On finitely Suslinian continua

by

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Abstract. This paper is concerned with the problem of finding in an arbitrary finitely Suslinian continuum that is not regular a canonical subcontinuum that is not regular. The problem is solved for the case of planar finitely Suslinian continua. The planar regular curves are characterized as those planar curves in which every sequence of pairwise disjoint connected sets forms a null sequence.

§ 1. Preliminaries. In a 1965 paper in Fundamenta Mathematicae R. Duda gave an example of a finitely Suslinian continuum that can be decomposed into the union of infinitely many, connected, mutually disjoint, dense subsets. In Section 2 of this paper we give a very simple such example.

In Section 3 we prove that if X is a finitely Suslinian continuum which is not regular then X contains a fairly simple subcontinuum at each of whose points X fails to be regular. A special case of this theorem was proved in [9] (see the proof of Theorem 6 in [9]). If X is planar we show that X contains a very nice subcontinuum which is also not regular. We use this to prove that a planar continuum is regular if and only if every sequence of pairwise disjoint connected sets in it is a null-sequence. This gives a partial solution to a problem that was stated in [2]. Finally, in Section 4 we give an example of a connected set which is not rim-compact but which can be embedded in a plane finitely Suslinian continuum. This complements a result in [7] Corollary 4.5.

Our notation will follow Whyburn [11]. A *continuum* is a non-degenerate, compact, connected, metric space. A continuum is said to be

(i) *hereditarily locally connected* if each of its subcontinua is locally connected.

(ii) *finitely Suslinian* if each sequence of pairwise disjoint subcontinua forms a null sequence, i.e., the diameters of the subcontinua in the sequence converge to zero.

(iii) *regular* if it has a basis of open sets with finite boundaries.

It is known that (iii)⇒(ii)⇒(i). The reader may consult [11] Chapter V for a discussion of hereditarily locally connected continua.

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A point p of a continuum X is said to be a *local cutpoint* of X if there is a connected open set U in X such that $U \setminus \{x\}$ is not connected. The set of local cutpoints of X is denoted by $L(X)$.

We denote the *closure* of a set A by $\mathrm{Cl}(A)$. By a *neighborhood* we always mean open neighborhood. The *boundary* of a set U is denoted by $\mathrm{Bd}(U)$.

Let X be a continuum and let $x, y \in X$. A closed set M in X which contains x and y is said to have *property* $P(x, y)$ if no finite set separates x and y in any neighborhood of M.

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§ 2. Two examples. The following two examples are prototypes of the objects that are discussed in the next section. The first example is of a finitely Suslinian plane continuum that can be decomposed into infinitely many, connected, mutually disjoint sets of diameter equal to one. It is homeomorphic to a subset of the continuum described in [5], p. 284.

EXAMPLE 1. Let $X = [0, 1] \cup A_1 \cup A_2 \cup \ldots$ where $[0, 1]$ denotes the line segment from $(0, 0)$ to $(1, 0)$ in the plane. A_{11} is the semi-circle in the upper half-plane with center $(\frac{1}{2}, 0)$ and radius $\frac{1}{4}$. The sets $A_1, A_2, \ldots$ are defined inductively to be pairwise disjoint sets such that $A_1 = A_{11}$ and for $i \geqslant 1$

$$A_i = A_{i1} \cup \ldots \cup A_{ij_i}$$

where the A_{ik} are pairwise disjoint semicircles. The endpoints of A_{ik} are $(a_{ik}, 0)$ and $(b_{ik}, 0)$ where $a_{ik} < b_{ik} < a_{ik} + 1/i$ and

$$a_{i1} \leqslant \frac{1}{i+1} < a_{i2} < b_{i1} < a_{i3} < b_{i2} < \ldots < a_{ij_i} < b_{i, j_i - 1} < 1 - \frac{1}{i+1} \leqslant b_{ij_i}.$$

Then X is a continuum since it is obtained by attaching a null sequence of arcs to the fixed arc $[0, 1]$. It is clear that X contains no continuum of convergence so X is hereditarily locally connected by [11], V. 2.1. By a theorem of Gehman ([5], p. 519) X is finitely Suslinian since it is planar.

The sequence $\mathscr{A} = \{A_1, A_2, \ldots\}$ can be decomposed into infinitely many pairwise disjoint sequences $\mathscr{B}_1, \mathscr{B}_2, \ldots$ where $\mathscr{B}_i = \{A_{1_i}, A_{2_i}, \ldots\}$ where $A_{j_i} \neq A_{k_r}$ if $(j, i) \neq (k, r)$. For each i let $X_i = [0, 1] \cup A_{1_i} \cup A_{2_i} \cup \ldots$ If $F = \{f_1, \ldots, f_r\}$ is a finite set in $X_i \setminus \{(0, 0), (1, 0)\}$ let k_i be an integer so large that $F \cap A_{k_i} = \emptyset$, $A_{k_i} \in \mathscr{B}_i$ and

$$1/k_i < \min\{\mathrm{distance}(f_p, f_q) |\ p, q \in \{1, \ldots, r\} \text{ and } p \neq q\}.$$

Then it is easy to see that $([0, 1] \cup A_{k_i}) \setminus F$ is connected between $(0, 0)$ and $(1, 0)$. Hence no finite set separates $(0, 0)$ from $(1, 0)$ in X_i and X_i is not regular. Also, $L(X_i) = A_{1_i} \cup A_{2_i} \cup \ldots$

By a slight extension of a theorem of F. Bernstein (see [3], p. 201) $[0, 1] = Q_1 \cup Q_2 \cup \ldots$ where the Q_i are pairwise disjoint sets and each Q_i meets each Cantor set in $[0, 1]$. We may suppose since $(A_{1_i} \cup A_{2_i} \cup \ldots) \cap [0, 1]$ is countable that $(A_{1_i} \cup A_{2_i} \cup \ldots) \cap [0, 1] \subset Q_i$ for each i.

For each i $P_i = Q_i \cup A_{1_i} \cup A_{2_i} \cup \ldots$ is connected by [11], III. 9.4 since $X_i \setminus P_i$ contains no points of $L(X_i)$ and no perfect set. The sets $P_1, P_2, \ldots$ are pairwise disjoint by construction. It is clear that $\mathrm{Cl}(P_i) = X_i$ so the diameter of each P_i is 1. Furthermore, $X = P_1 \cup P_2 \cup \ldots$

Note that if $a = (0, 0)$ and $b = (1, 0)$ then $[0, 1]$ is a continuum in X which is irreducible with respect to having $P(a, b)$.

Our next example is a finitely Suslinian plane continuum which can be decomposed into infinitely many, pairwise disjoint, connected, dense sets.

EXAMPLE 2. Let X be as in Example 1. Set $x \sim y$ in X if any only if $x = y$ or $x, y \in A_{ik}$ where i is a positive integer and $A_{ik} \subset A_i$. Then $\sim$ is an equivalence relation since the sets A_{ik}) are pairwise disjoint. The equivalence classes of $\sim$ are points and arcs. By Lemma 2.3 in [10] $\sim$ is upper semi-continuous and $X/\sim$ is a continuum. It is well-known (and easy to check) that the monotone image of a finitely Suslinian continuum is finitely Suslinian. Hence $X/\sim$ is finitely Suslinian. By Moore's theorem on upper semi-continuous decompositions of the 2-sphere (see [11], IX, 2.1′, p. 171) $X/\sim$ is also planar. If $\pi\colon X \to X/\sim$ is the natural projection then $\pi(P_1), \pi(P_2), \ldots$ are pairwise disjoint, dense, connected sets in $X/\sim$.

Note that if $a = \pi(0, 0)$ and $b = \pi(1, 0)$ $X/\sim$ is irreducible with respect to having $P(a, b)$.

§ 3. The main results. Parts of Theorems 1 and 2 were first proved in [9] Theorem 6 for X a finitely Suslinian continuum each subcontinuum of which has a free arc. The ideas in the proofs that are given here are similar to those that appear in [9] but the arguments that are given here are somewhat simpler and also more complete.

THEOREM 1. *Let X be a finitely Suslinian continuum and let $a, b \in X$. If X has $P(a, b)$ then*

(i) *There is a subcontinuum P_{ab} in X which is minimal with respect to having $P(a, b)$.*

(ii) *$P_{ab} \cap L(X)$ is at most countable.*

(iii) *If $x \in P_{ab}$ then there is a unique subcontinuum P_{ax} (resp. P_{bx}) in P_{ab} which is minimal with respect to having $P(a, x)$ (resp. $P(b, x)$).*

(iv) *If $x, y \in P_{ab}$ and $x \neq y$ then either $P_{ax} \subsetneqq P_{ay}$ or $P_{ay} \subsetneqq P_{ax}$.*

(v) *If $\mathscr{A}$ (resp. $\mathscr{B}$) is the closure of $\{P_{ax} |\ x \in P_{ab}\}$ (resp. $\{P_{bx} |\ x \in P_{ab}\}$) in the hyperspace of subcontinua of P_{ab} then there are homeomorphisms $h\colon [0, 1] \to \mathscr{A}$ and $g\colon [0, 1] \to \mathscr{B}$ such that for $0 < p < q < 1$*

$$h(0) = \{a\} \subset h(p) \subset h(q) \subset h(1) = P_{ab},$$

$$g(1) = \{b\} \subset g(q) \subset g(p) \subset g(0) = P_{ab}$$

such that if $h(r) = P_{ax}$ for some $x \in P_{ab} \setminus L(X)$ then $g(r) = P_{bx}$.

(vi) *If $x \in h(r)$ and $P_{ax} = h(s)$ then $s \leqslant r$.*

(vii) *There is a continuous selection* $f\colon [0,1]\to P_{ab}$ *for* h *such that if* $h(r)=P_{ax}$ *then* $f(r)=x$. *If* $x\in P_{ab}\setminus L(X)$ *then* $f^{-1}(x)$ *is a singleton.*

(viii) *If* $x\in (h(r)\cap g(r))\setminus L(X)$ *then* $h(r)=P_{ax}$ *and* $g(r)=P_{bx}$. *Hence* $(h(r)\cap g(r))\setminus L(X)$ *contains at most one point.*

(ix) *If* $A_i)$ *is a sequence of disjoint (possibly degenerate) continua in* $X\setminus\{f(r)\}$ *such that for each* i $A_i\cap h(r)\neq\emptyset$ *and* $A_i\cap g(r)\neq\emptyset$ *and if for each* i $r_i\in[0,1]$ *such that* $f(r_i)\in A_1$ *then* $\lim r_i=r$.

(x) *If* $r<s$ *in* $[0,1]$ *then* $h(r)\cup g(s)\neq P_{ab}$ *and* $h(r)\cap g(s)$ *is a finite set which is contained in* $L(X)$. *For* $r\in[0,1]$ $h(r)\cup g(r)=P_{ab}$.

(xi) *If* $r_i)$ *is a sequence which is increasing to* r *in* $[0,1]$ *and* U *is a neighborhood of* $f(r)$ *then there exists an integer* i *such that* $h(r)\subset h(r_i)\cup U$. *In particular* $h(r)=\big(\bigcup_i h(r_i)\big)\cup\{f(r)\}$.

(xii) *If* $r<s$ *in* $[0,1]$ *then* $f([r,s])$ *has* $P(x,y)$ *for all* $x,y\in f([r,s])$.

Proof. Parts of this theorem were proved in Theorem 6 Claims 1–8 of [9] for the special case in which each subcontinuum of X contains a free arc. The basic idea of the proof is the same as that given in [9].

Since the property $P(a,b)$ is inducible (see [11], p. 17) it follows by the Brouwer Reduction Theorem that X contains a closed set P_{ab} which is minimal with respect to having $P(a,b)$. It is easy to see that P_{ab} is a continuum. By Whyburn [11], III, 9.2 $P_{ab}\cap L(X)$ is at most countable. We have proved (i) and (ii).

I. If $c,d\in P_{ab}$ then P_{ab} has $P(c,d)$.

Just suppose U is a neighborhood of P_{ab} and A is a finite set such that $U\setminus A=U_1\cup U_2$ where U_1 and U_2 are disjoint open sets, $c\in U_1$ and $d\in U_2$. Let $K=P_{ab}\cap \mathrm{Cl}(U_1)$ and let $L=P_{ab}\cap\mathrm{Cl}(U_2)$. Then $P_{ab}=K\cup L$. If $a,b\in K$ (resp. $a,b\in L$) there exists a neighborhood V (resp. W) of K (resp. L) and a finite set B (resp. C) such that B (resp. C) separates a and b in V (resp. W) by the minimality of P_{ab}. If at least one of a and b is not in K (resp. L) let $V=U$ (resp. $W=U$) and let $B=\emptyset$ (resp. $C=\emptyset$). Then $(A\cup B\cup C)\setminus\{a,b\}$ separates a and b in $(V\cup W)\cap U$. This is a contradiction since $(V\cup W)\cap U$ is a neighborhood of P_{ab}.

II. If $x\in P_{ab}$ and U is a neighborhood of x such that $b\notin\mathrm{Cl}(U)$ then there exists $c\in\mathrm{Bd}(U)$ such that $P_{ab}\setminus U$ has $P(b,c)$.

Just suppose that for each $c\in\mathrm{Bd}(U)$ there is a neighborhood V_c of $P_{ab}\setminus U$ and a finite set A_c such that $V_c\setminus A_c=U_c\cup W_c$ where U_c and W_c are disjoint open sets, $c\in U_c$ and $b\in W_c$. Since $\mathrm{Bd}(U)$ is compact there exists a finite set $\{c_1,\dots,c_n\}\subset\mathrm{Bd}(U)$ such that $\mathrm{Bd}(U)\subset U_{c_1}\cup\dots\cup U_{c_n}$.

Let $V=V_{c_1}\cap\dots\cap V_{c_n}$, $A=A_{c_1}\cup\dots\cup A_{c_n}$, $R=(U_{c_1}\cup\dots\cup U_{c_n})\cap V$ and $S=W_{c_1}\cap\dots\cap W_{c_n}$. Then V is a neighborhood of $P_{ab}\setminus U$, $\mathrm{Bd}(U)\subset R$, $b\in S$, and R and S are disjoint open sets. Since $\mathrm{Bd}(U)\subset R$ and R is disjoint from S, $S\setminus U=S\setminus\mathrm{Cl}(U)$ is an open set. Now $(V\cup U)\setminus A\subset(U\cup R)\cup(S\setminus U)$. The sets $U\cup R$ and $S\setminus U$ are disjoint open sets, $x\in U\cup R$ and $b\in S\setminus U$. Thus, A is a finite set which separates x and b in the neighborhood $V\cup U$ of P_{ab}. This contradicts I.

III. If $x\in P_{ab}\setminus L(X)$ and P and Q are closed sets in P_{ab} such that P has $P(a,x)$ and Q has $P(b,x)$, respectively, then $P\cup Q=P_{ab}$.

Let U be a neighborhood of $P\cup Q$. Just suppose some finite set separates U between a and b. Let A be an irreducible finite set which separates U between a and b. By [11], III, 9.3 $x\notin A$. Since P has $P(a,x)$ x and a lie in the same component of $U\setminus A$. Since Q has $P(b,x)$ x and b lie in the same component of $U\setminus A$. Thus, a and b lie in the same component of $U\setminus A$. This is a contradiction. We have proved that $P\cup Q$ has $P(a,b)$. By the minimality of P_{ab} $P\cup Q=P_{ab}$.

IV. If $b_i)$ is a sequence in P_{ab} which converges to b and for each i P_i is a continuum in P_{ab} which has $P(a,b_i)$ then $\lim P_i=P_{ab}$.

Notice that $\limsup P_i$ is a continuum in P_{ab} which has $P(a,b)$. By the minimality of P_{ab} $\limsup P_i=P_{ab}$.

V. If A and B are continua in P_{ab} such that A has $P(a,x)$ for each $x\in A$ and B has $P(b,y)$ for each $y\in B$ and if $A_i)$ is a sequence of disjoint (possibly degenerate) continua in X such that for each i $A\cap A_i\neq\emptyset$ and $B\cap A_i\neq\emptyset$ then $A\cup B=P_{ab}$.

Let U be a neighborhood of $A\cup B$ and let F be a finite set in $U\setminus\{a,b\}$. Since X is finitely Suslinian the $A_i)$ form a null sequence. Hence, almost all of the A_i are contained in U. Let i be a natural number so large that $A_i\subset U$ and $A_i\cap F=\emptyset$. Let $x\in A_i\cap A$ and let $y\in A_i\cap B$. Since A has $P(a,x)$ a and x lie in the same component of $U\setminus F$. Similarly b and y lie in the same component of $U\setminus F$. Since A_i is a continuum in $U\setminus F$ which contains x and y, a and b lie in the same component of $U\setminus F$. We have proved that $A\cup B$ has $P(a,b)$. By the minimality of P_{ab} $A\cup B=P_{ab}$.

VI. If $x\in P_{ab}\setminus\{b\}$ then there exists a subcontinuum of $P_{ab}\setminus\{b\}$ which has $P(a,x)$.

Let $x\in P_{ab}\setminus\{b\}$ and suppose no subcontinuum of $P_{ab}\setminus\{b\}$ has $P(a,x)$. Let V be a neighborhood of x in P_{ab} such that $b\notin\mathrm{Cl}(V)$ and $\mathrm{Bd}(V)\subset X\setminus L(X)$. For each natural number i let U_i be a neighborhood of b in P_{ab} such that $\mathrm{Cl}(U_i)$ is disjoint from $\mathrm{Cl}(V)$, the diameter of U_i is less than $1/i$ and $\mathrm{Bd}(U_i)\subset X\setminus L(X)$.

By II there exists for each i $b_i\in\mathrm{Bd}(U_i)$ such that $P_{ab}\setminus U_i$ has $P(a,b_i)$. For each i let P_i be a continuum in $P_{ab}\setminus U_i$ which is minimal with respect to having $P(a,b_i)$. Then $\lim P_i=P_{ab}$ by IV. We may suppose, therefore, that for each i $P_i\cap V\neq\emptyset$. Since $x\notin P_i$ by I and $b_i\in X\setminus L(X)$ it follows from III that $P_{ab}\setminus V$ does not have $P(b,b_i)$. Since the sequence $b_i)$ converges to b it follows that for each $i=1,2,\dots$ there exists N_i such that $j\geqslant N_i$ implies $P_{ab}\setminus V$ does not have $P(b_i,b_j)$. We may suppose without loss of generality that the sequence $b_i)$ was chosen in such a way that $i\neq j$ implies $P_{ab}\setminus V$ does not have $P(b_i,b_j)$. By II there exists for each i $x_i\in\mathrm{Bd}(V)$ such that $P_i\setminus V\subset P_{ab}\setminus V$ has $P(b_i,x_i)$.

There is a neighborhood W_1 of $P_{ab}\setminus V$ and an irreducible finite set A_1 such that A_1 separates b_1 from each of $b,b_2,b_3,\dots$ in W_1. (It is possible to find one finite set A_1 which separates each of these points from b_1 because $\lim b_i=b$.) By [11], III, 9.3 $A_1\subset L(X)$. Let B_1 be an arc in $W_1\setminus A_1$ with endpoints x_1 and b_1.

Similarly, there exists a neighborhood $W_2 \subset W_1$ of $P_{ab} \setminus V$ and an irreducible finite set A_2 such that A_2 separates b_2 from each of $b, b_3, b_4, \ldots$ in W_2. Let B_2 be an arc in $W_2 \setminus (A_1 \cup A_2)$ with endpoints x_2 and b_2. Then $B_1 \cap B_2 = \emptyset$. In this way we construct inductively a sequence $B_i)$ of pairwise disjoint arcs where B_i has endpoints x_i and b_i. This sequence is not null since $b \notin \mathrm{Cl}(V)$. This contradicts the assumption that X is finitely Suslinian.

VII. For each $x \in P_{ab}$ there exists a unique subcontinuum P_{ax} (resp. P_{bx}) of P_{ab} which is minimal with respect to having $P(a, x)$ (resp. $P(b, x)$). If $x, y \in P_{ab}$ then either $P_{ax} \subset P_{ay}$ or $P_{ay} \subset P_{ax}$.

Suppose A and B are subcontinua of P_{ab} such that A (resp. B) is minimal with respect to having $P(a, x)$ (resp. $P(a, y)$) and such that $A \not\subset B$ and $B \not\subset A$. Let $p \in A \setminus (B \cup L(X))$ and let $q \in B \setminus (A \cup L(X))$. By I A has $P(a, p)$ and B has $P(a, q)$. Let C be a continuum in P_{ab} which is minimal with respect to having $P(b, p)$. By III $A \cup C = P_{ab}$ so $q \in C$. By VI there exists a continuum $D \subset C \setminus \{p\}$ such that D has $P(b, q)$. By III $D \cup B = P_{ab}$. This is a contradiction since $D \cup B \subset P_{ab} \setminus \{p\}$. Since the points x and y are not necessarily distinct VII is proved.

We have proved (iii) and (iv).

VIII. $\mathscr{A}$ is an arc and if $z \in C \in \mathscr{A}$ then $P_{az} \subset C$.

Inclusion is a partial ordering with closed graph on the hyperspace of subcontinua of X. By VII $\mathscr{A}$ is the closure of a set of continua totally ordered under inclusion so $\mathscr{A}$ is itself totally ordered under inclusion.

Let $z \in C \in \mathscr{A}$ and suppose $P_{az} \not\subset C$. Let $q \in P_{az} \setminus C$. Since $\mathscr{A}$ is totally ordered by inclusion and $P_{aq} \not\subset C$ we have by VI $C \subset P_{aq} \subset P_{az} \setminus \{z\}$ which is a contradiction.

Since $\mathscr{A}$ is totally ordered under inclusion and compact we need only show that $\mathscr{A}$ is connected to prove that $\mathscr{A}$ is an arc. To prove that $\mathscr{A}$ is connected it suffices to prove that if $C, D \in \mathscr{A}$ with $C \subsetneqq D$ then there exists $E \in \mathscr{A}$ with $C \subsetneqq E \subsetneqq D$.

Since $C \subsetneqq D$ there exist two points $y, z \in D \setminus C$. By VII either $P_{ay} \subsetneqq P_{az} \subset D$ or $P_{az} \subsetneqq P_{ay} \subset D$. Suppose $P_{ay} \subsetneqq P_{az}$. Then $P_{ay} \subsetneqq D$. Since $y \notin C$ and $\mathscr{A}$ is totally ordered by inclusion $C \subsetneqq P_{ay}$.

IX. If $z \in P_{ab}$ then $(P_{az} \cap P_{bz}) \setminus L(X) \subset \{z\}$.

Let $w \in (P_{az} \cap P_{bz})$. If $w \neq z$ then by VI $P_{aw} \subset P_{az} \setminus \{z\}$ and $P_{bw} \subset P_{bz} \setminus \{z\}$. Since $P_{aw} \cup P_{bw} \neq P_{ab}$ $w \in L(X)$ by III.

X. If $x, y \in P_{ab} \setminus L(X)$ and $y \notin P_{ax}$ then $P_{ax} \subsetneqq P_{ay}$, $P_{bx} \supsetneqq P_{by}$, $P_{ax} \cup P_{by} \neq P_{ab}$ and $P_{ax} \cap P_{by}$ is a finite subset of $L(X)$.

By VII $P_{ax} \subset P_{ay}$. By VI $P_{ax} \subset P_{ay} \setminus \{y\}$. Let $z \in P_{ay} \setminus (P_{ax} \cup L(X) \cup \{y\})$. By VII and VI $P_{ax} \subset P_{az} \setminus \{z\}$ and $P_{az} \subset P_{ay} \setminus \{y\}$. By III $P_{az} \cup P_{bz} = P_{ab}$. Hence, $y \in P_{bz}$. By VI $P_{by} \subset P_{bz} \setminus \{z\}$. Thus $z \notin P_{ax} \cup P_{by}$ and $P_{ax} \cup P_{by} \neq P_{ab}$. By I and III $P_{ax} \cap P_{by} \subset L(X)$. By V $P_{ax} \cap P_{by}$ is finite.

The existence of the homeomorphism h of $[0, 1]$ onto $\mathscr{A}$ is guaranteed by VIII. It also follows from VIII that (vi) holds. The existence of the homeomorphism g of

$[0, 1]$ onto $\mathscr{B}$ satisfying the condition that if $h(r) = P_{ax}$ where $x \in P_{ab} \setminus L(X)$ then $g(r) = P_{bx}$ follows from X and the fact that $\{P_{ax} | \ x \in P_{ab} \setminus L(X)\}$ is dense in $\mathscr{A}$.

XI. If $x_i)$ is a sequence in P_{ab} such that the sequence $P_{ax_i})$ converges in $\mathscr{A}$ then $x_i)$ converges in P_{ab}.

Just suppose that $x_i)$ and $y_i)$ are sequences in P_{ab} which converge to the points x and y respectively and the sequences $P_{ax_i})$ and $P_{ay_i})$ both converge to $C \in \mathscr{A}$. By IV we may suppose that for each i $x_i, y_i \in P_{ab} \setminus L(X)$ and $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$.

Since the sets P_{ax} for $x \in P_{ab}$ are nested we may assume that one of the following three cases holds:

(1) for each i $P_{ax_i} \subset P_{ax_{i+1}}$ and $P_{ay_i} \subset P_{ay_{i+1}}$,

(2) for each i $P_{ax_i} \supset P_{ax_{i+1}}$ and $P_{ay_i} \supset P_{ay_{i+1}}$,

(3) for each i $P_{ax_i} \subset P_{ax_{i+1}}$ and $P_{ay_i} \supset P_{ay_{i+1}}$.

Let us suppose first that case (1) holds. If for each i and j $P_{ax_i} \subset P_{ay_i}$ then $C = P_{ay}$ and the sequence $y_i)$ is a constant sequence. This is a contradiction. We suppose, therefore, that for each i $P_{ax_i} \subset P_{ay_i} \subset P_{ax_{i+1}}$.

For each $i = 1, 2, \ldots$ and each $j = 0, 1, \ldots, i-1$ P_{ax_i} has $P(a, x_{i-j})$ and $P(x_{i-j}, y_{i-j-1})$. For each $j, i = 1, 2, \ldots$ P_{y_ib} has $P(y_{i+j}, b)$ and $P(y_{i+j}, x_{i+j+1})$. By X $P_{ax_i} \cup P_{by_i}$ does not have $P(x_i, y_i)$ since it does not have $P(a, b)$.

There exists a neighborhood W_1 of $P_{ax_2} \cup P_{by_2}$ and an irreducible finite set A_1 such that A_1 separates x_2 and y_2 in W_1. By [11], III, 9.3 $A_1 \subset L(X)$. Let B_1 be an arc in $W_1 \setminus A_1$ with endpoints x_1 and y_1. For each $i = 1, 2, \ldots$ let $P_{x_{2i+2}y_{2i}}$ be a continuum in $P_{by_{2i}}$ which is minimal with respect to having $P(x_{2i+2}, y_{2i})$. There exists a neighborhood $W_2 \subset W_1$ of $P_{x_4y_2} \cup P_{by_4} \subset P_{by_2}$ and an irreducible finite set $A_2 \subset L(X)$ such that A_2 separates x_4 and y_4 in W_2. Let B_2 be an arc in $W_2 \setminus (A_1 \cup A_2)$ with endpoints x_3 and y_3. Then $B_1 \cap B_2 = \emptyset$.

Inductively, there exists for each $n = 2, 3, 4, \ldots$ a neighborhood $W_n \subset W_{n-1} \subset \ldots \subset W_1$ of $P_{x_{2n}y_{2n-2}} \cup P_{by_{2n}} \subset P_{by_{2n-2}}$ and an irreducible finite set $A_n \subset L(X)$ which is minimal with respect to separating x_{2n} and y_{2n} in W_n. Also, there exists an arc $B_n \subset W_n \setminus (A_1 \cup \ldots \cup A_n)$ with endpoints x_{2n-1} and y_{2n-1}. The arcs $B_n)$ form a non-null sequence of pairwise disjoint continua in X. This contradicts the assumption that X is finitely Suslinian.

If Case (2) holds then a similar argument leads to a contradiction.

If Case (3) holds let U be a neighborhood of x in P_{ab} such that $\mathrm{Bd}(U) \subset P_{ab} \setminus L(X)$ and $y \notin \mathrm{Cl}(U)$. For each i and j $P_{ax_i} \subset P_{ay_j}$. For each i let $P_{x_iy_i}$ be a continuum in P_{ay_i} such that $P_{x_iy_i}$ is minimal with respect to having $P(x_i, y_i)$. Since the sequence $x_i)$ converges to x we may suppose $x_i \in U$ for each i. Since the sequence $y_i)$ converges to y we may suppose $y_i \notin U$ for each i. For each i let $z_i \in P_{x_iy_i} \cap \mathrm{Bd}(U)$. By X $P_{ax_i} \subset P_{az_i} \subset P_{ay_i}$ hence $\lim P_{az_i} = \lim P_{ax_i} = \lim P_{ay_i} = C$.

If for some subsequence $z_{i_j})$ of $z_i)$ $P_{az_{i_j}} \subsetneqq P_{az_{i_{j+1}}}$ for each j then Case (1) applies for the sequences $x_i)$ and $z_{i_j})$. If for some subsequence $z_{i_j})$ of $z_i)$ $P_{az_{i_j}} P_{az} \supsetneqq_{i_{j+1}}$ for each j then Case (2) applies for the sequences $z_{i_j})$ and $y_i)$. If the sequence $z_i)$ is eventually constant then $C = P_{az}$ for some $z \in P_{ab}$. Let $w_i)$ be a sequence in $P_{az} \setminus \{z\}$ which converges to z. By IV $\lim P_{aw_i} = P_{az} = C$. We can now apply Case (1) for the sequences $w_i)$ and $x_i)$.

The existence of the continuous selection f for h follows from XI.

XII. If $r_i)$ is a sequence which is increasing to r in $[0, 1]$ and if U is a neighborhood of $f(r)$ then there exists a natural number i such that $h(r) \subset h(r_i) \cup U$. In particular $h(r) = (\bigcup_i h(r_i)) \cup \{f(r)\}$.

Just suppose that for each i $x_i \in h(r) \setminus (U \cup h(r_i))$. Let $s_i \in [0, 1]$ such that $h(s_i) = P_{ax_i}$. By (vi) $s_i \leqslant r$. Since $x_i \notin h(r_i)$ we have $r_i < s_i$ by (v). Hence $\lim s_i = \lim r_i = r$. Since f is continuous $\lim f(s_i) = f(r)$. This is a contradiction since for each i $f(s_i) = x_i \notin U$.

XIII. For $r \in [0, 1]$ $h(r) \cup g(r) = P_{ab}$.

By III this is true for a dense set in $[0, 1]$, namely for $\{r \in [0, 1] | f(r) \in P_{ab} \setminus L(X)\}$. Since h and g are continuous it is true for all $r \in [0, 1]$.

XIV. If $x \in (h(r) \cap g(r)) \setminus L(X)$ then $h(r) = P_{ax}$ and $g(r) = P_{bx}$.

Let $s \in [0, 1]$ such that $h(s) = P_{ax}$. By (vi) $s \leqslant r$. Similarly, $r \leqslant s$. Hence, $r = s$ and $f^{-1}(x)$ is a singleton.

XV. If $r < s$ in $[0, 1]$ then $h(r) \cup g(s) \neq P_{ab}$ and $h(r) \cap g(s)$ is a finite set which is contained in $L(X)$.

Let $t \in [0, 1]$ such that $r < t < s$ and $f(t) \notin L(X)$. Then $f(t) \notin h(r) \cup g(s)$ so $h(r) \cup g(s) \neq P_{ab}$. By III $h(r) \cap g(s) \subset L(X)$. By V $h(r) \cap g(s)$ is at most finite.

XVI. If $r \in [0, 1]$ and $A_i)$ is a sequence of pairwise disjoint (possible degenerate) continua such that each A_i meets both $h(r)$ and $g(r)$ and if for each i $r_i \in [0, 1]$ such that $f(r_i) \in A_i$ then $\lim r_i = r$.

If for each i $r_i \leqslant s \leqslant r$ then $h(s) \cup g(r) = P_{ab}$ by V. By XV $s = r$.

XVII. If $r < s$ in $[0, 1]$ then $f([r, s])$ has $P(x, y)$ for all $x, y \in f([r, s])$.

Notice that $f([0, s]) = h(s)$ for $s \in [0, 1]$. By VIII the above statement is true for $r = 0$ and $s \in [0, 1]$. If $r > 0$ and $f(r) \in P_{ab} \setminus L(X)$ then $f([r, s])$ plays the same role in $g(r, 1) = P_{bf(r)}$ that $f([0, s])$ plays in P_{ab}. Thus, the above statement is true for $\{r \in [0, 1] | f(r) \notin L(X)\}$ and for all $s \in [0, 1]$. Since XVII is true for a dense set of r and s it is easy to see that it is true for all $r, s \in [0, 1]$.

This completes the proof of Theorem 1.

Let X be a finitely Suslinian continuum and let $a, b \in X$ such that X has $P(a, b)$. Let P_{ab} and f be as in Theorem 1. If $s \in [0, 1]$ and $\varepsilon > 0$ then a set A is called an *ε-bridge* over s if $A \subset S(f(s), \varepsilon) \setminus \{f(s)\}$ is a (possibly degenerate) continuum which is irreducible with respect to meeting both $h(s)$ and $g(s)$. It follows that either A is a point of $h(s) \cap g(s)$ or A is an arc with one endpoint in $h(s)$, the other endpoint in $g(s)$ and with no other points in P_{ab}. The bridge A is said to have *ends* x and y in $[0, 1]$ if $\{f(x)\} = \{f(y)\} = A$ or if A is an arc with endpoints $f(x)$ and $f(y)$. (Note that the ends of A need not be unique since f need not be one to one.)

If $x < y$ in $[0, 1]$ the family of bridges $\{A_1, \ldots, A_k\}$ is said to be *interlaced* from x to y if for each $i = 1, \ldots, k$ there exists ends x_i, y_i of A_i such that

$$[x, y] \subset (x_1, y_1) \cup (x_2, y_2) \cup \ldots \cup (x_k, y_k).$$

The family of bridges $\{A_1, \ldots, A_k\}$ is said to be *irreducibly interlaced* from x to y if no proper subfamily is interlaced from x to y.

THEOREM 2. *Let X be a finitely Suslinian continuum that is not regular. Let $a, b \in X$ such that no finite set separates a and b in X. Let P_{ab} be a continuum in X which is irreducible with respect to having $P(a, b)$. Let f: $[0, 1] \to P_{ab}$ be as in Theorem* 1. *Then there exists a continuum Y in X such that*

(1) $Y = P_{ab} \cup B_1 \cup B_2 \cup \ldots$,

(2) *for each positive integer i B_i is the union of a finite family of $1/i$-bridges that is irreducibly interlaced between $1/i$ and $1-1/i$,*

(3) $B_i \cap B_j = \emptyset$ *for* $i \neq j$,

(4) $L(Y) \subset (L(X) \cap P_{ab}) \cup B_1 \cup B_2 \cup \ldots$

Proof. Let P_{ab}, f, g, and h be as in Theorem 1. Let $0 < r < 1$ and let $\varepsilon > 0$ be given. A bridge A over r is said to be *avoidable* if for each s such that $0 < s < 1$ there exists $\delta > 0$ such that each δ-bridge B over s is disjoint from A. We prove first that there is an avoidable ε-bridge over r.

Since f is continuous there exist $p, q \in [0, 1]$ such that $p < r < q$ and $f([p, q]) \subset S(f(r), \varepsilon)$. By (xii) of Theorem 1 there is an ε-bridge C over r. If C is a point then C is an avoidable bridge. We may suppose, therefore, that $S(f(r), \varepsilon) \cap g(r) \cap h(r) \subset \{f(r)\}$ and C is an arc. For each $\eta > 0$ let C_η be the component of $(S(P_{ab}, \eta) \cup C) \setminus P_{ab}$ which meets C. It follows easily from the fact that X is finitely Suslinian that $\lim_{\eta \to 0} \mathrm{Cl}(C_\eta) = C$. Let $\mu > 0$ such that $\mathrm{Cl}(C_\mu) \subset S(f(r), \varepsilon) \setminus \{f(r)\}$. Let $D = C_\mu$. Let $M = \mathrm{Cl}(D) \cap h(r)$ and let $N = \mathrm{Cl}(D) \cap g(r)$. If M (resp. N) has an isolated point let c (resp. d) be in $[0, 1]$ such that $f(c)$ (resp. $f(d)$) is an isolated point of M (resp. N). If M (resp. N) has no isolated points let

$$c = \inf\{s \in [0, 1] | \ \mathrm{Cl}(D) \cap h(s) \text{ is uncountable}\}$$

(resp. $d = \sup\{s \in [0, 1] | \ \mathrm{Cl}(D) \cap g(s) \text{ is uncountable}\}$). In this case $h(c) \cap \mathrm{Cl}(D)$ (resp. $g(d) \cap \mathrm{Cl}(D)$) is at most countable.

If $f(c)$ (resp. $f(d)$) is not an isolated point of M (resp. N) let $a_k)$ (resp. $b_k)$) be a sequence in $[c, 1]$ (resp. in $[0, d]$) which is strictly decreasing to c (resp. strictly increasing to d) such that for each k $f(a_k) \in M \setminus \{f(c)\}$ resp. $f(b_k) \in N \setminus \{f(d)\}$). Since D is an open, connected subset of the finitely Suslinian continuum $\mathrm{Cl}(D)$, D is arcwise connected. By [10] Corollary 2.2 every subset of $\mathrm{Cl}(D)$ which contains D is arcwise connected. For each $k = 1, 2, \ldots$ let $p_k \in D$ (resp. $q_k \in D$) such that there is an arc T_k in $D \cup \{f(a_k)\}$ from p_k to $f(a_k)$ (resp. an arc S_k in $D \cup \{f(b_k)\}$ from q_k to $f(b_k)$) such that T_k (resp. S_k) has diameter less than $1/k$.

By [11], V, 2.7 D has property S (see [11], p. 20). For each $j = 1, 2, \ldots$ let $\{V_{j1}, \ldots, V_{jt_j}\}$ be a finite cover of D by connected sets of diameter $<1/j$ which are open in D. We may suppose that for each j the sequence $p_k)$ is eventually in V_{j1} and the sequence $q_k)$ is eventually in V_{jt_j}. We may also suppose that for each $k, j = 1, 2, \ldots$ $p_{k+j} \in V_{k1}$ and $q_{k+j} \in V_{kt_k}$. For each k let P_k (resp. Q_k) be an arc in V_{k1} (resp. V_{kt_k}) with endpoints p_k and p_{k+1} (resp. q_k and q_{k+1}). Let $P = \bigcup_{k=1}^{\infty} P_k$ and let $Q = \bigcup_{k=1}^{\infty} Q_k$. Then $P \cup \{f(c)\}$ and $Q \cup \{f(d)\}$ are continua in $D \cup \{f(c)\}$ and $D \cup \{f(d)\}$ respectively.

Let E be an arc in $D \cup \{f(c), f(d)\}$ with endpoints $f(c)$ and $f(d)$ such that if $f(c)$ (resp. $f(d)$) is not an isolated point of M (resp. N) then some neighborhood in E of $f(c)$ (resp. $f(d)$) is contained in P (resp. Q). Then E is clearly an ε-bridge over r. It remains to prove that E is avoidable.

Let $s \in [0, 1]$ such that $f(s) \in E$. Without loss of generality we may suppose $f(s) = f(c)$. Since

$$S(f(r), \varepsilon) \cap g(r) \cap h(r) \subset \{f(r)\}, \qquad s \leqslant r .$$

We may suppose without loss of generality that there exists $\eta > 0$ such that

$$h(s) \cap g(s) \cap S(f(s), \eta) \subset \{f(s)\} .$$

For each natural number i such that $i > 1/\eta$ let F_i be a $1/i$-bridge over s with ends u_i and v_i. By Theorem 1 (ix), we may suppose $|v_i - u_i| < 1/i$. We may suppose the $F_i)$ are pairwise disjoint. By Theorem 1 (ix) $\lim u_i = s$ so $\lim f(u_i) = f(s)$. If $F_i \cap E \neq \emptyset$ then $f(u_i) \in M$. Since f is continuous $\lim f(u_i) = f(s) = f(c)$. If $f(c)$ is an isolated point of M then it follows that for all sufficiently large j $F_j \cap E = \emptyset$. Let us suppose, therefore, that M has no isolated points. Just suppose that for some subsequence $F_{j_m})$ of $F_j)$ $F_{j_m} \cap E \neq \emptyset$ for each m. For each m $f(u_{j_m}) \in M$. By the choice of c and by the assumption that M is perfect, each neighborhood of $f(u_{j_m})$ contains uncountably many points of $g(c)$. Since $g(c)$ is compact $f(u_{j_m}) \in g(c)$. If $s \leqslant c$ then $g(c) \subset g(s)$ and so

$$f(u_{j_m}) \in \big(h(s) \cap g(s) \cap S(f(s), \eta)\big) \setminus \{(c)\}$$

which is a contradiction. If $c < s$ then there exists a sequence of pairwise disjoint arcs $K_i)$ in $P \cup \bigcup F_{j_m} \cup \bigcup T_j$ such that for each i K_i has endpoints $f(a_{k_i})$ and $f(v_{l_i})$ for some integers k_i and l_i. It follows by V of the proof of Theorem 1 that for each i $h(a_i) \cup g(s)$ has $P(a, b)$. Since $\lim a_i = c$ $h(c) \cup g(s)$ has $P(a, b)$ contrary to Theorem 1 (x). This completes the proof that E is an avoidable ε-bridge over r.

Let $B_{11} = B_1$ be an avoidable 1-bridge over $\frac{1}{2}$. Let n be a natural number. Suppose $B_1, \ldots, B_{n-1}$ have been defined and satisfy conditions (2) and (3) of the theorem.

For each $x \in [1/n, 1-1/n]$ let E_x be a $1/n$-bridge over x which is avoidable and disjoint from $B_1 \cup \ldots \cup B_{n-1}$. This is possible since $B_1 \cup \ldots \cup B_{n-1}$ is a compact set which meets P_{ab} in a finite set and each of the bridges in $B_1, \ldots, B_{n-1}$ is avoidable. For each x let a_x and b_x be ends of E_x such that $a_x < x < b_x$ and such that $b_x - a_x$ is as large as possible. By Theorem 1 (ix) we may suppose $b_x - a_x < 1/n$. The set of open intervals $\{(a_x, b_x) | \; x \in [1/n, 1-1/n]\}$ is an open cover for the closed interval $[1/n, 1-1/n]$ hence it has a finite subcover. A family of $1/n$-bridges associated with this finite open cover is interlaced from $1/n$ to $1-1/n$. It contains a subfamily $B_{n1}, \ldots, B_{nk_n}$ which is irreducibly interlaced from $1/n$ to $1-1/n$. Let $B_n = B_{n1} \cup \ldots \cup B_{nk_n}$. By induction Y is defined.

Let $z \in P_{ab}$ such that $z \notin L(X)$ and $z \notin B_i$ for any i. By Theorem 1 (vii) $f^{-1}(z)$ is a singleton. If $c, d \in [0, 1]$ such that $c < f^{-1}(z) < d$ then $f([c, d]) \setminus \{z\}$ has at most two components. If $f([c, d]) \setminus \{z\}$ has two components they are $f([c, z))$ and $f((z, d])$. It is now easy to see that $z \notin L(Y)$ so Y satisfies condition (4).

LEMMA 3. *Let X, f, g, h and P_{ab} be as in Theorem 1. If $p < q$ in $[0, 1]$ and $x, y \in P_{ab} \setminus (h(p) \cup g(q))$ then for each neighborhood U of $f([p, q])$ and each finite set F of $X \setminus \{x, y\}$ there is an arc in $U \setminus (h(p) \cup g(q) \cup F)$ with endpoints x and y.*

Proof. Just suppose there exists a neighborhood U of $f([p, q])$ and a finite set F such that

$$U \setminus (h(p) \cup g(q) \cup F) = A \cup B$$

where A is separated from B and $x \in A$ and $y \in B$. Since A and B are open sets we may suppose by Theorem 1 (i) that $x, y \in P_{ab} \setminus L(X)$. By Theorem 1 (vii) $f^{-1}(x) < f^{-1}(y)$. Let

$$\omega = \sup\{t \in [f^{-1}(x), f^{-1}(y)] | \; f(t) \in A\} .$$

Then $f^{-1}(x) < \omega < f^{-1}(y)$. Let $x_i)$ be a sequence strictly increasing to ω such that for each i $f(x_i) \in A \setminus \{f(\omega)\}$. By Theorem 1 (ix) $(h(p) \cup g(q) \cup F) \cap f([p, q])$ is at most countable so there is a sequence $y_i)$ which is strictly decreasing to ω such that for each i $f(y_i) \in B \setminus \{f(\omega)\}$.

Let G_1 be an arc in $X \setminus \{f(\omega)\}$ with endpoints $f(x_1)$ and $f(y_1)$. Since $\lim f([x_i, y_i]) = \{f(\omega)\}$ there is an integer i_2 such that $G_1 \cap f([x_{i_2}, y_{i_2}]) = \emptyset$. By Theorem 1 (xii) there is an arc G_2 in $X \setminus (G_1 \cup \{f(\omega)\})$ with endpoints $f(x_{i_2})$ and $f(y_{i_2})$. Inductively there is a sequence $G_j)$ of pairwise disjoint arcs such that G_j has endpoints $f(x_{i_j})$ and $f(y_{i_j})$. If infinitely many of the G_j meet $h(p)$ then $h(p) \cup g(\omega)$ has $P(a, b)$ by V of the proof of Theorem 1. This contradicts Theorem 1 (x). Similarly, if infinitely many of the $G_j)$ meet $g(q)$ then $h(\omega) \cup g(q)$ has $P(a, b)$ which is a contradiction. Since $h(p) \cup g(q) \cup F$ separates $f(x_i)$ from $f(y_i)$ and F is finite at least one of these two cases must hold. The lemma is proved.

If C is a subset of a topological space we let $C^{(1)} = C^d$, the derived set of C. If α is an ordinal we let $C^{(\alpha+1)} = (C^{(\alpha)})^d$. If λ is a limit ordinal we let $C^{(\lambda)} = \bigcap_{\alpha<\lambda} C^{(\alpha)}$. The set $C^{(\alpha)}$ is called the αth *derived set of* C (see [4], p. 261).

LEMMA 4. *Let X be a finitely Suslinian continuum and let $a, b \in X$ such that no finite subset of X separates a and b. Let P_{ab} and f be as in Theorem 1 and let Y be as in*

Theorem 2. *Suppose* $C \subset Y$ *and* $x \in [0,1]$ *such that* $f(x)$ *is an isolated point of* C. *If* X *is planar or if* f *is at most countable to one then there exists a neighborhood* U *of* x *in* $[0,1]$ *such that* $f(U) \setminus C$ *is contained in one component of* $Y \setminus C$.

Proof. Suppose first that X is planar. Let $y = \inf f^{-1}(f(x))$ and let $z = \sup f^{-1}(f(x))$. If $y = z$ let U be any open interval in $[0,1]$ such that $x \in U$ and such that $f(U) \cap C = \{f(x)\}$. By the construction of Y $f(U)\setminus C$ is contained in one component of $Y\setminus C$. Let us suppose, therefore, that $y<z$. By Theorem 1 (x) $h(y) \cap g(z)$ is finite. We may assume X is embedded in the plane in such a way that $[0,1]\times\{0\} \subset h(y)$ and $[1,2]\times\{0\}\subset g(z)$. It follows that $f(y) = f(x) = (1,0) = f(z)$. Let U be an open interval about x in $[0,1]$ such that $f(U)\cap C = \{f(x)\}$ and the diameter of $f(U)$ is less than $\frac{1}{2}$. Let $U_1, U_2, \ldots$ be the components of $U \setminus f^{-1}(f(x))$. For each i let p_i and q_i be the endpoints of U_i. Suppose for each i $p_i<q_i$. Let $r, s \in U\setminus f^{-1}(C)$ such that $r<s$. Suppose $r \in U_1$. If $s \in U_1$ then $f(r)$ and $f(s)$ lie in the same component of $Y\setminus C$. Let us suppose, therefore, that $s \in U_2$.

Let n be a natural number such that $f(x) \notin B_n$ and $1/n<\min\{q_1-p_1, q_2-p_2\}$. Suppose $B_n = A_1 \cup \ldots \cup A_{k_n}$ where each A_j is a $1/n$-bridge with ends $a_j<b_j$. By the way B_n was chosen in the proof of Theorem 2 $b_j-a_j<1/n$ for $j = 1, \ldots, k_n$. Let $j \in \{1, \ldots, k_n\}$ be the largest integer such that $a_j \in U_1$ and let k be the smallest integer in $\{1, \ldots, k_n\}$ such that $b_k \in U_2$. By the proof of Theorem 2 $j \leqslant k$. We wish to show that

$$\bigcup_{i=j}^{k} A_i \cup \bigcup\{f(U_r) \mid U_r \cap \{a_i, b_i\} \neq \emptyset \text{ for some } i \in \{j, \ldots, k\}\}$$

is connected. This is clear if $k = j$. We shall show that

$$A_j \cup A_{j+1} \cup \bigcup\{f(U_r) \mid U_r \cap \{a_r, b_r\} \neq \emptyset \text{ for some } r \in \{j, j+1\}\}$$

is connected. The rest of the argument will be similar and we shall leave it to the reader.

Let U_3 be the component of $U\setminus f^{-1}(C)$ such that $b_j \in U_3$. Just suppose $f(a_{j+1}) \notin f(U_3)$. We may suppose without loss of generality that $a_{j+1} \in U_4$. Then $q_4<p_3$. By Theorem 1 (x) $f([a_{j+1}, q_4]) \cap f([a_j, q_1])$ is finite. Thus, there exist arcs C_1 in $f([a_j, q_1])$ and $C_4 \subset f([a_{j+1}, q_4])$ such that $C_1 \cap C_4 = \{f(x)\}$. Suppose U_5 is the component of $U\setminus f^{-1}(C)$ such that $b_{j+1} \in U_5$. We shall assume $U_5 \neq U_3$ since otherwise there is nothing to prove. As above there exist arcs $C_5 \subset f([b_{j+1}, q_5])$ and $C_3 \subset f([a_{j+1}, q_3])$ such that $C_t \cap C_s = \{f(x)\}$ for $t, s \in \{1, 3, 4, 5\}$ and $t \neq s$.

We may suppose since $(1,0) \in f(U)$, $f(U) \not\subset [0,2]\times\{0\}$ and the diameter of $f(U)<\frac{1}{2}$ that $f(U)$ meets the open upper half-plane. By Lemma 3 $f(U)$ is contained in the closed upper half-plane. We wish to show that there is a homeomorphism of the plane onto itself which carries $[0,2]\times\{0\}$ by the identity onto itself and which carries C_4 into the ray with parametric equation $\theta = \frac{1}{2}\pi$ and carries C_3 into the ray with parametric equation $\theta = \frac{1}{3}\pi$. If the above is not true then there exists a homeomorphism of the plane onto itself which carries $[0,2]\times\{0\}$ by the identity onto itself and which carries C_4 (resp. C_3) into the ray with parametric equation $\theta = \frac{1}{3}\pi$ (resp. $\theta = \frac{1}{2}\pi$).

By Lemma 3 $g(q_3)$ does not separate a from $C_4\setminus\{f(x)\}$ in any neighborhood of P_{ab}. It follows that every neighborhood of $h(q_4)$ separates $C_3\setminus\{f(x)\}$ from b. Thus $h(q_4)$ separates $C_3\setminus\{f(x)\}$ from b. This contradicts Lemma 3. We may suppose therefore that C_3 (resp. C_4) is contained in the ray with parametric equation $\theta = \frac{1}{3}\pi$ (resp. $\theta = \frac{1}{2}\pi$). By a similar argument we may assume that C_1 (resp. C_5) is contained in the sector $\frac{1}{2}\pi<\theta\leqslant\pi$ (resp. $0\leqslant\theta<\frac{1}{3}\pi$).

Since $C_1 \subset f(U_1)$, $C_3 \subset f(U_3)$, $C_4 \subset f(U_4)$ and $C_5 \subset f(U_5)$, A_j is an arc in $X\setminus\{f(x)\}$ from $f(U_1)$ to $f(U_3)$ and A_{j+1} is an arc in $X\setminus\{f(x)\}$ from $f(U_4)$ to $f(U_5)$ we have

$$f(U_1) \cup f(U_3) \cup f(U_4) \cup f(U_5) \cup A_j \cup A_{j+1}$$

is connected.

Suppose now that f is at most countable to one. Let U be a neighborhood of $f^{-1}(f(x))$ such that $f(U) \cap C = \{f(x)\}$. Notice that by the construction of Y the lemma holds at each point $y \in f^{-1}(f(x))$ such that y is an isolated point of $f^{-1}(f(x))$. It follows that if V is an interval in U such that V contains no points of $\big(f^{-1}(f(x))\big)^{(1)}$ then $f(V)\setminus C$ is contained in one component of $Y\setminus U$. By induction on the countable ordinal α it follows that the lemma holds for each $y \in \big(f^{-1}(f(x))\big)^{(\alpha)}\setminus\big(f^{-1}(f(x))\big)^{(\alpha+1)}$. Since $f^{-1}(x)$ is compact and countable there exists a countable ordinal β such that

$$f^{-1}(x) = \bigcup\{\big(f^{-1}(f(x))\big)^{(\alpha)}\setminus\big(f^{-1}(f(x))\big)^{(\alpha+1)} \mid \alpha\leqslant\beta\} \cup \big(f^{-1}(f(x))\setminus\big(f^{-1}(f(x))\big)^{(1)}\big).$$

THEOREM 5. *Let* X *be a finitely Suslinian continuum. Suppose* $a, b \in X$ *such that no finite subset of* X *separates* a *from* b. *Let* P_{ab} *and* f *be as in Theorem* 1 *and let* Y *be as in Theorem* 2. *If* X *is planar or if* f *is at most countable to one then no finite subset of* Y *separates* a *and* b *in* Y. *In particular,* Y *is not regular.*

Proof. The theorem follows immediately from Lemma 4 and the compactness of $[0,1]$.

It is well-known (see [1]) and easy to show that if X is a regular continuum then every sequence of pairwise disjoint connected sets in X is a null sequence. Theorem 6 asserts that the converse is also true for plane continua. Theorem 7 gives another condition under which the converse holds.

THEOREM 6. *A plane continuum* X *is regular if and only if every sequence of pairwise disjoint connected sets in* X *forms a null sequence.*

Proof. Let X be a plane continuum that is not regular. We shall prove that X contains a non-null sequence of pairwise disjoint connected sets. We suppose X is finitely Suslinian for otherwise there is nothing to prove. Let $a, b \in X$ such that no finite subset of X separates a and b. Let $Y = P_{ab} \cup B_1 \cup B_2 \cup \ldots$ be as in Theorem 2.

Decompose the family $\{B_1, B_2, \ldots\}$ into infinitely many pairwise disjoint infinite families $R_i' = \{B_{1_i}, B_{2_i}, \ldots\}$ where $i = 1, 2, \ldots$ Then $B_{j_i} \cap B_{r_s} = \emptyset$ $(j,i) \neq (r,s)$. For each i let $R_i = B_{1_i} \cup B_{2_i} \cup \ldots$ For each i $P_{ab} \cup R_i$ is not regular by Theorem 5. We prove first that no countable set in $P_{ab}\setminus R_i$ separates $P_{ab} \cup R_i$.

Let C be a countable set in $P_{ab}\setminus R_i$. Since X is completely normal we may assume C is closed. It suffices to prove since $[0, 1]$ is connected and $f^{-1}(C)$ is nowhere dense in $[0, 1]$ that for each $x \in [0, 1]$ there is a neighborhood U_x of x in $[0, 1]$ such that $f(U_x)\setminus C$ is contained in one component of $(P_{ab} \cup R_i)\setminus C$. By Lemma 4, such a neighborhood exists for each $x \in [0, 1]$ such that $f(x)$ is an isolated point of C.

Let $x \in [0, 1]$ such that $f(x) \in C^{(1)}\setminus C^{(2)}$. Let U be an open interval about x such that $f(U) \cap C^{(1)} = \{f(x)\}$. Let $U_1, U_2, \ldots$ be the components of $U\setminus f^{-1}(f(x))$. For each i $f(U_i)\setminus C$ is contained in one component of $Y\setminus C$ by Lemma 4. By the argument of Lemma 4 $f(U)\setminus C$ is contained in one component of $Y\setminus C$.

By induction on the countable ordinal α the theorem holds for each $x \in [0, 1]$ such that $f(x) \in C^{(\alpha)}\setminus C^{(\alpha+1)}$. Since C is countable and compact there is a countable ordinal β with

$$C = \bigcup \{C^{(\alpha)}\setminus C^{(\alpha+1)} |\ \alpha \leqslant \beta\} \cup (C\setminus C^{(1)}) .$$

This completes the proof that no countable subset of $P_{ab}\setminus R_i$ separates $P_{ab} \cup R_i$.

By a slight extension of a theorem of F. Bernstein (see [3], p. 201).

$$P_{ab}\setminus \bigcup R_i = Q_1 \cup Q_2 \cup \ldots$$

where the Q_i are pairwise disjoint sets such that for each i Q_i meets each Cantor set in P_{ab}. For each i $Q_i \cup R_i$ is connected since no Cantor set of $P_{ab} \cup R_i$ is contained in $P_{ab}\setminus(Q_i \cup R_i)$ and no countable subset of $P_{ab}\setminus R_i$ separates $P_{ab} \cup R_i$. Thus, $Q_1 \cup R_1, Q_2 \cup R_2, \ldots$ form a sequence of pairwise disjoint connected sets in X. This sequence is not null since for each i a and b are limit points of Q_i.

THEOREM 7. *Let X be a finitely Suslinian continuum and let a and b be two points of X such that X has $P(a, b)$. Let P_{ab} be a continuum in X which is minimal with respect to having $P(a, b)$. Let $f\colon [0, 1]\to P_{ab}$ be as in Theorem 1. If f is at most countable to one then X contains a non-null sequence of pairwise disjoint connected sets.*

Proof. The proof is exactly the same as that of Theorem 6.

§ 4. **Rim compactness.** It was proved in [6] Corollary 4.5 that a hereditarily locally connected, rim compact, separable, metric space has a hereditarily locally connected, metric compactification. The main purpose of this section is to show that the converse of that result fails. In particular, we give an example of a finitely Suslinian plane continuum that contains a subset that is not rim compact. The following proposition shows that the construction in Theorem 2 cannot yield such an example.

PROPOSITION 8. *Let X be a finitely Suslinian continuum that is not regular. Let Y be as in Theorem 2. Then every connected subset of Y is rim compact.*

Proof. Let C be a connected subset of Y and let $c \in C$. If $c \notin P_{ab}$ then there is a basis of neighborhoods of c in Y with two point boundaries. Let us suppose, therefore, that $c \in P_{ab}$. Let U be a neighborhood of c in Y. There exist disjoint intervals $[x_1, y_1], \ldots, [x_n, y_n]$ in $[0, 1]$ such that $c \in P_{ab}\setminus \bigcup_{i=1}^{n} f([x_i, y_i]) \subset U$ and $c \in f([y_i, x_j])$ for $i \neq j \in \{1, \ldots, n\}$.

Let $z = \inf f^{-1}(c)$ and suppose $y_1 < z$. Then either a finite set separates $f([x_1, y_1]) \cap C$ from c in C or there exists $\omega \in [0, 1]$ such that $y_1 < \omega < z$ and $f(\omega) \in C$. Suppose $\omega \in [0, 1]$ such that $y_1 < \omega < z$ and $f(\omega) \in C$. Then by the construction of Y and by Theorem 1 (ix)

$$K_1 = g(\omega) \cap \big(h(\omega) \cup \bigcup \{A|\ A \text{ is a bridge in } Y \text{ which meets } h(\omega)\}\big)$$

is a set with at most one limit point namely $f(\omega)$. If $c \notin K_1$ then K_1 separates $f([x_1, y_1])$ from c in Y. If $c \in K_1$ then c lies in a non-degenerate bridge of Y which meets $h(\omega)$. Let d be a point other than c in that bridge. Then $(K_1 \cup \{d\})\setminus\{c\}$ separates $f([x_1, y_1])$ from c in Y.

Suppose $e, d \in [0, 1]$ such that $f(e) = f(d) = c$, if $x \in [0, 1]$ such that $e < x < d$ then $f(x) \neq c$ and $e < x_i < y_i < d$. By an argument similar to the one above there is a compact subset of C which has at most two accumulation points and which separates $f([x_i, y_i])$ from c in C.

A similar argument may be used if $\sup f^{-1}(c) < x_1$. Thus, there is a compact subset K of C which separates $P_{ab}\setminus U$ from c in C. Since $C\setminus P_{ab}$ consists of a null sequence of free arcs it follows that a compact subset of C separates $C\setminus U$ from c in C.

EXAMPLE 3. Let S be the Sierpiński triangular curve (see [5], p. 276). It is defined as follows: Let T be the equilateral triangle in the plane with vertices $(0, 0)$, $(1, 1)$ and $(\sqrt{2}, 0)$. Partition T into four congruent triangles T_0, T_1, T_2 and T_3. Let T_0, T_1, and T_2 be the triangles which have a vertex in common with T. The triangles T_0, T_1, and T_2 are numbered clockwise and T_0 is the leftmost triangle of the three. Let v_0, v_1, and v_2 be the vertices of T_3 where v_0 is the leftmost vertex of the three and the numbering is clockwise. In a similar way, partition each of the triangles T_i for $i = 0, 1, 2$ into four congruent triangles T_{i0}, T_{i1}, T_{i2}, T_{i3}. Let v_{i0}, v_{i1}, v_{i2} be the vertices of T_{i3}. The triangles and vertices are ordered clockwise starting with the leftmost one. Continue inductively in this manner. Let

$$S = \mathrm{Cl}\big(\bigcup \mathrm{Bd}(T_{\alpha_1\alpha_2\ldots\alpha_k})\big)$$

where the subscripts α_i take the values 0, 1 and 2 and $k = 1, 2, \ldots$ The local cutpoints of S are the vertices $v_{\alpha_1\ldots\alpha_k}$ where the subscripts α_i take the values 0, 1 and 2 and $k = 1, 2, \ldots$

Let $X = S \cup K_1 \cup K_2 \cup \ldots$ where

1) the sets K_i are pairwise disjoint arcs in the plane whose diameters converge to zero,

2) $L(X) = K_1 \cup K_2 \cup \ldots$

3) for each i $K_i \cap S \subset S\setminus L(S)$ and $K_i \cap S$ consists of the two endpoints of K_i,

4) for each i there is a $k \in \{1, 2, \ldots\}$ and $\alpha_1, \ldots, \alpha_k \in \{0, 1, 2\}$ such that the endpoints of K_i are contained on one side of $T_{\alpha_1\alpha_2\ldots\alpha_k}$ or on one side of T.

It is clear that X is a Peano continuum. We shall prove that X is hereditarily locally connected. We proceed by contradiction. Suppose $A_1, A_2, \ldots$ is a sequence of pairwise disjoint continua in X such that $\lim A_i = A$ where A is a non-degenerate continuum which is disjoint from $A_1 \cup A_2 \cup \ldots$ Notice that $A \subset S$. We may suppose that each A_i is an arc which meets both of the horizontal lines $y = \frac{1}{4}$ and $y = \frac{1}{2}$. We may also suppose that no proper subarc of A_i meets both of the horizontal lines $y = \frac{1}{4}$ and $y = \frac{1}{2}$. Then A is contained in the closed horizontal strip bounded by the lines $y = \frac{1}{4}$ and $y = \frac{1}{2}$ and A meets both the top and bottom components of the boundary of that strip. Thus, either $v_0 \in A$ or $v_1 \in A$. We may suppose without loss of generality that $v_0 \in A$. Then $A \subset T_{01}$ and for each i

$$A_i \subset T_{01} \cup \bigcup \{K_j \mid K_j \cap T_{01} \neq \emptyset\}.$$

For each i A_i is disjoint from the line segment $\overline{v_0 v_{01}}$ with endpoints v_0 and v_{01} otherwise A_i would separate A between v_0 and $A \cap \overline{v_{00} v_{01}}$. By the minimality of the arcs A_i) it follows that $A \subset T_{010} \cup T_{011}$. Since A_i does not separate A between v_0 and $\overline{v_{00} v_{01}}$ $A_i \cap \overline{v_{010} v_{011}} = \emptyset$. Similarly $A_i \cap \overline{v_{010} v_{012}} = \emptyset$ for each i. Hence,

$$A \subset T_{0111} \cup T_{0110} \cup T_{0100} \cup T_{0101}.$$

In this way one can prove inductively that for each $k = 1, 2, \ldots$

$$A \subset \bigcup T_{\alpha_1 \alpha_2 \ldots \alpha_k}$$

where $\alpha_1, \ldots, \alpha_k \in \{0, 1\}$. Thus, $A = \overline{v_0 v_{00}}$. This is a contradiction since each A_i meets $\overline{v_0 v_{00}}$.

We have proved that X contains no continuum of convergence. By [11], V, 2.1 X is hereditarily locally connected. By a theorem of Gehman (see [5], p. 519) X is finitely Suslinian.

By a slight extension of a theorem of F. Bernstein (see [3], p. 201) $S = Q_0 \cup Q_1 \cup \ldots$ where the Q_i are pairwise disjoint sets and each Q_i meets each Cantor set in S. Since $L(S)$ is countable we may suppose $L(S) \subset Q_0$.

The family $\{K_1, K_2, \ldots\}$ can be decomposed into infinitely many pairwise disjoint families $R'_j = \{K_{1_j}, K_{2_j}, \ldots\}$ such that $L(R_j \cup S) = R_j$ where $R_j = \bigcup_{i=1}^{\infty} K_{i_j}$ for $j = 1, 2, \ldots$ Since $R_j \cap S$ is countable we may suppose that $R_j \cap S \subset Q_j$ for each $j = 1, 2, \ldots$

Now Q_0 is connected since any set in $S \setminus Q_0 \subset S \setminus L(S)$ which separates S contains a Cantor set by [11], III, 9.4. Similarly $Q_i \cup R_i$ is connected for each $i = 1, 2, \ldots$ since any set in

$$(S \cup R_i) \setminus (Q_i \cup R_i) \subset S \setminus Q_i \subset (S \cup R_i) \setminus L(S \cup R_i)$$

which separates the continuum $S \cup R_i$ contains a Cantor set.

Finally we shall show that $Q_j \cup R_j$ is not rim compact for each $j = 1, 2, \ldots$ Let a and b be two points of Q_j. Just suppose K is a compact set in $Q_j \cup R_j$ which separates a and b in $Q_j \cup R_j$. By a theorem of Mazurkiewicz (see [5], p. 244) we may suppose no proper subset of K separates a and b in $S \cup R_j$. There exists an irreducible compact set K^* in $K \cup (S \setminus Q_j)$ such that K^* separates a and b in $S \cup R_j$. Now $K^* \setminus K$ is topologically complete. Since $L(R_j \cup S) \subset R_j$ K^* has no isolated points by [11], III, 9.3. Since K is countable $K^* \setminus K$ has no isolated points. Thus, $K^* \setminus K \subset S \setminus Q_j$ contains a Cantor set. This is a contradiction since Q_j meets each Cantor set in S. We conclude that no compact subset of $R_j \cup Q_j$ separates a and b in $R_j \cup Q_j$ and so $R_j \cup Q_j$ is not rim compact.

We have proved that X is a finitely Suslinian continuum which can be decomposed into the pairwise disjoint connected sets $Q_0, Q_1 \cup R_1, Q_2 \cup R_2, \ldots$ Each of the sets Q_i is dense in S. For each $j = 1, 2, \ldots$ the set $Q_j \cup R_j$ is not rim compact.

Remark. T. Nishuira has pointed out that by modifying slightly the argument in the above example X can be decomposed into countably many large disjoint connected sets no one of which is rim compact.

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