Some remarks on well-ordered models *

by

H. Kotlarski (Warszawa)

Abstract. We give some information on well-ordered models. More exactly, assume that the language L contains a binary predicate $<$, T is a theory in L, and $T \vdash$ "$<$ is a linear ordering". In § 1 we give a syntactical characterization of those theories T which have a model which is well-ordered by the interpretation of $<$. In § 2 we give some conditions which imply that a given well-ordered model $\mathfrak{A} = \langle A, <, \ldots \rangle$ has a proper well-ordered elementary extension. In § 3 we connect our previous results with the notion of constructibility.

Our main tool is that of atomic models (cf. e.g. Chang–Keisler [3]). In § 2 we develop the theory of Skolem ultrapowers in the version we need and we use it to construct elementary extensions.

We use standard model-theoretic terminology and notation as in Chang–Keisler [3].

§ 1. Theories which have well-ordered model. Let L be a first order language and let $<$ be a binary predicate of L. Let T be a theory in L. Assume that $T \vdash$ "$<$ is a linear ordering".

To state our first characterization of those T which have a well-ordered model, we need some special notation. Let Φ denote the set of all sequences of formulas of L of the form $\varphi = \{\varphi_n(v_1, \ldots, v_n)\}_{n=1}^{\infty}$. For each sequence $\varphi \in \Phi$ we define a sequence $\tilde{\varphi} = \{\tilde{\varphi}_n\}_{n=1}^{\infty}$ of sentences of L as: $\tilde{\varphi}_1$ is $\exists v_1 \varphi_1(v_1)$ and $\tilde{\varphi}_{n+1}$ is

$$\forall v_1, \ldots, v_n \big(\varphi_n(v_1, \ldots, v_n) \to \exists v_{n+1}(v_{n+1} < v_n \wedge \varphi_{n+1}(v_1, \ldots, v_n, v_{n+1}))\big).$$

THEOREM 1.1. *Under the above notation, T has a well-ordered model iff there exists a function $h\colon \Phi \to \omega$ such that*

$$\forall k \in \omega\, \forall \varphi^1, \ldots, \varphi^k \in \Phi \text{ non } T \vdash \bigvee_{j=1}^{k} \tilde{\varphi}^j_{h(\varphi^j)}.$$

Note. This generalizes the main theorem of Kotlarski [5], where a version which holds for complete theories T (cf. 1.2 below) was stated and proved.

---

* This paper contains parts of the author's doctoral thesis [6], written under the direction of professor C. Ryll-Nardzewski.

The author would like to thank professors A. Mostowski and C. Ryll-Nardzewski and also W. Marek, J. Onyszkiewicz, K. Apt and M. Srebrny for valuable suggestions.

Proof of Theorem 1.1. $\Rightarrow$ Let $\mathfrak{A} = \langle A, <, \ldots\rangle$ be well ordered model of T. We claim that

(1) $$\forall \varphi \in \Phi \, \exists n \in \omega \, \mathfrak{A} \models \neg \tilde{\varphi}_n .$$

Indeed, otherwise $\forall n \, \mathfrak{A} \models \tilde{\varphi}_n$, which allows us to construct a descending sequence of elements of A in a trivial way, a contradiction.

Now let $h(\varphi)$ be any $n \in \omega$ such that $\mathfrak{A} \models \neg\tilde{\varphi}_n$. Obviously the function h constructed above satisfies our demand.

$\Leftarrow$ Let $h\colon \Phi \to \omega$ be such that

$$\forall k \in \omega \, \forall \varphi^1, \ldots, \varphi^k \in \Phi \text{ non } T \vdash \bigvee_{j=1}^{k} \tilde{\varphi}^j_{h(\varphi^j)} .$$

Let $T_1 := T \cup \{\neg \tilde{\varphi}_{h(\varphi)} |\ \varphi \in \Phi\}$. By the deduction theorem and the assumption about h,

(1) T_1 is consistent.

Now we claim that:

(2) the scheme of minimum (without parameters) is provable in T_1.

Indeed, let $A(v)$ be any formula of L. If

$$\text{non } T_1 \vdash \exists v A(v) \to \big(\exists v A(v) \wedge (\forall v_1 < v \neg A(v_1))\big) ,$$

then the sequence $\{\psi_n\}$ defined by the conditions

$$\psi_1(v_1) = A(v_1),$$
$$\psi_{n+1}(v_1, \ldots, v_{n+1}) = \bigwedge_{i=1}^{n+1} A(v_i) \wedge \bigwedge_{i=1}^{n} v_{i+1} < v_i$$

has the property that $T_1 \vdash \tilde{\psi}_n$ for all n, which contradicts the definition of T_1. So (2) is proved.

Next we claim that

(3) the scheme of minimum with parameters is provable in T_1

(this uses only (2), and not the assumption of the theorem). This is proved by induction on the number of parameters. Let $A(v_0, \ldots, v_n)$ be any formula of L. Let B be the following sentence:

$$\forall v_1, \ldots, v_n \big(\exists v_0 A(v_0, \ldots, v_n)$$
$$\to \big(\exists v_0 A(v_0, v_1, \ldots, v_n) \wedge \forall v_{n+1} < v_0 \neg A(v_{n+1}, v_1, \ldots, v_n)\big)\big) .$$

To prove (3) we only need to show that $T_1 \cup \{\neg B\}$ is inconsistent. Assume the contrary, i.e. that $T_1 \cup \{\neg B\}$ is consistent. Then "the smallest v_1 such that for some $v_2, \ldots, v_n$, the scheme of minimum fails for A" is definable in $T_1 \cup \{\neg B\}$ by a formula, say $C(v_1)$.

Now the formula $\forall v_1 C(v_1) \to A(v_0, v_1, \ldots, v_n)$ has fewer parameters than A, and so, by the inductive assumption, the scheme of minimum for $\forall v_1 C(v_1) \to A(v_0, \ldots, v_n)$ is provable in T, and so it is provable in $T_1 \cup \{\neg B\}$ thus $T_1 \cup \{\neg B\}$ is inconsistent.

By (3), T_1 is skolemized in the natural way: namely, for any formula $A(v_0, v_1, \ldots, v_n)$ let $t_A(v_1, \ldots, v_n)$ be "the smallest v_0 such that $A(v_0, v_1, \ldots, v_n)$ if there exists such a v_0, or the smallest v_0 such that $v_0 = v_0$ otherwise".

Obviously, t_A's are definable in T_1, and the usual Skolem axioms are provable in T_1.

Now, let T_2 be any complete and consistent extension of T_1. T_2 is also skolemized, and so it has a pointwise definable model, whence it has an atomic model: namely, the closure of the empty set under the Skolem functions is pointwise definable, and hence atomic.

Let $\mathfrak{A}$ be any atomic model of T_2. Obviously, $\mathfrak{A} \models T$; indeed, $T \subseteq T_1 \subseteq T_2$, and so we only need to show that $\mathfrak{A}$ is well ordered. Assume the contrary, i.e. let $\{a_n\}_{n=1}^{\infty}$ be any descending sequence of elements of A.

Let $\varphi_n(v_1, \ldots, v_n)$ be any atomic formula such that $\mathfrak{A} \models \varphi_n[a_1, \ldots, a_n]$. One easily verifies that $\mathfrak{A} \models \tilde{\varphi}_n$ for all $n \in \omega$, so $T_2 \vdash \tilde{\varphi}_n$ for all $n \in \omega$, and so non $T_1 \vdash \neg\, \tilde{\varphi}_n$, but this contradicts the definition of T_1. ■

Some consequences of Theorem 1.1 should be mentioned:

COROLLARY 1.2 ([5]). *If T is complete, then T has a well-ordered model iff $\forall \varphi \in \Phi \, \exists n$ non $T \vdash \tilde{\varphi}_n$.*

Proof. $\Rightarrow$ obvious.

$\Leftarrow$ by the completeness of T, any function $h\colon \Phi \to \omega$ such that non $T \vdash \tilde{\varphi}_{h(\varphi)}$ satisfies the condition from Theorem 1.1. ■

COROLLARY 1.3. *If T is complete, then T has a well-ordered model iff every countable $T_0 \subseteq T$ has a well-ordered model.*

Proof. Obvious from Corollary 1.2. ■

If we assume from the very beginning not only $T \vdash$ "$<$ is a linear ordering", but also $T \vdash$ "the scheme of minimum", then our theorem and its proof become slightly simpler. Namely, if $T \vdash$ "the scheme of minimum", then T is skolemized exactly as in the proof of 1.1, and we may assume that the symbols for these Skolem functions are in L. Denote by Term = the set of all constant terms of L.

COROLLARY 1.4. *Under the above assumptions, T has a well-ordered model iff there exists a function $h\colon \mathrm{Term}^{\omega} \to \omega$ such that for all $k \in \omega$ and all $t^1, \ldots, t^k \in \mathrm{Term}^{\omega}$*

$$\textit{non } T \vdash \bigvee_{j=1}^{k} t^j_{h(t^j)+1} < t^j_{h(t^j)} .$$

Proof. Slightly simpler than that of Theorem 1.1. ■

Other conditions for T to have a well-ordered model are also known (cf. [6]); they are based on other ideas, but they are even more complicated. It should be noted that such conditions must be complicated because of the following theorem, due to K. R. Apt.

THEOREM 1.5. *Assume that* L *is arithmetized. Then the formula "T has a well-ordered model" is Σ_2^1 but not Π_2^1.*

Proof. Obviously our formula is Σ_2^1, since it is equivalent to "there exists a consistent $T_1 \supseteq T$ such that for all $\varphi \in \Phi$ there exists an $n \in \omega$ such that $T_1 \vdash \neg\tilde{\varphi}_n$" (cf. 1.2) and this is a Σ_2^1 formula.

The second part of the proof needs some knowledge of second order arithmetic, as collected in Apt-Marek [2].

Let L be the language of the second order arithmetic and let $T = A_2 + \mathrm{V} = \mathrm{L}$, let $<$ be the usual constructible well-ordering. Assume our formula, call it $H(\cdot)$ is Π_2^1: we derive a contradiction.

First remark that each Π_2^1 formula is absolute downwards with respect to β-models, i.e. if $\mathfrak{B}$ is a β-model for A_2, and $P(\omega) \models H$, then $\mathfrak{B} \models H$.

Let $\mathfrak{B}_0$ be the minimal β-model for A_2. We have $T \in \mathfrak{B}_0$, since T is recursively enumerable. Now $P(\omega) \models H(T)$, hence $\mathfrak{B}_0 \models H(T)$, and so $\mathfrak{B}_0 \models$ "there exists a well-ordered (i.e. β-) model for A_2". But this contradicts the minimality of $\mathfrak{B}_0$ and the absoluteness of the notion of β-model with respect to β-models. ■

Similar negative results, concerning the β-rule rather than having a well-ordered model, can be found in Apt [1] (¹).

§ 2. **Well-ordered elementary extensions.** Let $\mathfrak{A} = \langle A, <, \ldots\rangle$ be a well-ordered model. We shall study the problem for $\mathfrak{A}$ having a proper well-ordered elementary extension.

An obvious necessary and sufficient condition is that the theory

$$T := \mathrm{Th}\,\mathfrak{A}_A \cup \{d \neq a \mid a \in A\},$$

where d is a new constant, should have a well-ordered model.

Analogies with the Omitting types theorem suggest the following: "if T is consistent under the rule

infer $\neg C$ from $\bigwedge\!\!\!\bigwedge_n C \to \tilde{\psi}_n$ for some $\psi \in \Phi$ and a sentence C

then T has a well-ordered model".

But the statement in quotation marks is easily refuted. Indeed, let $\mathfrak{A}$ be any infinite well-ordered model with no well-ordered elementary extension (e.g. the standard model of Peano arithmetic, or the minimal model of ZF). Then T is consistent and closed under this rule, but has no well-ordered model.

To state some positive results, let us give the following definition.

A function $f\colon A^{<\varkappa} \to A$ is a $\varkappa$-RZM function for $\mathfrak{A}$ iff

(i) $\forall x \in A^{<\varkappa}\; f(x) \notin \mathrm{range}\, x$,

(¹) Added in proof: $H(\cdot)$ is "almost" Π_1^{ZF}; see [6a] for the precise formulation.

(ii) if x is a subsequence of y, then $f(x)$ and $f(y)$ realize the same type in the model $\mathfrak{A}_{\mathrm{range}\,x}$, where $\mathfrak{A} = \langle A, \ldots\rangle$ is any model (not necessarily well ordered) and $A^{<\varkappa} = \bigcup_{\alpha<\varkappa} A^{\alpha}$ is the set of all sequences of elements of A of length less than $\varkappa$.

(Remark that if we wanted to prove Theorem 2.3 below for linearly ordered models only, we could consider a slightly simpler notion, namely that of functions with domain $\{X \subseteq A \mid \mathrm{card}\, X < \varkappa\}$ which satisfy (i) $f(X) \notin X$ and (ii) $X \subseteq Y \Rightarrow f(X)$ and $f(Y)$ realize the same type over X; we have chosen a more complicated version for generality.)

THEOREM 2.1. *If $\mathfrak{A} = \langle A, <, \ldots\rangle$ is a well-ordered model, which has an ω_1-RZM function, then $\mathfrak{A}$ has a proper well-ordered elementary extension.*

Proof. Let $T := \mathrm{Th}\,\mathfrak{A}_A \cup \{d \neq a \mid a \in A\}$, where d is a new constant. Obviously, we only need to construct a well-ordered model for T, i.e. (by Theorem 1.1) we only need to construct a function $h\colon \Phi \to \omega$ such that

$$\forall k \in \omega\, \forall \varphi^1, \ldots, \varphi^k \in \Phi \text{ non } T \vdash \bigvee_{j=1}^{k} \tilde{\varphi}^j_{h(\varphi^j)}, \tag{1}$$

where we use the same notation as in 1.1 (with respect to T of course).

So fix any $\varphi \in \Phi$ e.g. say that φ is $\{\varphi_n(d, v_1, \ldots, v_n)\}$. Let x_φ denote the sequence of parameters $a \in A$ which occur in some φ_n. Obviously, the length of x_φ is countable.

We claim that there exists an $l \in \omega$ such that

$$\mathfrak{A}_A \models \neg\tilde{\varphi}_l[f(x_\varphi)] \tag{2}$$

(i.e. $f(x_\varphi)$ is a "good" interpretation for d).

Indeed, otherwise we would have $\mathfrak{A}_A \models \tilde{\varphi}_l[f(x_\varphi)]$ for all $l \in \omega$, which allows us to construct a descending sequence of elements of A exactly as in 1.1, a contradiction. Now let $h(\varphi)$ be any $l \in \omega$ such that (2) holds. We claim that the function h constructed in this way satifies (1).

Fix any $k \in \omega$ and $\varphi^1, \ldots, \varphi^k \in \Phi$. By construction,

$$\mathfrak{A}_A \models \neg\tilde{\varphi}^j_{h(\varphi^j)}[f(x_{\varphi^j})] \quad \text{for} \quad j = 1, \ldots, k,$$

and so by (ii) from the definition of an RZM function

$$\mathfrak{A}_A \models \bigwedge\!\!\!\bigwedge_{j=1}^{k} \neg\tilde{\varphi}^j_{h(\varphi^j)}[f(x_{\varphi^1}{}^\frown \ldots {}^\frown x_{\varphi^k})]$$

($^\frown$ denotes concatenation). It follows that non $T \vdash \bigvee_{j=1}^{k} \tilde{\varphi}^j_{h(\varphi^j)}$. ■

Before going further, let us develop the construction of Skolem ultrapowers in the way we need it.

Let $\mathfrak{A} = \langle A, \ldots\rangle$ be any model (not necessarily well ordered). We denote by $\mathrm{Def}\,\mathfrak{A}$ = the Boolean algebra of all subsets of A definable in $\mathfrak{A}$ with parameters from A. Let $p \in S_1(\mathfrak{A})$ = the set of all ultrafilters in $\mathrm{Def}\,\mathfrak{A}$. Let Tm denote the set of all terms of the language of $\mathfrak{A}_A$ with at most one variable v free.

Define

$$t_1 \sim t_2 \quad \text{iff} \quad t_1(v) = t_2(v) \in p \quad \text{for} \quad t_1, t_2 \in \mathrm{Tm}.$$

One easily verifies that $\sim$ is an equivalence relation and the following definition of relations in $\mathrm{Tm}/_\sim$ makes sense:

$$R(t_1^\sim, \ldots, t_n^\sim) \quad \text{iff} \quad R(t_1(v), \ldots, t_n(v)) \in p.$$

Functions and constants are treated similarly. We shall denote by $\mathfrak{A}/_p$ the model constructed above. The appropriate version of the lemma of Łoś is

LEMMA 2.2 *If* Th$\mathfrak{A}$ *is skolemized, then for each formula* $\varphi(v_1, \ldots, v_n)$ *we have*

$$\mathfrak{A}/_p \models \varphi(t_1^\sim, \ldots, t_n^\sim) \quad \textit{iff} \quad \varphi(t_1(v), \ldots, t_n(v)) \in p.$$

Proof. Almost like the original one. One uses skolemization in the quantifier step of induction. ■

This gives as usual the following facts:

The function $r\colon A \to \mathrm{Tm}$ given by $r(a) = a^\sim$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{A}/_p$; r is proper iff, for all $a \in A$, $v \neq a \in p$.

Using this method of constructing elementary extensions, we shall give a slightly modified proof of Theorem 2.1.

Let $\mathfrak{A} = \langle A, <\ldots,\rangle$ be well ordered. Obviously we may assume that Th$\mathfrak{A}$ is skolemized by the scheme of minimum as in the proof Theorem 1.1. Let f be an ω_1-RZM function for $\mathfrak{A}$. Define

$$p = \{\varphi(v, a_1, \ldots, a_n) |\ \mathfrak{A} \models \varphi[f(\langle a_1, \ldots, a_n\rangle), a_1, \ldots, a_n]\}.$$

One easily verifies that p is an ultrafilter in Def$\mathfrak{A}$ ((ii) of the definition of an RZM function for finite sequences only is used to verify that $\varphi \in p$, $\psi \in p \Rightarrow \varphi \wedge \psi \in p$). Consider the model $\mathfrak{A}/_p$. This is a proper extension of by (i) from the definition of an RZM function. Thus we need only to verify that $\mathfrak{A}/_p$ is well-ordered.

Assume the contrary, i.e. let $\{t_n^\sim\}$ be a descending sequence, let t_n be $t_n(v, a_1^n, \ldots, a_{k_n}^n)$. Consider the sequence x containing all the parameters which occur in some t_n, $x = \langle a_1^0, \ldots, a_{k_0}^0, a_1^1, \ldots, a_{k_1}^1, \ldots\rangle$. Now we have $\mathfrak{A}/_p \models t_{n+1}^\sim < t_n^\sim$, so $t_{n+1}(v) < t_n(v) \in p$ for all n by 2.2; so $\mathfrak{A}_A \models t_{n+1}[f(x)] < t_n[f(x)]$ for all n by the definition of p and (ii) from the definition of on RZM function, but this contradicts the fact that $\mathfrak{A}$ is well-ordered. ■

The notion of a $\varkappa$-RZM function is rather strong: namely, we prove

THEOREM 2.3. *Let* $\mathfrak{A}$ *be a skolemized model. Then* $\mathfrak{A}$ *has a* $\varkappa$-RZM *function iff* $\mathfrak{A}$ *is* $\varkappa$ *saturated relative to some of its proper elementary extensions.*

(The notion of relative saturation is due to Simpson, see Chang-Keisler [3], p. 429).

Proof. $\Leftarrow$ (this direction does not use the skolemization of $\mathfrak{A}$). Let $\mathfrak{A}$ be $\varkappa$ saturated relative to $\mathfrak{B}$. Fix any $b_0 \in B - A$. For any sequence x with domain$(x) \in \varkappa$ the type of b_0 in the model $\mathfrak{B}_{\mathrm{range}\,x}$ is realised in $\mathfrak{A}_{\mathrm{range}\,x}$ (by relative saturativity) by an element, call any such element $f(x)$. Obviously the function f constructed in this way is a $\varkappa$-RZM function.

$\Rightarrow$ Exactly like the second proof of 2.1. ■

Let $\mathfrak{A} = \langle A, <, \ldots\rangle$ be a linearly ordered model. An ultrafilter p in Def$\mathfrak{A}$ is said to be *realized arbitrarily high in* $\mathfrak{A}$ iff

$$\forall a \in A\, \exists b \in A\, \mathfrak{A} \models a < b \text{ and } b \text{ realizes } p \cap \{\varphi(v, a_1, \ldots, a_n) |\ a_1, \ldots, a_n \leqslant a\}.$$

THEOREM 2.4. *Assume that* $\mathfrak{A} = \langle A, <, \ldots\rangle$ *is linearly ordered and has an ultrafilter* p *which is realized arbitrarily high. Then* $\mathfrak{A}$ *has an end elementary extension. Moreover, we may require* $\mathfrak{A}$ *to be* cf$(A, <)$ *saturated relative to that extension.*

Proof. One easily verifies that $\mathfrak{B} := \mathfrak{A}/_p$ has the required properties. ■

COROLLARY 2.5. *If* $\mathfrak{A}$ *is well ordered, not cofinal with* ω *and has an ultrafilter realized arbitrarily high, then* $\mathfrak{A}$ *has an end well-ordered elementary extension.* ■

In fact, we can weaken the assumption about skolemization in 2.3 and 2.4. Let us call a complete theory T *strongly atomic* iff, for all $\mathfrak{A} \models T$ and all $X \subseteq A$, the theory Th$\mathfrak{A}_X$ has an atomic model.

It is well known that every ω-stable theory is strongly atomic (Chang Keisler [3], Corollary 7.1.12). Obviously every skolemized theory is strongly atomic.

Now we can generalize 2.3 and 2.4 by requiring Th$\mathfrak{A}$ to be strongly atomic rather than skolemized. One considers any atomic model of Th$\mathfrak{A}_A \cup \{\varphi(d) |\ \varphi \in p\}$; the rest of the proofs are the same.

The following theorem, due to Silver, easily follows from Corollary 1.3:

THEOREM 2.6 (Silver [8]). *If* $\mathfrak{A} = \langle A, <, \ldots\rangle$ *is a well-ordered model generated by a set* X *of order indiscernibles with* cf$(X) > \omega$, *then* $\mathfrak{A}$ *has arbitrarily large well-ordered elementary extensions.*

Proof. Fix a cardinal $\varkappa$. Add to the language of Th$\mathfrak{A}$ constants c_α for $\alpha \in \varkappa$. Let

$$T := \mathrm{Th}\mathfrak{A} \cup \{\varphi(c_{\alpha_1}, \ldots, c_{\alpha_n}) |\ \alpha_1 \in \alpha_2 \in \ldots \in \alpha_n \text{ and } \exists \langle x_1, \ldots, x_n\rangle \in [X]^n\ \mathfrak{A} \models \varphi[x_1, \ldots, x_n]\}.$$

By the definition of indiscernibles, T is complete and consistent.

We claim that every countable $T_0 \subseteq T$ has a well-ordered model. Indeed, a countable T_0 contains only countably many c_α, their set is well-ordered by the relation $c_\alpha < c_\beta$ iff $T \vdash c_\alpha < c_\beta$ and so it can be order embedded into X; this gives an interpretation of T_0 in $\mathfrak{A}$.

It follows from Corollary 1.3 that T has a well-ordered model. Thus to prove Theorem 2.6 we only need to show that $\mathfrak{A}$ may be elementarily embedded into the reduct of any model of T to the language of Th$\mathfrak{A}$. Fix any $a \in A$, a being of the form $a = t(x_1, \ldots, x_n)$ for some term t of the language of Th$\mathfrak{A}$ and some sequence $\langle x_1, \ldots, x_n\rangle$ of elements of X. Embed X into $\varkappa$, and let j be this embedding. Define $r(a) = t(c_{jx_1}, \ldots, c_{jx_n})$: this is the required embedding. ■

§ 3. **Connections with set theoretic concepts.** Let $\mathfrak{A} = \langle L_{\omega_1}, \in, < \rangle$, where $<$ is Gödel's ordering. Does $\mathfrak{A}$ have an ω_1-RZM function? We show that this is independent of ZFC set theory.

PROPOSITION 3.1 *If* V = L *then* $\mathfrak{A}$ *has no* ω_1-RZM *function.*

Proof. One can prove directly that if $\mathfrak{A}$ has a proper well-ordered (i.e. well-founded) elementary extension, then $\omega_1^L < \omega_1$; hence V $\neq$ L; we shall give a more elegant proof, using some ideas of Mostowski, cf. Marek [7].

Let us call a model $\mathfrak{B}$ *typewise definable* iff the function $B \ni a \to \mathrm{type}(a) = \{\varphi(v) | \ \mathfrak{B} \models \varphi[a]\}$ is 1-1 (i.e. different members of B realize different types in $\mathfrak{B}$).

It is obvious that no typewise definable model has any RZM-function, and so to prove 3.1 we only need to show that

(1) the model $\langle L_{\omega_1^L}, \in \rangle$ is typewise definable.

To prove this, remark first that we only need to show

(2) $$\forall \alpha \in \beta \in \omega_1^L \ \langle L_{\omega_1^L}, \in, \alpha \rangle \not\equiv \langle L_{\omega_1^L}, \in, \beta \rangle$$

because of the definability of Gödel's function F in $\langle L_{\omega_1^L}, \in \rangle$. So assume that (2) does not hold, and let α_0 be the smallest ordinal such that $\exists \beta \alpha_0 \in \beta \in \omega_1^L$ and

$$\langle L_{\omega_1^L}, \in, \alpha_0 \rangle \equiv \langle L_{\omega_1^L}, \in, \beta \rangle .$$

We claim that

(3) α_0 is definable in $\langle L_{\omega_1^L}, \in, \beta \rangle$.

Indeed, since $\langle L_{\omega_1^L}, \in \rangle \models$ V = HC (V = HC means "everything is countable") $\langle L_{\omega_1^L}, \in \rangle \models \exists g$ (g is a function) and (domain$(g) = \omega$) and (range$(g) = \beta$), the first such g (in Gödel's ordering) is definable, but then α_0 is definable as $g(n)$ for some n. So (3) is proved.

Let $A(v)$ be the definition of α_0 in $\langle L_{\omega_1^L}, \in, \beta \rangle$. Thus $\langle L_{\omega_1^L}, \in, \alpha_0 \rangle \models \exists ! v A(v)$. Let α_1 be the element defined by A in $\langle L_{\omega_1^L}, \in, \alpha_0 \rangle$. It is routine to verify that $\alpha_1 \in \alpha_0$ and $\langle L_{\omega_1^L}, \in, \alpha_0 \rangle \equiv \langle L_{\omega_1^L}, \in, \alpha_1 \rangle$. But this contradicts the choice of α_0. ■

Numerous beautiful results using V = HC can be found in Polsilver [9].

PROPOSITION 3.2. *If there exists* $O^{\#}$, *then* $\mathfrak{A}$ *has an* ω_1-RZM *function.*

Proof. This is an obvious consequence of

PROPOSITION 3.3. *Let* $\mathfrak{A}$ *be a model generated by a set* X *of order indiscernibles, with* $\mathrm{cf}(X) \geqslant \varkappa$. *Then* $\mathfrak{A}$ *has a* $\varkappa$-RZM *function.*

Proof. Fix any $y \in A^{<\varkappa}$, each y_α being of the form $y_\alpha = t_\alpha(x_1^\alpha, \ldots, x_n^\alpha)$, $x_j^\alpha \in X$. Let $f(y)$ be any $x \in X$ which exceeds all x_j^α. Obviously f is a $\varkappa$-RZM function for $\mathfrak{A}$. ■

Now we shall assume $\exists O^{\#}$, but consider only constructible models, since this situation can easily be described. Our remarks were suggested by

LEMMA 3.4 (Glöde [4]). *Assume* $\exists O^{\#}$. *Let* $\mathfrak{A} \in$ L *and let the similarity type of* $\mathfrak{A}$ *be countable in* L. *If* $\mathfrak{A}$ *is uncountable* (*in* V, *not only in* L!), *then* $\mathfrak{A}$ *has an uncountable set of order indiscernibles.*

This can be used to produce an elementary tower $\{\mathfrak{A}_\alpha\}$ of well-ordered models elementary equivalent to $\mathfrak{A}$ exactly as Silver [8] has done, but this does not give us additional information that we may require each $\mathfrak{A}_\alpha$ to be constructible. However, this is easily proved.

THEOREM 3.5. *Assume* $\exists O^{\#}$. *Let* $\mathfrak{A} = \langle A, <, \ldots \rangle$ *be a constructible well-ordered model for a constructibly countable language. Then*

(a) *if the order type of* $\mathfrak{A}$ *is uncountable* (*in* V!), *then there exists an elementary tower* $\{\mathfrak{A}_\alpha\}_{\alpha \in On}$ *of proper well-ordered elementary extensions of* $\mathfrak{A}$ *such that* $\forall \alpha \in On \ \mathfrak{A}_\alpha \in L$,

(b) *if the order type of* $\mathfrak{A}$ *is an uncountable cardinal, then we may require the order type of each* $\mathfrak{A}_\alpha$ *to be a cardinal and each cardinal greater than the order type of* $\mathfrak{A}$ *to be the order type of some* $\mathfrak{A}_\alpha$.

Proof. Since $\mathfrak{A} \in L$, we have $\mathfrak{A} = t(c_1, \ldots, c_n)$ for some term t of set theory and $\langle c_1, \ldots, c_n \rangle \in [C]^n$, where C denotes the class of ordinals generating L and indiscernible in L. Now fix any function $G\colon On \to [C]^n$ such that G is increasing in the following sense: $\alpha < \beta \Rightarrow$ any coordinate of $G(\alpha)$ is less than any coordinate of $G(\beta)$, and let $G(0) = \langle c_1, \ldots, c_n \rangle$. Put $\mathfrak{A}_\alpha := t(G(\alpha))$. One easily verifies that all the conditions are satisfied since they can be written as formulas of set theory and follow from the indiscernibility of C. So (a) is proved. If the order type of $\mathfrak{A}$ is a cardinal $\varkappa$, then $\mathfrak{A} = t(c_1, \ldots, c_{m-1}, \varkappa, c_{m+1}, \ldots, c_n)$. $\varkappa$ must occur, since otherwise it would be definable from other elements of C as the order type of $t(c_1, \ldots, c_n)$, which contradicts the indiscernibility of C.

Now (b) is proved exactly as (a) only need the additional assumption on G, namely that the mth coordinate of $G(\alpha)$ is a cardinal and each cardinal exceeding the order type of $\mathfrak{A}$ is $G(\alpha)$ for some α; the rest is the same. ■

References

[1] K. R. Apt, Doctoral Dissertation, Institute of Mathematics, Polish Academy of Sciences 1974.

[2] — and W. Marek *Second order arithmetic and related topics*, Ann. Math. Logic 6 (1974), pp. 177–229.

[3] C. C. Chang and H. J. Keisler, *Model Theory*, Amsterdam 1973.

[4] K. Glöde, *Ordinals with partition properties and constructible hierarchy*, Zeitschr. Math. Log. 18 (1972), pp. 135–164.

[5] H. Kotlarski, *On the existence of well ordered models*, Bull. Acad. Polon. Sci. 22 (5) (1974), pp. 459–462.

[6] — *On well ordered models* (Polish), Thesis, University of Wroclaw, 1976.

[6a] — *A note on countable well-ordered models*, Bull. Acad. Polon. Sci. (to appear).

[7] W. Marek, *Fraisse hypothesis*, preprint. French version of this paper has appeared in C. R. Acad. Sc. Paris 276 (1973), pp. 1169–1172.
[8] J. H. Silver, *Some applications of model theory in set theory*, Ann. Math. Logic 3 (1971), pp. 45–110.
[9] M. Srebrny, *β-models and constructible reals*, Doctoral Dissertation, Institute of Mathematics, Polish Academy of Sciences 1973. (Parts of this paper are published as W. Marek and M. Srebrny, *Gaps in constructible universe*, Ann. Math. Logic 6 (1974), pp. 359–394).

Accepté par la Rédaction le 17. 11. 1975

The axiom of choice for linearly ordered families

by

J. Truss (Oxford)

Abstract. We study the statement (1) Any linearly ordered family of non-empty sets has a choice function. (1) implies AC in ZF but not in ZF without foundation. We show that a weaker form of (1), namely "every family of non-empty sets indexed by $P(\omega)$ has a choice function", does not imply AC even in ZF; in fact it is consistent with the existence of a partition of $P(\omega)$ without a choice function. We study further properties of the model used to prove this, and also of Feferman's model.

§ 1. The axiom of choice for linearly ordered families is the following statement.

(1) Any linearly ordered family of non-empty sets has a choice function.

We prove in this paper that in the presence of the axiom of foundation, (1) implies AC, the axiom of choice. However this is false in set theory without the axiom of foundation. (1)+AC is therefore an example of what Pincus [8, pp. 740–741] calls a "non-transferable" consistency, i.e. it holds in an appropriate Fraenkel–Mostowski model (where the axiom of foundation may be violated) but not in any model of Zermelo–Fraenkel (ZF) set theory.

Our interest in this proposition was prompted by a question of A. Zalc. She asked whether (2) implies (3), where (2) and (3) are as follows.

(2) Every family of non-empty sets indexed by $P(\omega)$ has a choice function.

(3) Every partition of $P(\omega)$ into non-empty subsets has a choice function.

That the answer is "no" follows from consideration of one of the models $\mathfrak{N}_1$ of [12], of "Feferman type". We thought at first that it would be enough to consider Feferman's original model, $\mathfrak{N}$ [2]. However it turns out that both (2) and (3) are false there, the reason being connected with the fact that (1)→AC in ZF. (2) follows from (1), of course, but not conversely, as we shall show.

We include further information about $\mathfrak{N}$ (and similar results hold for the other models discussed in [12]). We show that for each ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, $2^{\aleph_0}$, and moreover that for any set X of $\mathfrak{N}$ which can be linearly ordered, there is an α such that $|X| \leqslant 2^{\aleph_\alpha}$. This is a "Kinna–Wagner ordering principle" for orderable sets. In fact the proof will show that this conclusion holds for any set such that

$$[X]^2 = \{x \subseteq X\colon |x| = 2\}$$