that the inverse sequence

$$C(\Gamma_1^*)\xleftarrow{g_1^*}C(\Gamma_2^*)\xleftarrow{g_2^*}\ldots$$

satisfies the conditions of the Approximation Lemma, and therefore $C(X)\approx Q$.

§ 5. The relative hyperspaces 2_A^X and $C_A(X)$. Recall that for $A\in 2^X$, $2_A^X = \{B\in 2^X: B\supset A\}$, and for $A\in C(X)$, $C_A(X) = \{B\in C(X): B\supset A\}$.

LEMMA 5.1. *Let X be a nondegenerate Peano continuum, and $p\in X$. Then $2_p^X\approx Q$, $C_p(X)\times Q\approx Q$, and $C_p(X)\approx Q$ if X contains no free arcs.*

Proof. We may assume that p is in some element of G_i, for each partition G_i considered above. Moreover, we may suppose that p is a vertex of each nerve Γ_i, and that $\varphi_i(p) = \{p\}$ for each map $\varphi_i: \Gamma_{i+1}\to C(\Gamma_i)$.

Application of the Approximation Lemma to the inverse sequence $2_p^{\Gamma_1}\xleftarrow{f_1}2_p^{\Gamma_2}\xleftarrow{f_2}\ldots$, where the maps $\{f_i\}$ are the induced near-homeomorphisms, gives the result $2_p^X\approx Q$. The same procedure (via the proof of Theorem 4.1) works for $C_p(X)$.

THEOREM 5.2. *If X is a Peano continuum, with $X\neq A\in 2^X$, then $2_A^X\approx Q$. If $X\neq A\in C(X)$, then $C_A(X)\times Q\approx Q$, and $C_A(X)\approx Q$ if $X\setminus A$ contains no free arcs.*

Proof. The quotient $X^* = X/A$ is a Peano continuum. Let A^* be the point of X^* corresponding to A. Then $2_A^X\approx 2_{A^*}^{X^*}$, and for $A\in C(X)$, $C_A(X)\approx C_{A^*}(X^*)$.

§ 6. Refinements and nerves of S-partitions. In the remaining sections we consider some increasingly technical conditions and theorems on S-partitions and their refinements, leading ultimately to the proof of the Partition Refinement Theorem 2.3.

The *boundary* of a partition G is defined by

$$\operatorname{Bd} G = \bigcup\{\operatorname{Bd} g: g\in G\} = X\setminus\bigcup\{g: g\in G\}.$$

We shall assume that each element of G is a maximal open (in X) subset of its closure, so that $p\in \operatorname{Bd} G$ if and only if $p\in \overline{g_1}\cap\overline{g_2}$ for distinct elements g_1 and g_2 of G.

Let H be a refinement of a partition G. The collection of *border elements* of H is defined by

$$\operatorname{Bord}_G H = \{h\in H: \bar{h}\cap \operatorname{Bd} G\neq\emptyset\},$$

and the *core elements* by $\operatorname{Core}_G H = H\setminus \operatorname{Bord}_G H$. The subscript G will be omitted when it is clear from the context.

The refinement H is a *core-connected refinement* of G if the following conditions are satisfied:

(i) each border element of H is adjacent to a core element;

(ii) for each element g of G, the union of the closures of the core elements of H contained in g is connected.

THEOREM 6.1 ([1], [2]). *If G is an S-partition of X, then for each $\varepsilon>0$ there exists an ε-S-partition H of X which is a core-connected refinement of G.*

Since each Peano space X has Property S, we may use the above theorem to construct a sequence $[G_i]$ of partitions of X such that each G_{i+1} is a core-connected refinement of G_i and $\operatorname{mesh} G_i\to 0$.

LEMMA 6.2. *Each boundary point of an element g of an S-partition of X is accessible from g.*

Proof. There exists a core-connected S-refinement G_0 of G. Let p be a boundary point of $g\in G$, g_0 a border element of G_0 containing p in its boundary, and p_0 any common boundary point of g_0 and a core element of G_0. For $\varepsilon_1 = d(p_0, \operatorname{Bd} g)$, take an ε_1-S-core-connected refinement G_1 of G_0, and let g_1 be a border element of G_1 containing p in its boundary. Pick any common boundary point p_1 of g_1 and a core element of G_1. Since each connected union of closures of elements of G_1 is a Peano space, and therefore arc-connected, there exists an arc α_0 from p_0 to p_1 which is contained in the union of the closures of elements of G_1 which are contained in g_0 and do not have boundary points in $\operatorname{Bd} g$. Thus $\operatorname{diam}\alpha_0<\operatorname{mesh} G_0$ and $\alpha_0\cap \operatorname{Bd} g = \emptyset$.

Inductively, continuing this procedure, we construct partitions $\{G_i\}$ and arcs $\{\alpha_i\}$ with $\operatorname{diam}\alpha_i<\operatorname{mesh} G_i\to 0$, such that $\dot{\alpha}_i = \{p_i, p_{i+1}\}$, $\alpha_i\cap \operatorname{Bd} g = \emptyset$, and $p_i\to p$. Thus $\bigcup\{\alpha_i\}$ is a path from p_0 to p hitting $\operatorname{Bd} g$ only at p, and since $\bigcup\{\alpha_i\}$ is itself a Peano space, it contains an arc between p_0 and p.

LEMMA 6.3. *There exists a nerve for every S-partition G of X.*

Proof. Let B be a (finite) subset of $\operatorname{Bd} G$ formed by selecting exactly one point from each maximal nonempty intersection $\overline{g_1}\cap\ldots\cap\overline{g_k}$. For each $g\in G$ there exists a compact connected graph (in fact, a tree) $T_g\subset\bar{g}$ such that $T_g\cap \operatorname{Bd} g = B\cap \operatorname{Bd} g$ and each point of $T_g\cap \operatorname{Bd} g$ is an endpoint of T_g. (We are using here the accessibility of points of $\operatorname{Bd} g$ from g). Then $\Gamma = \bigcup\{T_g: g\in G\}$ is a nerve, with $\Gamma\cap \operatorname{Bd} G = B$.

For a nerve Γ of a partition G, we shall always consider a triangulation in which $\mathrm{Bd}\,\Gamma = \Gamma \cap \mathrm{Bd}\,G$ is contained in the vertex set (we say that each $p \in \mathrm{Bd}\,\Gamma$ is a *boundary vertex*), and such that for each $g \in G$ there is at least one vertex (an *interior vertex*) of Γ in g.

§ 7. **C-monotone maps on partition nerves.** The Partition Refinement Theorem is proved by the simultaneous inductive construction of a refinement H of G *and* a function Φ: $H \to C(\Gamma)$ such that, for any nerve Λ of H, Φ induces in a well-specified manner a C-monotone piecewise-linear map φ: $\Lambda \to C(\Gamma)$ having the properties called for by the theorem. In this section, we describe the necessary properties of the function Φ (Definitions 7.1–7.3), state the Existence Theorem 7.4 for the pair (H, Φ), and show how Φ induces the map φ (Theorem 7.5).

Let H be a partition of X, Γ a compact connected graph in X (not necessarily a nerve of H), and Φ: $H \to C(\Gamma)$ a function, with $H_0 = \{h \in H\colon \Phi(h) \text{ is degenerate}\}$.

DEFINITION 7.1. The function Φ: $H \to C(\Gamma)$ is *C-monotone* if the following conditions are satisfied:

(i) For each vertex v of Γ, the set $\bigcup\{\bar{h}\colon \Phi(h) = v\}$ is nonempty and connected;

(ii) For each edge e of Γ, the subset $\{h \in H_0\colon \Phi(h) \in \mathrm{int}\, e\}$ is nonempty, and Φ is 1-1 on this subset;

(iii) For adjacent elements h and k of H_0, either $\Phi(h) = \Phi(k)$ or $\Phi(h)$ and $\Phi(k)$ are adjacent points in the subdivision of Γ determined by $\Phi(H_0)$, and conversely, each pair of adjacent points of $\Phi(H_0)$ arises in this way;

(iv) There exists a relation $\searrow$ in H such that:

a) if $h \searrow k$, then h and k are adjacent and $\Phi(h) \supset \Phi(k)$;

b) if $h_1 \searrow k_1$, $h_2 \searrow k_2$, and $\overline{h_1} \cap \overline{k_1} \cap \overline{h_2} \cap \overline{k_2} \neq \emptyset$, then $\Phi(k_1) = \Phi(k_2)$;

c) if $h \searrow k$ and h is adjacent to an element h_0 of H_0, then $k \in H_0$, and $\Phi(k) = \Phi(h_0)$ if $\Phi(h_0)$ is a vertex of Γ;

d) for each element h of H there exists a chain $(h_j, h_{j-1}, \ldots, h_0)$ with $h_j = h$, $h_0 \in H_0$, and $h_{i+1} \searrow h_i$ for $i = 0, \ldots, j-1$.

DEFINITION 7.2. The function Φ: $H \to C(\Gamma)$ is *ε-continuous* if for each pair of adjacent elements h and k of H, $d^*(\Phi(h), \Phi(k)) \leqslant \varepsilon$ and either $\Phi(h) \cap \Phi(k) \neq \emptyset$ or $h, k \in H_0$.

DEFINITION 7.3. If the partition H refines the partition G, we say that the function Φ: $H \to C(\Gamma)$ is *limited by* G if, for each $h \in H$, $\Phi(h) \subset \mathrm{St}(g_h; G)$, where $h \subset g_h \in G$.

THEOREM 7.4. *Let G be an S-partition of X, with a nerve Γ. Then for each $\varepsilon > 0$ there exists an ε-S-refinement H of G, and a function Φ: $H \to C(\Gamma)$ such that:*

(i) *Φ is limited by G;*

(ii) *Φ is ε-continuous;*

(iii) *Φ is C-monotone.*

Constructions leading to the proof of Theorem 7.4 are begun in § 8.

THEOREM 7.5. *Let the function Φ: $H \to C(\Gamma)$ be C-monotone and ε-continuous, and let Λ be a nerve of H. Then there exists a C-monotone piecewise-linear map φ: $\Lambda \to C(\Gamma)$ such that:*

(i) *for each $x \in \Lambda$, either $\varphi(x) \subset \Phi(h) \cup \Phi(k)$ for some adjacent pair $h, k \in H$ with $x \in \bar{h}$, or $\varphi(x) \subset [\Phi(h), \Phi(k)]$ for some adjacent pair $h, k \in H_0$ with $x \in \bar{h}$;*

(ii) *for each $h \in H$,* $\mathrm{diam}\,\varphi(\Lambda \cap \bar{h}) \leqslant 2\varepsilon$.

Proof. We first define φ on the vertices of Λ subject to the following conditions:

1) for each vertex v, $\varphi(v) = \Phi(h)$ for some $h \in H$ such that $v \in \bar{h}$;

2) if $v \in \bar{h} \cap \bar{k}$ and $h \searrow k$, then $\varphi(v) = \Phi(k)$;

3) if $v \in \bar{h}$ for $h \in H_0$, then $\varphi(v)$ is degenerate, and $\varphi(v) = \Phi(h)$ if $\Phi(h)$ is a vertex of Γ.

The consistency of conditions 2) and 3) is guaranteed by conditions (iv), b), c) of Definition 7.1.

For each pair p, q of adjacent points of $\Phi(H_0)$, pick adjacent elements h_p, h_q of H_0 with $\Phi(h_p) = p$ and $\Phi(h_q) = q$, and pick $b_{pq} \in \mathrm{Bd}\,\Lambda \cap \overline{h_p} \cap \overline{h_q}$. Let $B_0 \subset \mathrm{Bd}\,\Lambda$ be the collection of all such choice b_{pq}. Note that either $\varphi(b_{pq}) = p$ or $\varphi(b_{pq}) = q$.

Let $H = \{h_1, \ldots, h_n\}$. For each $b \in \Lambda \cap \mathrm{Bd}\,h_i$ let b_i be an interior point of the unique edge of Λ in $\bar{h}_i$ containing b. We now define φ on the subset

$$\{b_i\colon b \in \Lambda \cap \mathrm{Bd}\,h_i,\ 1 \leqslant i \leqslant n\}$$

of Λ as follows:

1) $\varphi(b_i) = \Phi(h_i) \cup \varphi(b)$ if $h_i \notin H_0$ (note that $\Phi(h_i) \cap \varphi(b) \neq \emptyset$ by the second part of Definition 7.2);

2) $\varphi(b_i) = [\Phi(h_i), \varphi(b)]$ if $h_i \in H_0$ and $b \notin B_0$;

3) $\varphi(b_i) = \Phi(h_i)$ if $h_i \in H_0$ and $b \in B_0$.

Let Λ' be the subdivision of Λ obtained by the addition of the points $\{b_i\}$ to the vertex set. Thus φ has been defined on the vertex set of Λ', and there exists a piecewise-linear extension φ: $\Lambda' \to C(\Gamma)$. This map satisfies the conditions of Theorem 7.5.

We observe that the Partition Refinement Theorem 2.3 follows immediately from Theorems 7.4 and 7.5. The deferred proof of Theorem 7.4 requires some technical lemmas which we discuss in the following sections.

§ 8. **Refinements which chain partition nerves.**

DEFINITION 8.1. Let G be a partition of X, with a nerve Γ, and let H be a refinement of G such that to each edge e of Γ there corresponds a chain $\tilde{e} = \{h_i^e\colon 1 \leqslant i \leqslant i_e\}$ of elements of H satisfying the following conditions:

(i) elements h_i^e and h_j^e of $\tilde{e}$ are adjacent if and only if $|i-j| = 1$;

(ii) the closures of the endlinks h_1^e and $h_{i_e}^e$ of each chain $\tilde{e}$ contain the endpoints of e;

(iii) elements of distinct chains $\tilde{e}$ and $\tilde{f}$ are adjacent (or identical) only if they are the endlinks whose closures contain a common endpoint of e and f;

(iv) an element of a chain $\tilde{e}$ is a border element of H only if it is the endlink whose closure contains an endpoint of e in $\mathrm{Bd}\,G$. Then we say that the refinement H *chains* the nerve Γ, with the *chaining set* $\bigcup\{\tilde{e}\colon e$ an edge of $\Gamma\}\subset H$.

LEMMA 8.2. *Let G be an S-partition of X, with a nerve Γ. Then for each $\varepsilon>0$ there exists a core-connected ε-S-refinement H of G which chains Γ.*

Proof. There exists by Theorem 6.1, a core-connected ε-S-refinement F of G. We may assume mesh F is small enough that the closure of a border element of F meets an edge e of Γ only if $e\cap \mathrm{Bd}\,G\neq\emptyset$, and meets distinct edges e_1 and e_2 only if $e_1\cap e_2\cap\mathrm{Bd}\,G\neq\emptyset$. If p is an interior vertex of Γ such that $p\in\mathrm{Bd}\,F$, let $f_p=\mathrm{int}(\bigcup\{\bar{f}\colon f\in F$ and $p\in\mathrm{Bd}\,f\})$. Then f_p is a connected open neighborhood of p with Property S, and the collection $\{f_p\}\cup\{f\in F\colon p\notin\mathrm{Bd}\,f\}$ is a core-connected S-refinement of G. Thus we may assume that each interior vertex of Γ is contained in an element of F.

Let $F_\Gamma=\{f\in\mathrm{Bord}\,F\colon \bar{f}\cap\Gamma\neq\emptyset$ and $\bar{f}\cap\mathrm{Bd}\,\Gamma=\emptyset\}$. (Recall that $\mathrm{Bd}\,\Gamma=\Gamma\cap\mathrm{Bd}\,G$ is the set of boundary vertices of Γ). Let $C=\bigcup\{\bar{f}\colon f\in\mathrm{Core}\,F\}$, and for each $f\in F_\Gamma$, let $\delta_f>0$ be the minimum distance between points of $\bar{f}\cap(\Gamma\cup C)$ and $\mathrm{Bd}\,G$. With $\delta=\min\{\delta_f\colon f\in F_\Gamma\}$, let F' be a core-connected $\frac{1}{2}\delta$-S-refinement of F, and let $F'(F_\Gamma)=\{f'\in F'\colon f'\subset f$ for some $f\in F_\Gamma\}$. Then clearly the partition $F^*=(F\setminus F_\Gamma)\cup F'(F_\Gamma)$ is a core-connected ε-S-refinement of G such that $F^*_\Gamma=\emptyset$. Thus we may assume that $F_\Gamma=\emptyset$.

For each edge e of Γ with a boundary vertex b as one endpoint, let f_b^e be a border element of F with closure containing b for which $d^*(b,\overline{f_b^e}\cap e)>0$ is a maximum. (Note that for any $f\in F$ with $\bar{f}\cap e\neq\emptyset$ and $d^*(b,\bar{f}\cap e)>d^*(b,\overline{f_b^e}\cap e)$, we must then have $f\in\mathrm{Core}\,F$.) For each interior vertex p of Γ, let f_p be the element of F containing p.

The partition elements $\{f_b^e\}$ and $\{f_p\}$ thus selected will serve as the endlinks of the required chains for edges of Γ. It is now easily seen that for any S-refinement $\tilde{F}$ of F with sufficiently small mesh, $H=\mathrm{Bord}\,F\cup\{f_p\colon p$ an interior vertex of $\Gamma\}\cup\cup\tilde{F}(\mathrm{Core}\,F\setminus\{f_p\})$ is a core-connected ε-S-refinement of G which chains Γ.

§ 9. Refinements with border element parameters. Let $G=\{g_1,\ldots,g_n\}$ be an S-partition of X, with a nerve Γ. For $\delta>0$ and $g\in G$, let $N_\delta(g)$ be the open δ-neighborhood of g in X, and $\overline{N_\delta}(g)$ its closure. There exists $\varepsilon_0>0$ such that if $N_{2\varepsilon_0}(g_{i_1})\cap\ldots\cap N_{2\varepsilon_0}(g_{i_k})\neq\emptyset$, then $\overline{g_{i_1}}\cap\ldots\cap\overline{g_{i_k}}\neq\emptyset$. Since the boundary vertices of Γ are contained in the *maximal* nonempty intersections of the closures of partition elements, it follows that the distance between distinct boundary vertices is at least $2\varepsilon_0$.

LEMMA 9.1. *For each $0<\varepsilon<\varepsilon_0$, there exists a core-connected ε-S-refinement H of G which chains Γ, and functions $f\colon \mathrm{Bord}\,H\to\Pi\{I_i\colon 1\leqslant i\leqslant n\}\times\Pi\{I_i^b\colon b\in\Gamma\cap\mathrm{Bd}\,g_i,$ $1\leqslant i\leqslant n\}$ and $\pi\colon\mathrm{Bord}\,H\to\mathrm{Bord}\,H$ with the following properties (f_i and f_i^b are the compositions of f with the projections on the $I_i=[0,1]$ and $I_i^b=[0,1]$ coordinates, respectively):*

(i) *for each pair of adjacent elements h_1 and h_2 of* $\mathrm{Bord}\,H$, *we have $|f_i(h_1)-f_i(h_2)|\leqslant\varepsilon$ for each i, $|\max_i\{f_i^b(h_1)\}-\max_i\{f_i^b(h_2)\}|\leqslant\varepsilon$ for each $b\in\mathrm{Bd}\,\Gamma$, and if $h_1,h_2\subset g_i$ or if $f_i(h_1),f_i(h_2)>0$, then $|f_i^b(h_1)-f_i^b(h_2)|\leqslant\varepsilon$ for each $b\in\Gamma\cap\mathrm{Bd}\,g_i$;*

(ii) *for each $b\in\mathrm{Bd}\,\Gamma$, the subcollection $H^b=\{h\in\mathrm{Bord}\,H\colon f_i(h)=0=f_i^a(h)$ for all i and all $a\neq b\}$ is connected along $\mathrm{Bd}\,G$, i.e., if $H^b=H_1^b\cup H_2^b$ where $H_1^b\neq\emptyset\neq H_2^b$, then $\overline{h_1}\cap\overline{h_2}\cap\mathrm{Bd}\,G\neq\emptyset$ for some $h_1\in H_1^b$ and $h_2\in H_2^b$. Also, $\{h\colon b\in\bar{h}\}\subset H^b\subset\{h\colon \bar{h}\subset N_\varepsilon(b)\}$;*

(iii) *if $f_i(h)>0$, then $\bar{h}\subset N_\varepsilon(g_i)$;*

(iv) *if $h\subset g_i$ or if $f_i(h)>0$, then $f_i^b(h)=1$ for some $b\in\bigcap\{\overline{g_j}\colon \bar{h}\cap\overline{N_\varepsilon}(g_j)\neq\emptyset\}$;*

(v) *if $f_i^b(h)>0$, then either $f_j(h)=1$ for some j such that $b\in\mathrm{Bd}\,g_j$, or $f_j^a(h)=0$ for all j and all $a\neq b$;*

(vi) *for each $h\in\mathrm{Bord}\,H$, $\bar{h}\cap\overline{\pi(h)}\neq\emptyset$ and $f(h)\geqslant f(\pi(h))$ (in each coordinate);*

(vii) *$\{h\colon\pi(h)=h\}=\{h\colon f_i(h)=0$ for all $i\}\cup\{h\colon f_i(h)=1$ for $h\subset g_i\}$, and some iterate of π is a function onto this set of fixed elements;*

(viii) *if $\overline{h_1}\cap\overline{h_2}\cap\overline{\pi(h_1)}\cap\overline{\pi(h_2)}\neq\emptyset$, with $\pi(h_1)\neq h_1$ and $\pi(h_2)\neq h_2$, then $\pi(h_1)=\pi(h_2)$;*

(ix) *if h_1 and h_2 are adjacent and $f_i(h_2)=0$ for all i, then $f_i(\pi(h_1))=0$ for all i.*

Proof. By the chaining Lemma 8.2 there exists a core-connected ε-S-refinement H of $G=\{g_1,\ldots,g_n\}$ which chains the nerve Γ. We construct the desired functions f and π by defining them inductively on the collections $\mathrm{Bord}\,H\cap H(g_i)$, $1\leqslant i\leqslant n$. (Recall that $H(g_i)=\{h\in H\colon h\subset g_i\}$). Thus, for $h\in\mathrm{Bord}\,H\cap H(g_i)$, we will have $\pi(h)\in\mathrm{Bord}\,H\cap H(g_j)$ for some $j\leqslant i$. This inductive construction will require, at the ith stage, that mesh $H(g_i)$ be sufficiently small. Note that if $H_1,\ldots,H_n$ are core-connected ε-S-refinements of G, each of which chains Γ, then $H=H_1(g_1)\cup\ldots\cup H_n(g_n)$ is also a core-connected ε-S-refinement of G which chains Γ. To simplify the notation, we therefore assume that for each i, mesh $H(g_i)$ is as small as may be required. Set $\mathrm{Bord}\,H(g_i)=\mathrm{Bord}\,H\cap H(g_i)$.

For an element h of $\mathrm{Bord}\,H(g_1)$ adjacent to a boundary vertex b of Γ, we set $f_1^b(h)=1$ and all other coordinates of $f(h)$ equal to 0. This definition is unambiguous and consistent with property (i), since for elements h_i and h_j adjacent to distinct boundary vertices a and b, respectively, we must have $\overline{h_i}\cap\overline{h_j}=\emptyset$, otherwise $d(a,b)<2\varepsilon<2\varepsilon_0$.

For each h and b as above, we obviously have

$$\{g_i\colon b\in\mathrm{Bd}\,g_i\}\subset\{g_i\colon \bar{h}\cap\overline{N_\varepsilon}(g_i)\neq\emptyset\}.$$

By choice of the set $\mathrm{Bd}\,\Gamma$ of boundary vertices, the collection $\{\overline{g_i}\colon b\in\mathrm{Bd}\,g_i\}$ must be maximal with respect to having a nonempty intersection. Since

$$\bigcap\{\overline{g_i}\colon \bar{h}\cap\overline{N_\varepsilon}(g_i)\neq\emptyset\}$$

is also nonempty (by our stipulation that $\varepsilon<\varepsilon_0$), we must have $\{g_i\colon b\in \operatorname{Bd} g\} = \{g_i\colon \bar{h}\cap \overline{N_\varepsilon}(g_i)\neq \emptyset\}$, and therefore property (iv) is satisfied.

For each boundary vertex $b\in \operatorname{Bd} g_1$, set $L^0(b) = \{h\in \operatorname{Bord} H(g_1)\colon b\in \operatorname{Bd} h\}$, and inductively define $L^{k+1}(b) = \{h\in \operatorname{Bord} H(g_1)\setminus \bigcup_{i\leqslant k} L^i(b)\colon h$ is adjacent to some element of $L^k(b)\}$. We may assume for convenience that $\varepsilon = 1/N$ for some integer N. We assume $\operatorname{mesh} H(g_1)$ is small enough that $L^i(a)\cap L^j(b) = \emptyset$ for $0\leqslant i, j\leqslant 2N$ and distinct boundary vertices a and b.

The function f has been defined above on elements of $L^0(b)$, for each $b\in \Gamma\cap \operatorname{Bd} g_1$. In general, for $h\in L^i(b)$ with $0\leqslant i\leqslant N$, we set $f_1^b(h) = 1$ and $f_1(h) = i/N$, with all other coordinates of $f(h)$ equal to 0. For $h\in L^i(b)$ with $N\leqslant i\leqslant 2N$, we set $f_1^b(h) = f_1(h) = 1$, $f_1^a(h) = i/N-1$ for each $a\in \Gamma\cap \operatorname{Bd} g_1$ with $a\neq b$, and all other coordinates of $f(h)$ equal to 0. Finally, for $h\in \operatorname{Bord} H(g_1)$ not in any layer $L^i(b)$, $b\in \Gamma\cap \operatorname{Bd} g_1$, $i\leqslant 2N$, we define $f_1^b(h) = f_1(h) = 1$ for all $b\in \Gamma\cap \operatorname{Bd} g_1$, with all other coordinates equal to 0.

We assume that $\operatorname{mesh} H(g_1)$ is small enough that every element of $L^i(b)$, $i\leqslant 2N$, is contained in $N_\varepsilon(b)$, for each $b\in \Gamma\cap \operatorname{Bd} g_1$. Thus

$$\{g\in G\colon b\in \operatorname{Bd} g\}\subset \{g\in G\colon \bar{h}\cap \overline{N_\varepsilon}(g)\neq \emptyset\}$$

for each $h\in L^i(b)$, $i\leqslant 2N$, and it follows as before that $\{g\in G\colon b\in \operatorname{Bd} g\} = \{g\in G\colon \bar{h}\cap \overline{N_\varepsilon}(g)\neq \emptyset\}$, and therefore property (iv) is satisfied.

We linearly order the elements of $\operatorname{Bord} H(g_1)$ in such a way that $h_i<h_j$ whenever $h_i\in L^i(b)$, $h_j\in L^j(b)$, $b\in \Gamma\cap \operatorname{Bd} g_1$, and $i<j\leqslant 2N$. Then for each $h\in L^i(b)$, $0<i<N$, we define $\pi(h)$ to be the first element of $\operatorname{Bord} H(g_1)$ adjacent to h. Thus $\pi(h)\in L^{i-1}(b)$. For all other $h\in \operatorname{Bord} H(g_1)$ we set $\pi(h) = h$. Clearly, this is consistent with properties (vi)-(ix).

Suppose now that functions f and π with properties (i)-(ix) have been defined on $\bigcup_{i<k} \operatorname{Bord} H(g_i)$, with each $\pi(h)$ an element of $\bigcup_{i<k} \operatorname{Bord} H(g_i)$, and $f_i(h) = f_i^b(h) = 0$ for all $i\geqslant k$ and all b. Suppose also that the elements of $\bigcup_{i<k} \operatorname{Bord} H(g_i)$ have been linearly ordered in such a way that $h<h'$ whenever $h\in \operatorname{Bord} H(g_i)$ and $h'\in \operatorname{Bord} H(g_j)$ with $i<j$, and the function π has the property that for each $h\in \bigcup_{i<k} \operatorname{Bord} H(g_i)$, either $\pi(h) = h$, or $f_i(\pi(h)) = 0$ for all i, or $\pi(h)<h$ and $\pi(h)$ is the first element adjacent to h.

We assume that $\operatorname{mesh} H(g_k)<\frac{1}{2}d(p,q)$, for any pair of points p and q in non-adjacent elements of $\bigcup_{i<k} \operatorname{Bord} H(g_i)$. Thus if adjacent elements h_k and h_k' of $\operatorname{Bord} H(g_k)$ are adjacent to elements h_j and h_j', respectively, of $\bigcup_{i<k} \operatorname{Bord} H(g_i)$, then h_j and h_j' must be adjacent.

For each $b\in \Gamma\cap \operatorname{Bd} g_k$ such that $b\in \operatorname{Bd} g_i$ for some $i<k$, let $M^0(b) = \{h\in \operatorname{Bord} H(g_k)\colon h$ is adjacent to an element of $H^b\cap (\bigcup_{i<k} \operatorname{Bord} H(g_i))\}$. (Recall that $H^b = \{h\in \operatorname{Bord} H\colon f_i(h) = f_i^a(h) = 0$ for all i and all $a\neq b\}$.) For each $h\in M^0(b)$ we select an adjacent element h' of $H^b\cap (\bigcup_{i<k} \operatorname{Bord} H(g_i))$ and define $f(h) = f(h')$ with the exception that $f_k^b(h) = 1$. Thus $h\in H^b$. Properties (ii) and (iv) are satisfied if $\operatorname{mesh} H(g_k)$ is small enough. We set $\pi(h) = h$.

For each boundary vertex b as above, we now consider the collection $M^1(b) = \{h\in \operatorname{Bord} H(g_k)\setminus M^0(b)\colon h$ is adjacent to an element of $M^0(b)\}$. Linearly order $M^0(b)$, and for each $h\in M^1(b)$ consider the first adjacent element h' in $M^0(b)$. We set $f(h) = f(h')$ with the exception that $f_k(h) = \varepsilon$, and we set $\pi(h) = h'$.

We next consider those elements of $\operatorname{Bord} H(g_k)$ which are not in $M^0(b)\cup M^1(b)$ for any boundary vertex b, but which are adjacent to some element of $\bigcup_{i<k} \operatorname{Bord} H(g_i)$. For each such element h, define $\pi(h)$ to be the first adjacent element in $\bigcup_{i<k} \operatorname{Bord} H(g_i)$. We define $f(h) = f(\pi(h))$ with the exception that, for each $b\in \Gamma\cap \overline{g_k}\cap \overline{g_i}$ for some $i<k$, we set $f_k^b(h) = \max_i\{f_i^b(\pi(h))\colon b\in \operatorname{Bd} g_i, i<k\}$.

Let $L^0(g_k) = \{h\in \operatorname{Bord} H(g_k)\colon h$ is adjacent to an element of $\operatorname{Bord} H(g_i)$ for some $i<k$, or $h\in M^1(b)$ for some boundary vertex $b\}$, and inductively define $L^{i+1}(g_k) = \{h\in \operatorname{Bord} H(g_k)\setminus \bigcup_{j\leqslant i} L^j(g_k)\colon h$ is adjacent to an element of $L^i(g_k)\}$.

We linearly order the elements of $\operatorname{Bord} H(g_k)$ such that $h_i<h_j$ whenever $h_i\in L^i(g_k)$, $h_j\in L^j(g_k)$, and $i\leqslant j$. (The previous orderings of the subcollections $M^0(b)$ of $\operatorname{Bord} H(g_k)$ are no longer relevant; they served only to insure property (viii).) For each $h\in L^i(g_k)$, $i>0$, let $\alpha(h)$ be the first adjacent element in $\operatorname{Bord} H(g_k)$; thus $\alpha(h)\in L^{i-1}(g_k)$.

For $h\in L^i(g_k)$ with $0<i<N$ (recall that $N\varepsilon = 1$), we set $\pi(h) = \alpha(h)$. If $i\geqslant N$, set $\pi(h) = h$. The function f is defined on the layers $L^i(g_k)$ as follows. For $h\in L^i(g_k)$, $0<i\leqslant N$, we set $f(h) = f(\alpha(h))$ with the exception that $f_k(h) = i\varepsilon$. For $N<i\leqslant 2N$, set $f(h) = f(\alpha(h))$ with the exceptions that $f_k^b(h) = \max\{f_k^b(\alpha(h)), (i-N)\varepsilon\}$ for each $b\in \Gamma\cap \operatorname{Bd} g_k$. For $2N<i\leqslant 3N$, set $f(h) = f(\alpha(h))$ with the exceptions that $f_j^b(h) = \min\{f_j^b(\alpha(h)), (3N-i)\varepsilon\}$ for each $b\in \Gamma\cap (\operatorname{Bd} g_j\setminus \operatorname{Bd} g_k)$, $j<k$. Finally, for $3N<i\leqslant 4N$, set $f(h) = f(\alpha(h))$ with the exceptions that $f_j(h) = \min\{f_j(\alpha(h)), (4N-i)\varepsilon\}$ for each $j<k$.

It can be verified that properties (i)-(v) of the function f (as it has been defined so far) are satisfied if $\operatorname{mesh} H(g_k)$ is small enough. Properties (vi)-(ix) of the function π are also satisfied.

For each $b\in \Gamma\cap (\operatorname{Bd} g_k\setminus \bigcup \{\operatorname{Bd} g_j\colon j<k\})$, set

$$L^0(b) = \{h\in \operatorname{Bord} H(g_k)\colon b\in \operatorname{Bd} h\},$$

and inductively define $L^{i+1}(b) = \{h\in \operatorname{Bord} H(g_k)\setminus \bigcup_{j\leqslant i} L^j(b)\colon h$ is adjacent to an element of $L^i(b)\}$. We assume $\operatorname{mesh} H(g_k)$ is small enough that $L^i(g_k)\cap L^j(b) = \emptyset$ for $0\leqslant i\leqslant 4N$, $0\leqslant j\leqslant 2N$, and each boundary vertex b as above, and that $L^i(a)\cap L^j(b) = \emptyset$ for $0\leqslant i, j\leqslant 2N$ and distinct boundary vertices a and b as above. The functions f and π are defined on the elements of each layer $L^i(b)$ in the fashion as previously defined on the layers in $H(g_1)$. Thus, for $h\in L^i(b)$, $0\leqslant i\leqslant N$, we set $f_k^b(h) = 1$;

$f_k(h) = i\varepsilon$, and set all other f-coordinates equal to zero. For $N \leqslant i \leqslant 2N$, we set $f_k(h) = f_k^b(h) = 1$, and $f_k^a(h) = (i-N)\varepsilon$ for each $a \in \Gamma \cap \mathrm{Bd}\, g_k$ with $a \neq b$, and set all other coordinates equal to 0.

We may assume that the previously constructed linear ordering on $\mathrm{Bord}\, H(g_k)$ is such that $h_i < h_j$ whenever $h_i \in L^i(b)$ and $h_j \in L^j(b)$, with $0 \leqslant i < j \leqslant N$. Then for $h \in L^i(b)$, $0 < i < N$, we define $\pi(h)$ to be the first adjacent element; thus $\pi(h) \in L^{i-1}(b)$. Otherwise, set $\pi(h) = h$.

Finally, for all elements h of $\mathrm{Bord}\, H(g_k)$ not contained in any layer $L^i(g_k)$ for $0 \leqslant i \leqslant 4N$, or in any layer $L^i(b)$ for $0 \leqslant i \leqslant 2N$ and b a boundary vertex as above, we set $f_k(h) = f_k^b(h) = 1$ for each $b \in \Gamma \cap \mathrm{Bd}\, g_k$, with all other coordinates equal to 0.

This completes the inductive step, and with it the proof of the lemma.

§ 10. Stratification of border elements of a refinement.

LEMMA 10.1. *Let H be a core-connected S-refinement of a partition G of X, and m a positive integer. Then there exists an S-refinement K of H such that $K \supset \mathrm{Core}\, H$, and $K(\mathrm{Bord}\, H) = \{k \in K\colon k \subset h$ for some $h \in \mathrm{Bord}\, H\}$ has a decomposition $S^0 \cup \ldots \cup S^m \cup R$ with the following properties:*

(i) *$S^i \cap S^j = \emptyset$ if $i \neq j$, and $S^i \cap R = \emptyset$ for each i;*

(ii) *there exist adjacent elements of S^i and S^j only if $|i-j| \leqslant 1$, and adjacent elements of S^i and R only if $i = m$;*

(iii) $\mathrm{Bord}_G K \subset S^0$;

(iv) $R = \{k \in K(\mathrm{Bord}\, H)$: *$k$ is adjacent to an element of* $\mathrm{Core}\, H\}$;

(v) *for each $k \in S_h^l = S^l \cap K(h)$, $h \in \mathrm{Bord}\, H$, $0 \leqslant l < m$, there exists a chain between k and an element of S_h^{l+1} such that, with the possible exception of k, each link of the chain is an element of* $\mathrm{Core}_H K$, *and each interior link is an element of S_h^l nonadjacent to every element of S_h^{l-1};*

(vi) *for each $k \in S_h^l$, $0 < l \leqslant m$, there exists a chain between k and an element of S_h^{l-1} such that, with the possible exception of k, each link of the chain is an element of* $\mathrm{Core}_H K$, *each link except the last is an element of S_h^l, and each link except k or one of the last two links is nonadjacent to every element of S_h^{l-1};*

(vii) *each pair of elements $k, k' \in S_h^0$ is connected by a chain in S_h^0 for which each interior link is an element of* $\mathrm{Core}_H K$;

(viii) *each element of $R_h = R \cap K(h)$, $h \in \mathrm{Bord}\, H$, is adjacent to an element of S_h^m;*

(ix) *each element of S_h^m is connected by a chain to an element of R_h, such that each interior link is an element of $S_h^m \cap \mathrm{Core}_H K$ and is nonadjacent to every element of S_h^{m-1}.*

Proof. Choose $\delta_0 > 0$ such that the δ_0-neighborhood of $\mathrm{Bd}\, G$ does not meet $\bigcup \{\bar{h}\colon h \in \mathrm{Core}\, H\}$, and let K^0 be a core-connected δ_0-S-refinement of H. Let $S^0 = \{k \in K^0(\mathrm{Bord}\, H)\colon k$ is nonadjacent to every element of $\mathrm{Core}\, H\}$. Choose $\delta_1 > 0$ such that the δ_1-neighborhood of $\bigcup \{\bar{k}\colon k \in S^0\}$ does not meet $\bigcup \{\bar{h}\colon h \in \mathrm{Core}\, H\}$ and the δ_1-neighborhood of $\mathrm{Bd}\, H$ does not meet any element), of $\mathrm{Core}_H K^0$. Let K^1 be a core-connected δ_1-S-refinement of K^0. With $R^0 = K^0(\mathrm{Bord}\, H) \setminus S^0$, set $S^1 = \{k \in K^1(R^0)\colon k$ is nonadjacent to every element of $\mathrm{Core}\, H\}$. Let $R^1 = K^1(R^0) \setminus S^1$, and choose $\delta_2 > 0$ such that the δ_2-neighborhood of $\bigcup \{\bar{k}\colon k \in S^1\}$ does not meet $\bigcup \{\bar{h}\colon h \in \mathrm{Core}\, H\}$ and the δ_2-neighborhood of $\mathrm{Bd}\, H$ does not meet any element of $\mathrm{Core}_{K^0} K^1$.

Continuing this procedure, we obtain the desired S-refinement

$$K = S^0 \cup \ldots \cup S^m \cup R \cup \mathrm{Core}\, H,$$

where at the last step we take K^m to be a core-connected δ_m-S-refinement of K^{m-1}, $S^m = \{k \in K^m(R^{m-1})\colon k$ is nonadjacent to every element of $\mathrm{Core}\, H\}$, and $R = R^m = K^m(R^{m-1}) \setminus S^m$. The verification of the listed properties is routine.

§ 11. Proof of Theorem 7.4. Let $\varepsilon_0 > 0$ be chosen with respect to the partition $G = \{g_1, \ldots, g_n\}$ as in § 9, and let M be a positive integer such that $1/M < \min\{\varepsilon, \varepsilon_0\}$. Then choose $\mu > 0$ such that $\mu < 1/2M$ and $M\mu < d(p, q)$, for every pair of adjacent vertices p and q of Γ. Let ϱ be the minimum path-length metric on Γ, where each edge of Γ is metrized linearly and has unit length.

Let H be a core-connected μ-S-refinement of G which chains Γ, and f and π the functions on $\mathrm{Bord}\, H$ (constructed with respect to μ), given by Lemma 9.1. Let K be an S-refinement of H given by Lemma 10.1, where we take $m = 3M$. (The desired C-monotone function Φ will eventually be defined on a refinement of K.)

Let $C \subset H$ be a chaining set for Γ. Since each chain $\tilde{e} \subset C$ has at least $M+1$ links, there exists a function $\Phi\colon C \to \Gamma$ such that $\Phi(c) = p$ if c is an endlink whose closure contains the vertex p, $\Phi(c)$ is an interior point of the edge e if c is an interior link in a chain $\tilde{e}$, and $\varrho(\Phi(c_1), \Phi(c_2)) \leqslant 1/M$ for adjacent links c_1 and c_2. (Actually, we are interested now only in the restriction $\Phi|_{C \cap \mathrm{Core}\, H}$, since we shall be considering the stratification $K(\mathrm{Bord}\, H)$, of the border elements of H.)

For each endlink c of a chain $\tilde{e}$ such that $\bar{c}$ contains a boundary vertex b of Γ, let r_c^b be an element of $R_c = R \cap K(c)$ (where $R \subset K(\mathrm{Bord}\, H)$ is given by Lemma 10.1) adjacent to the next link of the chain $\tilde{e}$. We set $\Phi(r_c^b) = b$.

For $b \in \Gamma \cap \mathrm{Bd}\, g_i$, let $L_i^0(b) = \{h \in \mathrm{Bord}\, H(g_i)\colon b \in \mathrm{Bd}\, h\}$, and inductively define $L_i^{m+1}(b) = \{h \in \mathrm{Bord}\, H(g_i) \setminus \bigcup_{j \leqslant m} L_i^j(b)\colon h$ is adjacent to an element of $L_i^m(b)\}$. We may assume that for each $h \in \bigcup \{L_i^m(b)\colon 0 \leqslant m \leqslant M+1\}$, $f_j^a(h) = 0$ if $a \neq b$, $f_j(h) = 0$ if $i \neq j$, and $f_i(h) < 1$. (This assumption is justified by our construction of the function f and the fact that $(M+1)\mu \leqslant 2M\mu < 1$, provided $\mathrm{mesh}\, H(g_i)$ is chosen small enough that no element of $L_i^m(b)$, $m \leqslant M+1$, is adjacent to an element of $\mathrm{Bord}\, H(g_j) \setminus H^b$, for any $j < i$. And since $H^b \supset \{h\colon b \in \mathrm{Bd}\, h\}$, it is certainly possible to do this.)

The function Φ is defined on the subcollection $S^0 \cup \ldots \cup S^{3M}$ of $K(\mathrm{Bord}\, H)$ by use of the parameter provided by the function f on $\mathrm{Bord}\, H$. The first step is to define on the subcollection $S^0 \cup \ldots \cup S^{3M}$ a parameter function with the same properties as f. Accordingly, there will be no confusion if we use the same notation — f, f_i, f_i^b — for this function. For $k \in S_h^l$, $0 \leqslant l \leqslant M$ and

$$h \in \operatorname{Bord} H(g_i) \setminus \bigcup \{L_i^m(b)\colon m \leqslant M, b \in \Gamma \cap \operatorname{Bd} g_i\},$$

we set $f(k) = f(h)$, with the exception that $f_i(k) = \max\{f_i(h), l/M\}$. For $M \leqslant l \leqslant 2M$, set $f(k) = f(h)$ with the exceptions that $f_i(k) = 1$ and $f_j^b(k) = \min\{f_j^b(h), 2 - l/M\}$ for each boundary vertex b not in $\operatorname{Bd} g_i$. And for $2M \leqslant l \leqslant 3M$, set $f(k) = f(h)$ with the exceptions that $f_i(k) = 1$, $f_j^b(k) = 0$ for each boundary vertex b not in $\operatorname{Bd} g_i$, and $f_j(k) = \min\{f_j(h), 3 - l/M\}$ for each $j \neq i$.

For elements of S_h^l, where $h \in L_i^m(b)$ for some $m \leqslant M$ and $b \in \Gamma \cap \operatorname{Bd} g_i$, we modify the above assignment of f-coordinates as follows. For $k \in S_h^l$, $0 \leqslant l \leqslant m$, $f(k)$ is defined as above. However, for $k \in S_h^l$, $l < m$, we set $f(k) = f(k')$ for some (any) $k' \in S_h^m$.

Note that the function f thus defined on $S^0 \cup \ldots \cup S^{3M}$ is constant on each subcollection S_h^l, $0 \leqslant l \leqslant 3M$ and $h \in \operatorname{Bord} H$. The desired C-monotone function Φ can now be defined on $S^0 \cup \ldots \cup S^{3M}$ by the use of expansion homotopies $e_i\colon C(T_i) \times I \to C(T_i)$ on each tree $T_i = \Gamma \cap \overline{g_i}$, $1 \leqslant i \leqslant n$. Specifically, let $D_i = \operatorname{diam} T_i$ (with respect to the minimum path-length metric ϱ on Γ), and define the expansion map e_i by $e_i(A, t) = \{x \in T_i\colon \varrho(x, A) \leqslant tD_i\}$. Then for each $k \in S^0 \cup \ldots \cup S^{3M}$, set $\Phi(k) = \bigcup \{e_i(b, f_i(k) f_i^b(k))\colon b \in \Gamma \cap \operatorname{Bd} g_i \text{ with } f_i^b(k) > 0,\ 1 \leqslant i \leqslant n\}$.

We verify that each $\Phi(k) \in C(\Gamma)$ by using the properties (iii)-(v) of the function f described in Lemma 9.1 Let $k \subset h \in \operatorname{Bord} H(g_i)$. By property (iv), $f_i^b(h) = f_i^b(k) = 1$ for some $b \in \bigcap \{\overline{g_j}\colon \overline{h} \cap \overline{N_\mu(g_j)} \neq \emptyset\}$, and thus $b \in \Phi(k)$. Suppose $f_j^a(k) > 0$ for some j and some $a \neq b$. Then $f_j^a(h) \geqslant f_j^a(k) > 0$, and by property (v), $f_{j'}(h) = 1$ for some j' with $a \in \operatorname{Bd} g_{j'}$. By property (iii), $\overline{h} \cap \overline{N_\mu(g_{j'})} \neq \emptyset$, and thus $b \in \overline{g_{j'}}$. By property (iv), $f_{j'}^{a'}(h) = 1$ for some $a' \in \overline{g_i} \cap \overline{g_{j'}}$. Then also $f_{j'}^{a'}(k) = 1$. If $f_{j'}(k) = 1$, then $\Phi(k) \supset T_j \supset \{a, b\}$. On the other hand, if $f_{j'}(k) < 1$, then we must have $k \in S_h^l$ for some $l > 2M$, and $a \in \operatorname{Bd} g_i$ (since $f_j^a(k) > 0$). Then $f_i(k) = 1$, so that $\Phi(k) \supset T_i \supset \{a, b\}$. In either case, there is a subcontinuum of $\Phi(k)$ containing both a and b, and since a was arbitrary with $f_j^a(k) > 0$, it follows that $\Phi(k)$ is connected.

Note that $f(k) = f(k')$, and hence $\Phi(k) = \Phi(k')$ for elements k and k' in the same subcollection S_k^l; if $l = 3M$ and $h \in \operatorname{Bord} H(g_i) \setminus \bigcup \{L_i^m(b)\colon m \leqslant M, b \in \Gamma \cap \operatorname{Bd} g_i\}$, then $\Phi(k) = T_i$; and in any case, if $l = 3M$ and $h \in \operatorname{Bord} H(g_i)$, then $\Phi(k) \subset T_i$.

The function Φ has now been defined on the subcollection K^0 of the μ-S-refinement K, where

$$K^0 = S^0 \cup \ldots \cup S^{3M} \cup$$
$$\cup \{r_c^b\colon b \in \Gamma \cap \operatorname{Bd} G \cap \operatorname{Bd} c,\ c \in C \cap \operatorname{Bord} H\} \cup (C \cap \operatorname{Core} H).$$

Note that $S^0 \cup \ldots \cup S^{3M}$ and $C \cap \operatorname{Core} H$ have no adjacent elements; they are linked only by the connecting elements $\{r_c^b\}$. We have $\operatorname{Bord}_G K \subset S^0 \subset K^0$, and for any element k of K^0 adjacent to an element of $K \setminus K^0$, we have $\Phi(k) \subset T_i$, where $k \subset g_i$.

The function Φ, as constructed to this point, satisfies the conditions (i)-(iii) of the C-monotone Definition 7.1. To verify condition (i) we use the fact that for a boundary vertex b of Γ, the subcollection H^b of $\operatorname{Bord} H$ is connected along $\operatorname{Bd} G$, given by property (ii) of Lemma 9.1, together with the following observation. For $k \in K^0$, with $k \subset h \in \operatorname{Bord} H(g_i)$ and $b \in \Gamma \cap \operatorname{Bd} g_i$, we have $\Phi(k) = b$ if and only if one of the following conditions holds: either $b \in \operatorname{Bd} h$, or $h \in H^b$ and $k \in S^0$.

Φ will be ε-continuous provided M is sufficiently large (exactly how large will depend not only on ε but also on the diameters D_i of the subtrees T_i, $i = 1, \ldots, n$, of Γ). The typical problem that might arise, for example, would be the case in which, for adjacent elements h and h' of $\operatorname{Bord} H$, $0 < \max_i f_i^b(h) < \mu$ while $\max_i f_i^b(h') = 0$, for some boundary vertex b. Then for adjacent elements $k \subset h$ and $k' \subset h'$ of S^0, we have $b \in \Phi(k)$, while $\Phi(k')$ conceivably could contain no points near b. However, properties (iv) and (v) of Lemma 9.1 imply that $f_i(h) = 1$ for some i such that $b \in \operatorname{Bd} g_i$. Hence $f_i(h') > 1 - \mu$, and by property (iv), $f_i^a(h') = 1$ for some $a \in \operatorname{Bd} g_i$. Then $f_i(h') f_i^a(h') > 1 - \mu$, and it follows that $\Phi(k')$ contains almost all of T_i, and therefore contains points near $b \in \operatorname{Bd} g_i$. The other similar verifications of this sort, for adjacent elements in $S^0 \cup \ldots \cup S^{3M}$, also use properties (i), (iii), (iv), and (v) of Lemma 9.1 and are left to the reader. If $k \in S_h^l$ is adjacent to an element r_c^b, then $l = 3M$ and either $h = c$ or h is adjacent to c. Thus either $\Phi(k) = b$ or $\Phi(k)$ is the $1/M$-neighborhood (with respect to the metric ϱ) of b in T_i, where $h \subset g_i$, so that $d^*(\Phi(k), \Phi(r_c^b)) < \varepsilon$ provided again that M is large enough. Finally, we must show that, for adjacent elements $k \subset h \in \operatorname{Bord} H(g_i)$ and $k' \subset h' \in \operatorname{Bord} H(g_{i'})$ of K^0, $\Phi(k) \cap \Phi(k') \neq \emptyset$. By property (iv) of Lemma 9.1, $f_i^b(k) = f_i^b(h) = 1$ for some $b \in \Gamma \cap \operatorname{Bd} g_i \cap \operatorname{Bd} g_{i'}$, and hence by property (i), $f_j^b(k') = f_j^b(h') > 1 - \mu > 0$ for some j with $b \in \operatorname{Bd} g_j$. Hence $\Phi(k) \cap \Phi(k') \supset \{b\} \neq \emptyset$.

Choose $\delta > 0$ less than the minimum distance between points of non-adjacent elements of K^0 and points of non-adjacent elements of H. Since we could take a $\delta/2M$-S-refinement $\tilde{K}$ of K, and then consider the refinement $K^* = \tilde{K}(K \setminus K^0) \cup K^0$ of K, simply assume that for each $k \in K \setminus K^0$, $\operatorname{diam} k < \delta/2M$. We now extend Φ over all of K by defining it on layers built up from K^0. Inductively define $K^{m+1} = \{k \in K \setminus \bigcup_{i \leqslant m} K^i\colon k \text{ is adjacent to an element of } K^m\}$. We linearly order the elements of K in such a way that $k_m < k_{m+1}$ for every $k_m \in K^m$, $k_{m+1} \in K^{m+1}$, and $k < k'$ for $k, k' \in K^0$ if $\Phi(k)$ is a vertex of Γ and $\Phi(k')$ is not. For $k \in K^{m+1}$ define $\alpha(k) \in K$ to be the first adjacent element; thus $\alpha(k) \in K^m$. We then inductively define $\Phi(k) = e_i(\Phi(\alpha(k)), 1/M) \in C(T_i)$, for $k \in \bigcup \{K^m\colon 0 < m \leqslant M\}$ with $k \subset g_i$. Thus $\Phi(k) = T_i$ if $k \in K^M$. Finally, for $k \in K(g_i) \setminus \bigcup \{K^m\colon 0 \leqslant m \leqslant M\}$, we set $\Phi(k) = T_i$.

The function $\Phi\colon K \to C(\Gamma)$ thus constructed is certainly limited by G (use property (iii) of Lemma 9.1), and is ε-continuous if M is large enough. Since $\{k \in K\colon \Phi(k) \text{ is degenerate}\} \subset K^0$, and the restriction $\Phi_{|K^0}$ satisfies conditions (i)-(iii) of the C-monotone Definition 7.1, so also does Φ. It remains only to describe a relation $\searrow$ in K satisfying the C-monotone condition (iv). Here we make use of the function $\pi\colon \operatorname{Bord} H \to \operatorname{Bord} H$ satisfying properties (vi)-(ix) of Lemma 9.1. We first define a function α_H: $\bigcup\{K^m: 0<m\leqslant M\}\to H$. For $k\in K^m$, consider the mth iterate of the previously defined function α; thus $\alpha^m(k)\in K^0$. Then let $\alpha_H(k)$ be the element of H containing $\alpha^m(k)$. Note that if $k\in K^m$, with $k\subset h\in H$, then by our assumption on the diameters of elements of $K\setminus K^0$, either $\alpha_H(k)=h$ or $\alpha_H(k)$ is adjacent to h.

We set $k\searrow k'$ if k and k' are adjacent elements of K, $\Phi(k)\supset\Phi(k')$, and at least one of the following situations occurs:

1) Either k or k' is an element of $K\setminus\bigcup\{K^m: 0\leqslant m\leqslant M\}$. (In this case $\Phi(k)=\Phi(k')=T_i$, for $k, k'\subset g_i$.)

2) Either k or k' is an element of

$$\alpha_H^{-1}(\mathrm{Bord}H\setminus\bigcup\{L_i^m(b): b\in\Gamma\cap\mathrm{Bd}g_i, 0\leqslant m\leqslant M, 1\leqslant i\leqslant n\}).$$

(In this case also $\Phi(k)=\Phi(k')=T_i$ for $k, k'\subset g_i$.)

3) $k\in\bigcup\{K^m: 0<m\leqslant M\}$ and $\alpha(k)=k'$.

4) $k, k'\in S_h^l$ for $0\leqslant l\leqslant 3M$, $h\in\mathrm{Bord}H$, and either k or k' is an element of $\mathrm{Core}_H K$ and is nonadjacent to every element of S_h^{l-1} (however, we will *not* set $k\searrow k'$ in this case if $k\in S_h^l$ is adjacent to an element k_0 of S_k^0 such that $\Phi(k_0)$ is degenerate — see situation 7 (below).

5) $k\in S_h^l$ and $k'\in S_h^{l+1}$ for $0\leqslant l\leqslant 3M$, $h\in\mathrm{Bord}H$, and either k or k' is an element of $\mathrm{Core}_H K$.

6) $k, k'\in\mathrm{Bord}_G K$, $k\subset h$ and $k'\subset h'$ for distinct $h, h'\in\mathrm{Bord}H$, and $\pi(h)=h'$.

7) $k\in S_h^1$, $k'\in S_h^0$, and $\Phi(k')$ is degenerate.

Condition (iv), a) of Definition 7.1 is automatic (if one were to define the relation $\searrow$ by condition (iv), a) alone, then (iv), b) and (iv), c) would not in general be satisfied). Condition (iv), b) is insured by the restrictions appearing in situations 3), 4), 5), and 6) above, together with the properties of the functions α and π. The restrictions of situations 3), 6), and 7) insure condition (iv), c).

We outline the lengthier argument for condition (iv), d). Suppose $k\in K(h)$, for $h\in\mathrm{Core}H$. Since H is a core-connected refinement of G, there exists a chain γ in $\mathrm{Core}H$ between h and an element c of $K^0\cap\mathrm{Core}H$ (i.e., a core element of the chaining set for Γ). We may assume c is the only element of γ in K^0. Let γ_K be a chain in K which "refines" γ and goes from k to c. Then c is the only element of γ_K in K^0. Let k' be the first element of γ_K which is in the set

$$\alpha_H^{-1}(\bigcup\{L_i^m(b): b\in\Gamma\cap\mathrm{Bd}g_i, 0\leqslant m\leqslant M+1, 1\leqslant i\leqslant n\}\cup\mathrm{Core}H).$$

Then the subchain of γ_K from k to k' is a chain in the relation $\searrow$, by situations 1) and 2) above. By 3) there exists a $\searrow$ chain (not necessarily a subchain of γ_K) from k' to an element k'' of K^0, and either $\Phi(k'')$ is degenerate or $k''\in S_h^{3M}$ with $h\in L_i^m(b)$, for some $b\in\Gamma\cap\mathrm{Bd}g_i$, $1\leqslant m\leqslant M+1$, and $1\leqslant i\leqslant n$. In the latter case $f_j^a(h)=0$ if $a\neq b, f_j(h)=0$ if $i\neq j$, and $f_i(h)<1$ (see the remark in the fifth paragraph of this section). By situations 4) and 5), by the properties (vi) and (vii) of the layers S_h^l, $0\leqslant l\leqslant 3M$, given in Lemma 10.1, and by the definition of the function Φ on the elements of S_h^l, there exists a $\searrow$ chain between k'' and every element k''' of $\mathrm{Bord}_G K(h)$. If $f_i(h)=0$, then $\Phi(k''')=b$ is degenerate for any such k'''. If $f_i(h)>0$, then $\pi(h)\neq h$, and $\pi^r(h)\in H^b$ for some iterate π^r of π. In this case there exists an element k''' of $\mathrm{Bord}_G K(h)$ adjacent to an element of $\mathrm{Bord}_G K(\pi(h))$. Then repeated occurrences of situations 4) and 6) yield a $\searrow$ chain from k''' to an element k^{iv} of $\mathrm{Bord}_G K(\pi^r(h))$, and $\Phi(k^{\mathrm{iv}})=b$. Thus condition (iv), d) is satisfied for elements of $K(\mathrm{Core}H)$.

Now consider $k\in K(h)$ for $h\in\mathrm{Bord}H$. Suppose $h\in L_i^m(b)$, for some $b\in\Gamma\cap\mathrm{Bd}g_i$, $0\leqslant m\leqslant M$, and $1\leqslant i\leqslant n$. If $k\notin\bigcup\{K^j: 0\leqslant j\leqslant M\}$, then since h is connected there exists a minimal chain in $K(h)$ from k to an element k' of K^M. By situation 1) this is also a $\searrow$ chain from k to k'. Since $\alpha_H(k')$ is either identical or adjacent to h, we have $\alpha_H(k')\in\bigcup\{L_i^m(b): m\leqslant M+1\}\cup\mathrm{Core}H$, and it follows by the same argument as in the previous paragraph that there exists a $\searrow$ chain from k' to an element of K whose Φ-image is degenerate. If $k\in\bigcup\{K^j: 0\leqslant j\leqslant M\}$ this argument applies perforce.

Now suppose that $k\in K(h)$ for $h\in\mathrm{Bord}H\setminus\bigcup\{L_i^m(b): m\leqslant M\}$. We consider first the case when $k\in S_h^l$ for $M\leqslant l\leqslant 3M$. If $l\neq 3M$, then by construction of Φ, and by situations 4) and 5), there exists a $\searrow$ chain from k to an element of S_h^{3M}. Hence we may assume $l=3M$. Recall that, since initially obtaining K by invoking Lemma 10.1, we have actually been considering a refinement K^* of K for which $K^0\subset K^*$. However, for simplicity of notation we have continued to refer to K^* as K. Thus the elements r of the subcollection R described in (10.1) will not in general be elements of K (unless $r\in\{r_c^b\}\subset K^0$), but we may consider instead the subcollection $K(r)=\{k\in K: k\subset r\}$, for each such r.

Resuming our argument for $k\in S_h^{3M}$, we have by property (ix) of (10.1) that k is chain-connected to an element r of R contained in h, such that each interior link of the chain is an element of $S_h^{3M}\cap\mathrm{Core}_H K$ and is nonadjacent to every element of S_h^{3M-1}. Let k' be the (interior) link of this chain adjacent to r. Then k is $\searrow$ chained to k' by 4). Since r is connected and is adjacent to an element of $\mathrm{Core}H$, there exists an element k'' of $K(r)$ adjacent to k' (thus $k'\searrow k''$ by 2)), and a chain of elements of $K(r)$ between k'' and an element k''' adjacent to an element h' of $\mathrm{Core}H$. Then by 1) and 2) there exists a $\searrow$ chain from k'' to k''', and by 1), $k'''\searrow k^{\mathrm{iv}}$ for any adjacent element k^{iv} of $K(h')$. By a previous argument k^{iv} is $\searrow$ chained to an element of K whose Φ-image is degenerate. Obviously the case where $k\in K(r)$ for some $r\in R$ is subsumed under the above argument (we then have $k=k''$).

Finally, we consider the remaining case where $k\in S_h^l$, for $0\leqslant l<M$ and $h\in\mathrm{Bord}H$. By situations 4) and 5), there exists a $\searrow$ chain from k to an element k' of $S_h^0\cap\mathrm{Bord}_G K$. If $\pi(h)\neq h$, let t be an integer such that $\pi^t(h)=\pi^{t+1}(h)$. Then by 4) and 6) there exists a $\searrow$ chain from k' to an element k'' of $S_{\pi^t(h)}^0\cap\mathrm{Bord}_G K$. Thus we may assume $\pi(h)=h$. If $f_i(h)=0$ for all i, then $h\in H^b$ for some $b\in\Gamma\cap\mathrm{Bd}G$, and $\Phi(k')=\Phi(h)=b$. Otherwise, we must have $f_i(h)=1$ for $h\subset g_i$ (which implies that $h\notin\bigcup\{L_i^m(b): m\leqslant M\}$). In this situation there exists, by 1) and 2), and the

construction of Φ, a $\searrow$ chain from k' to an element k'' of S_h^M (note that $T_i \subset \Phi(k') = \Phi(k'')$, since $f_i(k') = f_i(h) = 1$). By the argument of the previous paragraph k'' is $\searrow$ chained to an element of K with degenerate Φ-image. This completes the proof that condition (iv), d) of the C-monotone Definition 7.1 is satisfied, and with it the proof of Theorem 7.4

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Accepté par la Rédaction le 5. 4. 1976

A degree theory for almost continuous functions

by

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Abstract. A degree theory is developed for almost continuous functions. This theory is used to prove certain fixed point theorems as well as a generalization of the Borsuk–Ulam theorem.

I. Introduction. In recent years non-continuous functions have been studied and applied to fixed point theory.

Let $f\colon X \to Y$ be a function from a topological space X to a topological space Y. For $C \subseteq X$, the graph over C is defined to be $\{(x, f(x))\colon x \in C\}$, a subspace of the topological space $X \times Y$. The graph of f, denoted by Γf, is defined to be the graph over X. A function $f\colon X \to Y$ is called a *connectivity function* if the graph over each connected set is connected. O. H. Hamilton [7] initiated the study of connectivity functions when he proved the following theorem:

THEOREM 1. *Every connectivity function from the n-cell I^n to the n-cell has a fixed point.*

Let $\mathrm{bd}(N)$ denote the boundary of N. In order to prove Theorem 1, Hamilton defined an additional class of functions:

DEFINITION 1. If $f\colon X \to Y$ is a function, then f is *peripherally continuous* if for each $x \in X$, each open $V \subseteq X$ for which $x \in V$, and each open $U \subseteq Y$ for which $f(x) \in U$, there exists a neighborhood N of x such that $\overline{N} \subseteq V$ and $f(\mathrm{bd}(N)) \subseteq U$.

He then proceeded to show, for $n \geqslant 2$, that every connectivity function is peripherally continuous and every peripherally continuous function has a fixed point. John Stallings [11] discovered a gap in Hamilton's argument, corrected it, and generalized the result to polyhedra. In doing so he defined a third class of functions:

DEFINITION 2. A function $f\colon X \to Y$ is *almost continuous* if for every open subset U of $X \times Y$ with $\Gamma f \subseteq U$ there exists a continuous function $g\colon X \to Y$ with $\Gamma g \subseteq U$.

As a consequence of a key theorem in Stallings paper we have:

THEOREM 2. *If f is a peripherally continuous function from either I^n or S^n, $n \geqslant 2$, into R^n then f is almost continuous.*

* This paper is dedicated to the memory of William Carroll Chewning, a friend and a bright young mathematician who was a source of inspiration to the second author.