Dendroids and their endpoints

by

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Abstract. In this paper we introduce some non-negative real valued functions (which are of the 1st class of Baire) defined on dendroids. These functions can be used to characterize several properties of dendroids. We apply some of them to prove that in every dendroid X the minimal arcwise connected set spanning the set of all endpoints of X at which X is semi-locally connected is a dense subset of X. Moreover, the set of endpoints at which X is semi-locally connected is a G_δ-set in X and the remaining endpoints form a subset of the first category in X.

1. Introduction. All spaces under considerations are assumed to be metric. By a continuum we mean a compact connected space. As usual by a dendroid is understood an arcwise connected hereditarily unicoherent continuum (see [2]). A continuum X is said to be *semi-locally connected at a point* $x \in X$ provided that for every neighbourhood U of x there is a neighbourhood V of x contained in U whose complement consists of a finite number of components (see [8, p. 19]). By a neighbourhood we always mean an open set. We say that x is an *endpoint* of X if there is no arc in X containing x in its interior. We follow [7] in writing X^e for the set of endpoints of X. Denote by X^e_s the set of points of X^e at which X is semi-locally connected.

In this paper we study the class of dendroids. We introduce a collection of non-negative real valued functions α_{na} and α_n defined on dendroids. The functions α_n are of the 1st class of Baire (see [5]). In Section 3 these functions are used to characterize the following topological properties of dendroids: smoothness, semi-local connectedness, local connectedness and uniform arcwise connectedness. In the last two sections we apply these functions to prove some geometrical properties of dendroids. From now on by X is denoted an arbitrary dendroid (not a one-point set).

B. J. Fugate raised the question if the set X^e_s is not empty. Theorem 4.1 provides an affirmative answer to this questions. We show also that X^e_s is a G_δ subset of X (see 3.5). The set X^e_s can be a second category subset of X. In fact one easily checks that the dendroid X constructed in [7, p. 314] has this property, because the set X^e_s is dense in X. However the set $X^e \setminus X^e_s$ is always a first category subset of X (Theorem 5.1). Nevertheless, the example [7, p. 311] shows that $X^e \setminus X^e_s$ can be dense in X. The set X^e_s is always "large" in the sense that the minimal arcwise connected set spanning it is dense in X (Theorem 4.1). But the complement of the spanning set need not be of the first category in X (Example 4.7).

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2. On the functions α_n. An arc with endpoints a and b will be denoted by ab. The arc ab will be, in some cases, regarded as an ordered arc with a as the first and b as the last point. An order will be denoted by the symbol "$<$". Let ab be an arc in a space Y and let $U_1, U_2, \ldots$ be a sequence of subsets of Y. We say that the *ordered arc* $ab \subset Y$ *has type* $(U_1, U_2, \ldots)$, and write $ab \in (U_1, U_2, \ldots)$, if there exists a sequence of points $a_1, a_2, \ldots$ satisfying the conditions:

$$(*) \qquad a_n \in ab \cap U_n \quad \text{for each } n \geqslant 1 ,$$

$$(**) \qquad a < a_1 < a_2 < \ldots < b .$$

It is not assumed in the definition above that $U_1, U_2, \ldots$ is an infinite sequence nor that $U_i \neq U_j$ for $i \neq j$.

In the case where $U_1, U_2, \ldots$ is a finite sequence consisting of n terms we sometimes write $(U_1, U_2, \ldots)_n$ instead of $(U_1, U_2, \ldots)$. Similarly, if the sequence is infinite we sometimes write $(U_1, U_2, \ldots)_\infty$ instead of $(U_1, U_2, \ldots)$.

2.1. LEMMA. *Let $U_1, U_2, \ldots, U_n$ be a finite sequence of open sets in a hereditarily unicoherent continuum Y and let ab be an arc in Y of type $(U_1, U_2, \ldots, U_n)$. Then there are neighbourhoods U of a and V of b such that any arc in Y joining U and V has type $(U_1, \ldots, U_n)$.*

Proof. It is easy to check that the lemma holds for each arc of type (U), where U is open in Y. Assume we have proved the lemma for all arcs of type $(U_1, \ldots, U_{n-1})$, where $n \geqslant 2$. From the definition it follows that there exists a point $a_{n-1} \in ab \cap (U_{n-1} \setminus \{a, b\})$ such that the arc $aa_{n-1} \subset ab$ has type $(U_1, \ldots, U_{n-1})$ and the arc $a_{n-1}b$ has type (U_n). From the assumption it follows that there exist open sets $U' \ni a$, $G \ni a_{n-1}$ and $V' \ni b$ such that each arc joining U' and G has type $(U_1, \ldots, U_{n-1})$ and each arc joining G and V' is of type (U_n). Let $U \ni a$ and $V \ni b$ be such open sets that each arc joining them has type (G). Clearly, we may assume that $U \subset U'$ and $V \subset V'$. Now it is easily seen that each arc joining U and V is of type $(U_1, \ldots, U_n)$. This completes the proof.

2.2. LEMMA. *Let $U_1, U_2, \ldots$ be an infinite sequence of open sets in a space Y. If $ab \subset Y$ is an arc of type $(U_1, U_2, \ldots)$, then $\lim_n d(U_n, U_{n+1}) = 0$ (for two sets $A, B \subset Y$ we define $d(A, B) = \inf\{\varrho(x, y)\colon x \in A, y \in B\}$).*

Proof. Let $a_n \in U_n \cap ab$, $n = 1, 2, \ldots$, satisfy the relations:

$$a < a_1 < a_2 < \ldots < b .$$

Observe that $\{a_n\}$ is a convergent sequence. Hence the conclusion follows.

2.3. LEMMA. *Let $U_1, U_2, \ldots$ be an infinite sequence of open sets in a space Y such that $\overline{U}_n \cap \overline{U}_{n+1} = \emptyset$ for each $n \geqslant 1$. If ab is an arc in Y of type $(U_1, U_2, \ldots, U_n)$ for each $n \geqslant 1$, then ab is of type $(U_1, U_2, \ldots)_\infty$.*

Proof. Define a sequence $a_1', a_2', \ldots$ in the following way (recursively):

$$a_1' = \inf\{x \in ab\colon x \in U_1\} ,$$
$$a_{n+1}' = \inf\{x \in a_n'b\colon x \in U_{n+1}\}$$

(the infimum is taken with respect to the order on ab). Notice that the construction is possible. Observe that $a_n' \in \overline{U_n \cap a_n'b}$ for each $n \geqslant 1$. The points a_n' lie on the arc ab in the order:

$$a \leqslant a_1' < a_2' < \ldots < b$$

and $a_n'a_{n+1}' \subset ab$ contains some point $a_n \in U_n$ in its interior for each $n \geqslant 1$. The points $a_1, a_2, \ldots$ satisfy conditions $(*)$ and $(**)$, which completes the proof.

Recall that by X we always denote an arbitrary dendroid. For any two points $x, y \in X$ by the arc xy is meant the unique arc in X joining x and y provided $x \neq y$, and the one-point set $\{x\} = \{y\}$ in the case $x = y$.

Fix a point $a \in X$ and a natural number $n \geqslant 1$. For an arbitrary point $x \in X$ consider all the points $y \in ax$ such that for every neighbourhood U of x and every neighbourhood V of y there is an arc $az \in (V, U, V, U, \ldots)_{n+2}$. Observe that the points y constitute a subarc of ax with one endpoint at x. Denote this arc by $A_n(x, a)$ and define

$$\alpha_{na}(x) = \inf\{\varepsilon > 0\colon A_n(x, a) \subset K(x, \varepsilon)\} ,$$

where $K(x, \varepsilon)$ denotes the ε-ball around x. Let us note that $A_n(x, a) \supset A_m(x, a)$ for $n \leqslant m$. It follows that the intersection $A_\omega(x, a) = \bigcap_{n=1}^{\infty} A_n(x, a)$ is again an arc. Let us define analogously

$$\alpha_{\omega a}(x) = \inf\{\varepsilon > 0\colon A_\omega(x, a) \subset K(x, \varepsilon)\} .$$

2.4. PROPOSITION. *For each $a, x \in X$ and for natural numbers $n \leqslant m$ we have $\alpha_{na}(x) \geqslant \alpha_{ma}(x)$, and $\alpha_{\omega a}(x) = \inf\{\alpha_{na}(x)\}$.*

In the definitions below let us agree to denote by n a natural number or ω. The above formulas define a nonnegative real-valued functions

$$\alpha_{na}\colon X \to R .$$

Define also a function

$$\alpha_n\colon X \to R$$

obtained from the preceding ones as follows

$$\alpha_n(x) = \sup\{\alpha_{na}(x)\colon a \in X\} .$$

2.5. PROPOSITION. *For each $x \in X$ and for natural numbers $n \leqslant m$ we have $\alpha_n(x) \geqslant \alpha_m(x)$ and $\alpha_\omega(x) \leqslant \inf_n \alpha_n(x)$.*

We shall show later that the last inequality can be replaced by an equality.

2.6. LEMMA. *Let n be a natural number and let $x_1, x_2, \ldots$ be a sequence in X converging to a point x. If $a_1, a_2, \ldots$ is another sequence such that*

$$\alpha_{na_k}(x_k) \underset{n\to\infty}{\longrightarrow} r$$

for some $r \in R$, then there exists a point $a \in X$ such that $\alpha_{na}(x) \geqslant r$.

Proof. For each $k \geqslant 1$ there is a point $\tilde{x}_k \in A_n(x_k, a_k)$ such that $\varrho(\tilde{x}_k, x_k) = \alpha_{na_k}(x_k)$. We may assume that the sequence $\{\tilde{x}_k\}$ is convergent. Denote its limit by a and note that $\varrho(a, x) = r$. Take a neighbourhood U of x and a neighbourhood V of a. To complete the proof it suffices to construct an arc aq such that

$$aq \in (V, U, V, \ldots)_{n+2}. \tag{1}$$

We may assume that $a \neq x$. Then for some index k there exist a neighbourhood U_0 of x_k and a neighbourhood V_0 of $\tilde{x}_k$ such that:

$$\overline{U}_0 \subset U, \quad \overline{V}_0 \subset V, \tag{2}$$

$$\overline{U}_0 \cap (\overline{V}_0 \cup a_k\tilde{x}_k) = \emptyset. \tag{3}$$

Since $\tilde{x}_k \in A_n(x_k, a_k)$, there is an arc $a_k p \in (V_0, U_0, V_0 \ldots)_{n+2}$. Hence there exist $n+2$ points $v_1^0, u_1^0, v_2^0, \ldots$ such that

$$v_i^0 \in V_0 \cap a_k p, \quad u_j^0 \in U_0 \cap a_k p, \tag{4}$$

$$a_k < v_1^0 < u_1^0 < v_1^0 < \ldots < p. \tag{5}$$

There is a point b such that

$$ab \cap (a_k\tilde{x}_k \cup a_k p) = \{b\}. \tag{6}$$

Denote by M the union of all components of the set $a_k p \setminus U_0$ which intersect V_0 and do not contain a_k and define

$$c = \inf\{y \in a_k p \colon y \in M\}.$$

One easily checks that

$$c \in \overline{U_0 \cap ca_k}, \tag{7}$$

(8) $v', v'' \in V_0 \cap a_k p \wedge u \in U_0 \cap a_k p \wedge a_k < v' < u < v'' < p \Rightarrow v'' > c$ (with respect to the order on $a_k p$).

(9) the component of $X \setminus U_0$ containing a_k does not intersect pc.

We may assume that

$$a_k < c < b \leqslant p. \tag{10}$$

In fact, otherwise by (6) we have $b \in a_k\tilde{x}_k \cup a_k c$. By (3), (6) and (9) the point c cuts the continuum $a_k\tilde{x}_k \cup a_k p \cup ab$ between a and p. Hence $ap = ac \cup cp$. By (7) and (2) we have $ac \in (V, U)$. This fact together with (5) and (8) imply that ap satisfies (1), where q is replaced by p. Hence in the sequel we assume that (10) is fulfilled.

Conditions (10), (9), (7), (6) and (3) imply that the set $U_0 \setminus (ab \cup pc)$ separates X between $ab \cup pc$ and $\tilde{x}_k$. It follows that there exists a neighbourhood $V_1 \subset V_0$ of $\tilde{x}_k$ such that every continuum joining $ab \cup pc$ with V_1 meets U_0. Since $\tilde{x}_k \in A_n(x_k, a_k)$, there is an arc $a_k q$ and a sequence $v_1, u_1, v_2, \ldots$ of $n+2$ points such that

$$v_i \in V_1 \cap a_k q, \quad u_j \in U_0 \cap a_k q, \tag{11}$$

$$a_k < v_1 < u_1 < v_2 < \ldots < q. \tag{12}$$

We shall show that aq satisfies condition (1). Denote by d the last point on $a_k p$ belonging to $a_k q$. First assume $a_k \leqslant d \leqslant c$. Then

$$aq = ab \cup bd \cup dq.$$

By the assumption, (8), (11) and (12) we have $d < v_2 < \ldots < q$. Since $ab \in (V)$ and $bd \in (U)$, conditions (11), (12) and (2) imply (1). Next assume $c < d \leqslant p$. Let e be the first point on ad belonging to dq. Then $aq = ae \cup eq$. We claim that $e < v_2 < q$. Otherwise $v_2 \in ed \cup da_k$. But $d, e \in ad \subset ab \cup bd \subset ab \cup cp$ and the last set is disjoint from V_1. Hence (11) implies $v_2 \in a_k d = a_k c \cup cd$. It follows that $v_2 \in a_k c$ because $cd \subset cp$. However this is impossible by (8) and (12). Now, $e \in ab \cup cp$ and $v_2 \in V_1$, hence $ev_2 \in (U_0)$, by the construction of V_1. Finally, since $ae \in (V)$ and $ae \cap eq = \{e\}$, condition (1) follows from (12). This completes the proof of 2.6.

2.7. LEMMA. *Let $x_1, x_2, \ldots$ be a sequence in X converging to a point $x \in X$. Let $n_1, n_2, \ldots$ be either a strictly increasing sequence of natural numbers or a constant sequence $n_k = \omega$. If $a_1, a_2, \ldots$ is another sequence in X such that*

$$\alpha_{n_k a_k}(x_k) \underset{k\to\infty}{\longrightarrow} r$$

for some $r \in R$, then there exists a point $a \in X$ such that $\alpha_{\omega a}(x) \geqslant r$.

Proof. The proof is like the proof of the preceding lemma, but simpler. For each $k \geqslant 1$ there is a point $\tilde{x}_k \in A_{n_k}(x_k, a_k)$ such that $\varrho(\tilde{x}_k, x_k) = \alpha_{n_k a_k}(x_k)$. We may assume that $\{\tilde{x}_k\}$ converges to some point $a \in X$. Note that $\varrho(a, x) = r$. Take a neighbourhood U of x, a neighbourhood V of a, and a natural number m. To complete the proof it suffices to show that there is an arc $az \in (V, U, V, \ldots)_m$. Take k so large that $n_k > 2m$, $x_k \in U$ and $\tilde{x}_k \in V$. There is an arc $a_k b \in (V, U, \ldots, V, U)_{2m}$. It is easy to see that for $z = a_k$ or $z = b$ the arc az has the required properties. This completes the proof.

2.8. COROLLARY. *For each $x \in X$ and for each $n \in \{1, 2, \ldots, \omega\}$ there exists a point $a \in X$ such that*

$$\alpha_n(x) = \alpha_{na}(x).$$

Proof. Let $a_1, a_2, \ldots$ be a sequence such that

$$\alpha_{na_k}(x) \to \alpha_n(x).$$

By 2.6 or 2.1 there is a point $a \in X$ such that $\alpha_{na}(x) \geqslant \alpha_n(x)$. Hence the conclusion follows from the definition of α_n.

2.9. COROLLARY. *For each $x \in X$ we have*

$$\alpha_\omega(x) = \inf_n \alpha_n(x) .$$

Proof. By 2.5 we have $\alpha_\omega(x) \leqslant \inf_n \alpha_n(x)$. Let $r = \inf_n \alpha_n(x)$. Again by 2.5 the sequence $\alpha_1(x), \alpha_2(x), \ldots$ converges to r. By 2.8 for each $k \geqslant 1$ there is a point $a_k \in X$ such that $\alpha_k(x) = \alpha_{ka_k}(x)$. Hence

$$\alpha_{ka_k}(x) \underset{k\to\infty}{\longrightarrow} r .$$

Using 2.7 we get a point $a \in X$ such that $\alpha_{\omega a}(x) \geqslant r$. It follows that

$$\alpha_\omega(x) \geqslant \alpha_{\omega a}(x) \geqslant r = \inf_n \alpha_n(x) ,$$

which completes the proof.

2.10. THEOREM. *For each $n \in \{1, 2, \ldots, \omega\}$ the function $\alpha_n \colon X \to R$ is upper semi-continuous, i.e. for each $r \in R$ the set $\alpha_n^{-1}([r, \infty))$ is closed in X.*

Proof. Let $x_1, x_2, \ldots$ be a convergent sequence of points from $\alpha_n^{-1}([r, \infty))$ and let x denote its limit. We have to prove that $\alpha_n(x) \geqslant r$. For each $k \geqslant 1$ by 2.8 there is a point a_k such that $\alpha_n(x_k) = \alpha_{na_k}(x_k)$. Without loss of generality we may assume that the sequence $\{\alpha_{na_k}(x_k)\}$ converges. Denote its limit by s. Clearly, $s \geqslant r$. By 2.6 or 2.7 there is a point $a \in X$ such that $\alpha_{na}(x) \geqslant s$. Hence the theorem is proved because $\alpha_n(x) \geqslant \alpha_{na}(x)$.

Remark. Observe that if $\alpha_{\omega a}$ vanishes on X for some $a \in X$, then so does α_ω.

2.11. COROLLARY. *In every closed subset of X the function α_n attains its least upper bound, for $n \in \{1, \ldots, \omega\}$.*

Recall that a function $f\colon Y \to Z$ is of the 1st class of Baire if for every open subsubset U of Z the inverse image $f^{-1}(U)$ is an F_σ-set in Y (comp. [4, p. 373]).

In the next theorem we show that each function α_n, $n = 1, 2, \ldots, \omega$, is of the 1st class of Baire.

2.12. THEOREM. *For each $n \in \{1, 2, \ldots, \omega\}$ and for every open subset U of R the inverse image $\alpha_n^{-1}(U)$ is an F_σ-set in X. Consequently, the set of points of X at which α_n is not continuous is an F_σ-set of the first category in X.*

Proof. The set U can be represented as a union of open intervals, $U = \bigcup_{k=1}^{\infty} (r_k, s_k)$, where $r_k < s_k$. Then

$$\alpha_n^{-1}(U) = \bigcup_{k=1}^{\infty} \alpha_n^{-1}((r_k, s_k)) .$$

To complete the proof of the first part of 2.12 it suffices to show that the inverse image of an open interval (r, s), $r < s$, is an F_σ-set. But

$$\alpha_n^{-1}((r, s)) = X \setminus \big(\alpha_n^{-1}((-\infty, r]) \cup \alpha_n^{-1}([s, \infty))\big) .$$

Now,

$$\alpha_n^{-1}((-\infty, r]) = X \setminus \bigcup_{k=1}^{\infty} \alpha_n^{-1}([r+1/k, \infty))$$

and $\alpha_n^{-1}([s, \infty))$ are G_δ-sets by 2.10. Hence $\alpha_n^{-1}((r, s))$ is an F_σ-set, as required.

Let M denote the set of points at which α_n is not continuous. Let $U_1, U_2, \ldots$ be a base for open sets in R. It is easily seen that

$$M = \bigcup_{k=1}^{\infty} \alpha_n^{-1}(U_k) \setminus \operatorname{Int} \alpha_n^{-1}(U_k) .$$

By the first part of 2.12 we obtain the conclusion.

3. Some properties of dendroids characterized by means of the functions α_n. In this section we express some topological properties of dendroids using the functions α_{na} and α_n. Among them are: smoothness, local connectedness, semi-local connectedness and uniform arcwise connectedness.

Recall that a dendroid X is said to be *smooth with respect to a point $a \in X$* if for each $x \in X$ and for every sequence $\{x_n\}$, $x_n \in X$, coverging to x, the sequence of arcs $\{ax_n\}$ converges to ax in the Hausdorff metric dist (.,.) (see [6, p. 47] for the definition of dist (.,.), and [3] for the definition of smoothness). The dendroid is smooth if it is smooth with respect to some point.

The following theorem characterizes smooth dendroids.

3.1. THEOREM. *A dendroid X is smooth with respect to $a \in X$ if and only if the function α_{1a} vanishes on X.*

Proof. Assume X is smooth with respect to $a \in X$, and suppose $\alpha_{1a}(z) > 0$ for some $z \in X$. Take a point $x \in A_1(z, a) \setminus \{z\}$. Hence $z \notin ax$. By the definition of α_{1a} it is easy to construct a sequence of arcs $ax_1, ax_2, \ldots$ such that $\{x_n\}$ converges to x and $d(\{z\}, ax_n) < 1/n$ for each $n \geqslant 1$. Hence if the sequence $\{ax_n\}$ converges to some continuum C, then $z \in C$. By our assumption $\{ax_n\}$ converges to ax, hence $z \in ax$, a contradiction.

Next, assume α_{1a} vanishes on X and suppose X is not smooth with respect to a. Hence there is a point $x \in X$ and a sequence $\{x_n\}$ converging to x such that the sequence of arcs $\{ax_n\}$ does not converge to ax. Clearly, we may assume that $\{ax_n\}$ converges to some continuum $D \subset X$. Since $a, x \in D$, we have $ax \subset D$. Hence there is a point $y \in D \setminus ax$. Let z be a point such that $yz \cap ax = \{z\}$. Clearly, $z \neq y$ and $ay = az \cup zy$. To complete the proof it suffices to show that

$$z \in A_1(y, a) . \tag{1}$$

(in fact, since $z \neq y$ this will imply $\alpha_{1a}(y) \geqslant \varrho(y, z) > 0$, contrary to our assumption).

So, let U be a neighbourhood of y, and let V be a neighbourhood of z. Since $ay \in (V)$, by 2.1 there is a neighbourhood $U_1 \subset U$ of y such that for each $p \in U_1$ the arc $ap \in (V)$. Observe that $yx = yz \cup zx$, hence $yx \in (V)$. By the same argument as above, there is a neighbourhood $U_2 \subset U_1$ of y, and a neighbourhood G of x, such that every arc joining U_2 and G has type (V). Since $y \in D$, $y \in U_2$ and D is the topological limit of ax_n's (see [6, § 43, II]), U_2 intersects all ax_n's but a finite number of them. Since $\{x_n\} \to x$ and G is a neighbourhood of x, by the above remark there is an index m such that $x_m \in G$ and $ax_m \cap U_2 \neq \emptyset$. Let $p \in ax_m \cap U_2$. By the above constructions we have:

$$ap \in (V), \quad p \in U \quad \text{and} \quad px_m \in (V).$$

It follows that $ax_m \in (V, U, V)$. Hence for an arbitrary neighbourhood U of y and for an arbitrary neighbourhood V of z, there is an arc $aq \in (V, U, V)$. This proves (1), and completes the proof.

3.2. COROLLARY. *A dendroid X is smooth if and only if there exists a point $a \in X$ such that the function α_{1a} vanishes on X.*

The following theorem characterizes the points at which X is semi-locally connected.

3.3. THEOREM. *A dendroid X is semi-locally connected at $x \in X$ if and only if $\alpha_1(x) = 0$.*

Proof. Assume X is semi-locally connected at x and suppose $\alpha_{1a}(x) > 0$ for some $a \in X$. Take a point $\tilde{x} \in A_1(x, a) \setminus \{x\}$. Let G be a neighbourhood of x such that $a\tilde{x} \cap \overline{G} = \emptyset$. There is a neighbourhood $U \subset G$ of x such that $X \setminus U$ has finitely many components. Let C_U denote the component of $X \setminus U$ which contains a. Put $V = \operatorname{Int} C_U$. Clearly, $\tilde{x} \in V$. By our supposition there is an arc $az \in (V, U, V)$ with $z \in V$. This is impossible because $az \subset C_U \subset X \setminus U$. Now, assume $\alpha_1(x) = 0$ and suppose X is not semi-locally connected at x. Hence there is a neighbourhood G of x such that for every neighbourhood $U \subset G$ of x the set $X \setminus U$ has infinitely many components. It is easily seen that there exists a point $a \notin G$ such that $a \notin \operatorname{Int} C_U$, where U is any neighbourhood of x contained in G and C_U is the component of $X \setminus U$ containing a. Let V be a neighbourhood of a and let U be as above. Take $z \in V \setminus C_U$. Clearly, $az \in (V, U, V)$. It follows that $\alpha_1(x) \geqslant \alpha_{1a}(x) \geqslant \varrho(a, x) > 0$, a contradiction.

By a dendrite we mean a locally connected dendroid. Theorem 3.3 implies the following.

3.4. THEOREM. *A dendroid X is a dendrite if and only if the function α_1 vanishes on X.*

Proof. Assume X is a dendrite. Pick a point $x \in X$. To prove that $\alpha_1(x) = 0$ it suffices by 3.3 to show that X is semi-locally connected at x. But this follows from [8, (13.21), p. 20].

Now, assume α_1 vanishes on X. Let $x \in X$ and let U be a neighbourhood of x. To complete the proof it remains to show that there is a connected set $D \subset U$ such that $x \in \operatorname{Int} D$. By 3.3 the dendroid X is semi-locally connected at each of its points. Since $X \setminus U$ is compact, it follows that there is a finite number of open sets $V_1, \ldots, V_m$ such that $x \notin \bigcup_{j=1}^{m} \overline{V}_j$, $X \setminus U \subset \bigcup_{j=1}^{m} V_j$ and $X \setminus V_j$ consists of a finite number of components, for each $j = 1, \ldots, m$. Let D_j denote the component of $X \setminus V_j$ containing x. Clearly, $x \in \operatorname{Int} D_j$ and $D = \bigcap_{j=1}^{m} D_j \subset U$. It follows that $x \in \operatorname{Int} D$. Since X is hereditarily unicoherent, D is a continuum, which completes the proof.

We say that continuum Y is *colocally connected at a point* $y \in Y$ provided that for every neighbourhood U of y in Y there is a neighbourhood $V \subset U$ of y such that $Y \setminus V$ is connected.

3.5. THEOREM. *The set X_s^e is the subset of X consisting of all points at which X is colocally connected. Moreover, it is a G_δ-subset of X. For each $a \in X$ we have*

$$X_s^e \cup \{a\} = [X^e \cap \alpha_{1a}^{-1}(0)] \cup \{a\}.$$

Proof. Let $x \in X_s^e$. Pick an arbitrary neighbourhood G of x. To prove that X is colocally connected at x it suffices to show that there is a neighbourhood $V \subset G$ of x such that $X \setminus G$ is contained in some component of $X \setminus V$. Suppose it is not true. Pick a point $a \notin G$. There is a decreasing sequence $V_1 \supset V_2 \supset \ldots$ of neighbourhoods of x with diameters converging to 0 such that $\overline{V}_1 \subset G$ and no component of $X \setminus V_n$ contains $X \setminus G$, for each $n \geqslant 1$. Denote by C_n the component of $X \setminus V_n$ which contains a. By the supposition there is a component D_n of $X \setminus G$ disjoint from C_n. Let ab_n be an arc irreducible between a and D_n. Hence $ab_n \cap V_n \neq \emptyset$ and $b_n \notin G$. Let $b_n c_n \subset ab_n$ be an arc irreducible between b_n and $\overline{V}_n$. We can assume that $\{b_n c_n\}$ converges to a continuum A. Thus A contains x and intersects $X \setminus G$. We claim that $A \cap ax = \{x\}$. Otherwise there is a point $\tilde{x} \in A \cap ax \setminus \{x\}$. Consider arbitrary neighbourhoods U of x and V of $\tilde{x}$. Since $ax \in (V, U)$, $x\tilde{x} \in (U, V)$ and $c_n \to x$, there is an index n such that $ab_n \in (V, U, V)$. It follows that $\alpha_1(x) > 0$, contrary to 3.3. Now it is obvious that x is an interior point of some arc. This contradicts our assumption.

One can similarily prove the last assertion of the theorem.

Now, assume X is colocally connected at x. Clearly, $x \in X^e$ and X is semi-locally connected at x. By the definition $x \in X_s^e$.

Let $\mathcal{U}_n$ be the collection of all open subsets of X such that $U \in \mathcal{U}_n$ if and only if $\operatorname{diam} U < 1/n$ and $X \setminus U$ is connected. Denote by $\mathcal{U}_n^*$ the union of the sets from $\mathcal{U}_n$. Clearly,

$$X_s^e = \bigcap_n \mathcal{U}_n^*$$

which follows from the first part of 3.5. This ends the proof.

We finish this section showing some relationship between the function α_ω and the notion of uniform arcwise connectivity introduced in [2]. Recall that a dendroid X is called *uniformly arcwise connected*, briefly: u.a.c., provided that for each $\varepsilon > 0$

there is a natural number n such that for every arc $ab \subset X$, $a \neq b$, there exist $n+1$ points $a_1, a_2, \ldots, a_{n+1}$ $(a_i \in ab)$ such that

(i) $$a = a_1 < a_2 < \ldots < a_{n+1} = b\,,$$

(ii) $$\operatorname{diam} a_j a_{j+1} < \varepsilon \quad \text{for each } j = 1, \ldots, n\,.$$

3.6. Theorem. *A dendroid X is uniformly arcwise connected if and only if the function α_ω vanishes on X.*

Proof. The neccessity is obvious. Now we prove the sufficiency part of the theorem.

Suppose, to the contrary, that X is not u.a.c. Hence there is an $\varepsilon > 0$ and a sequence of arcs $L_1, L_2, \ldots$ in X such that for each n there exist $n+1$ points $x_n(1), x_n(2), \ldots, x_n(n+1)$ on L_n having the properties:

(1) $$L_n \supseteq x_n(1)x_n(n+1)\,,$$

(2) $x_n(1) < x_n(2) < \ldots < x_n(n+1)$ (with respect to an order on L_n),

(3) $$\varrho\big(x_n(j), x_n(j+1)\big) \geqslant \varepsilon \quad \text{for each } j = 1, \ldots, n\,.$$

Let N denote the set of natural numbers (0 is not considered as a natural number).

Since X is a compact metric space there exist a sequence of points $x(1), x(2), \ldots, x(j) \in X$, and a sequence of strictly increasing functions $p_1, p_2, \ldots, p_j$: $N \to N$, such that

(4) $$p_{j+1}(N) \subset p_j(N) \quad \text{for} \quad j \in N\,,$$

(5) $$\lim_n x_{p_j(n)}(j) = x(j) \quad \text{for} \quad j \in N\,.$$

Clearly, $\lim_n x_{p_j(n)}(i) = x(i)$ for $i \leqslant j$. Observe that the sequence $p_1(1), p_2(2), \ldots$ is strictly increasing, and for $j \geqslant i$ we have

$$p_j(j) \in p_i(N)\,,$$

which follows from (4). Now it is a consequence of (5) that

(6) $$\lim_j x_{p_j(j)}(i) = x(i)$$

for each $i \in N$ (it can happen that some of the symbols $x_{p_j(j)}(i)$ are not defined, but this can only happen for a finite number of j's). Without loss of generality we may assume that $L_{p_1(1)}, L_{p_2(2)}, \ldots$ is our original sequence $L_1, L_2, \ldots$ Hence condition (6) changes to

(7) $$\lim_n x_n(i) = x(i) \quad \text{for each } i \in N\,.$$

Observe also that conditions (1)-(3) are still valid. Conditions (3) and (7) imply

(8) $$\varrho\big(x(i+1), x(i)\big) \geqslant \varepsilon \quad \text{for each } i\,.$$

It follows from the compactness of X that there exist a strictly increasing function q: $N \to N$ and two points $a, x \in X$ such that

$$\lim_n x\big(q(n)\big) = a \quad \text{and} \quad \lim_n x\big(q(n)+1\big) = x\,.$$

Condition (8) implies that

$$\varrho(a, x) \geqslant \varepsilon\,.$$

Now we show that $\alpha_\omega(x) > 0$.

Let U be a neighbourhood of x and let V be a neighbourhood of a. Take an arbitrary natural number k. There exists an index m such that

$$x\big(q(n)\big) \in V \quad \text{and} \quad x\big(q(n)+1\big) \in U$$

for each $n \geqslant m$. Using (7) one can find an index r such that the symbols below make sense and satisfy the conditions:

$$x_r\big(q(m+j)\big) \in V \quad \text{for} \quad 1 \leqslant j \leqslant k$$

and

$$x_r\big(q(m+j)+1\big) \in U \quad \text{for} \quad 1 \leqslant j \leqslant k\,.$$

Let us note that the following inequalities hold true (with respect to the order on $x_r(1)x_r(r)$)

$$x_r(1) \leqslant x_r\big(q(m+1)\big) < x_r\big(q(m+1)+1\big) < x_r\big(q(m+2)\big) < \ldots < x_r\big(q(m+k)+1\big) \leqslant x_r(r)\,.$$

Let ab be an arc irreducible between a and $x_r(1)x_r(r)$. It is easily seen that either $ax_r(1)$ or $ax_r(r)$ is an arc of type $(V, U, V, U, \ldots)_k$.

This proves that $\alpha_\omega(x) \geqslant \alpha_{\omega a}(x) \geqslant \varrho(a, x) \geqslant \varepsilon$, and the proof is completed.

In his paper [4] W. Kuperberg proved that a dendroid X can be represented as a continuous image of the Cantor fan if and only if X is u.a.c. Combining this result with 3.6 we get the following.

3.7. Corollary. *A dendroid X is a continuous image of the Cantor fan if and only if the function α_ω vanishes on X.*

4. Nonemptness of the set X_s^e. The aim of this section is to prove the following.

4.1. Theorem. *The minimal arcwise connected set spanning the set X_s^e of all endpoints of a dendroid X at which X is semi-locally connected is a dense subset of X.*

Clearly this implies that $X_s^e \neq \varnothing$, and answers the mentioned question of B. J. Fugate.

This theorem will follow from several lemmas which we are now going to state and prove. In the lemmas we use the following fixed notations.

Fix a point a in the dendroid X. For any open subset U of X which does not contain a denote by C_U the component of $X\setminus U$ containing a, and let

$$D_U = X\setminus(U \cup C_U).$$

Denote

$$P(U) = \{x \in X\colon \text{there is a neighbourhood } V \not\ni a \text{ of } x \text{ such that } C_U \subset \operatorname{Int} C_V\}.$$

4.2. Lemma. *Let U be an open set not containing a and let $x \notin P(U) \cup \overline{U} \cup C_U$. Then for every neighbourhood V of x there exists an arc $ay \in (U, V, U)$.*

Proof. Take a neighbourhood G of x such that $G \subset V$ and $\overline{G} \cap (\overline{U} \cup C_U) = \emptyset$. Since C_U is a component of the compact space $X\setminus U$, hence U separates X between C_U and $\overline{G}$. There are disjoint closed subsets A and B in X such that

$$X\setminus U = A \cup B, \quad C_U \subset A \quad \text{and} \quad \overline{G} \subset B.$$

Since $x \notin P(U)$, we have $\overline{D}_G \cap C_U \neq \emptyset$. There is a point $y \in D_G \cap (A \cup U)$ because $A \cup U$ is open in X. One easily checks that ay has the required properties.

4.3. Lemma. *Let U be an open set in the dendroid X such that $a \notin U$ and $x \in P(U)$ for some $x \in X$. If $x \in ay$, then $y \in P(U)$.*

Proof. Let V be a neighbourhood of x such that $C_U \subset \operatorname{Int} C_V$. Clearly, $y \notin C_V$. Let W be a neighbourhood of y contained in $X\setminus C_V$. Then $C_V \subset C_W$ and therefore $C_U \subset \operatorname{Int} C_W$, which completes the proof.

4.4. Lemma. *Let U and V be disjoint open subsets of $X\setminus\{a\}$ and let $G_1, \ldots, G_n$ be arbitrary open subsets of X. Assume that*

(1) $$P(U) \cap \{x \in X\colon ax \in (V, U, V)\} = \emptyset,$$

(2) *there is a nonempty open set $V_1 \subset V$ such that every arc joining a and V_1 is both of the type (V, U, V) and $(G_1, \ldots, G_n)$.*

Then there is an open nonempty set $V_2 \subset \overline{V}_2 \subset V_1$ such that every arc joining a and V_2 is of the type $(G_1, \ldots, G_n, U, V)$.

Proof. Let $x_1 \in V_1$. It follows that $x_1 \notin P(U)$ because $ax_1 \in (V, U, V)$. Also $x_1 \notin \overline{U} \cup C_U$. From 4.2 we infer that there is a point $b \in U$ such that $ab \in (U, V_1, U)$. Clearly, $ab \in (G_1, \ldots, G_n, U)$ and $b \notin P(U)$. It follows that $b \notin P(V_1)$. Also it is clear that $b \notin \overline{V}_1 \cup C_{V_1}$. By 2.1 there is a neighbourhood $W \subset U$ of b such that every arc joining a and W is of the type $(G_1, \ldots, G_n, U)$. By 4.2 there is a point $x_2 \in V_1$ such that $ax_2 \in (V_1, W, V_1)$. Again by 2.1 there is a neighbourhood $V_2 \subset \overline{V}_2 \subset V_1$ of x_2 such that every arc joining a and V_2 is of the type (V_1, W, V_1). Let c be an arbitrary point of V_2. We shall show that $ac \in (G_1, \ldots, G_n, U, V)$.

There exist points $p \in ac \cap V_1$, $q \in ac \cap W$ and $r \in ac \cap V_1$ such that $a<p<q<r<c$. Clearly, $ap \in (G_1, \ldots, G_n)$, $pq \in (U)$ and $qc \in (V)$. This completes the proof.

4.5. Lemma. *Let $x_0, \tilde{x}_0 \in X$ be such that $\tilde{x}_0 \in A_1(x_0, a)$ (comp. the definition of α_{1a}). Let U, V be neighbourhoods of respectively x_0 and $\tilde{x}_0$, such that $\overline{U} \cap \overline{V} = \emptyset$. Then $P(U) \cap \{x \in X\colon ax \in (V, U, V)\} \neq \emptyset$.*

Proof. Suppose $P(U) \cap \{x \in X\colon ax \in (V, U, V)\} = \emptyset$. By the assumption there is a point $x \in V$ such that $ax \in (V, U, V)$. From 2.1 we infer that there is a neighbourhood $V_1 \subset \overline{V}_1 \subset V$ of x such that every arc joining a and V_1 is of the type (V, U, V). By 4.4 there is a nonempty open set $V_2 \subset \overline{V}_2 \subset V_1$ such that every arc joining a and V_2 is of the type (V, U, V, U, V). In particular, any such arc is of the type (V, U, V). Again using 4.4 we can construct a nonempty open set $V_3 \subset \overline{V}_3 \subset V_2$ such that every arc joining a and V_3 is of the type (V, U, V, U, V, U, V).

Repeating the argument we can construct a sequence of nonempty open sets $V_1, V_2, \ldots$ such that $\overline{V}_{n+1} \subset V_n$ for each $n \geqslant 1$ and every arc joining a and V_n is of the type $(V, U, V, U, \ldots, V)_{2n+1}$. The intersection $\bigcap_n V_n$ is a nonempty set and every arc joining a with that intersection is of the type $(V, U, V, \ldots)_\infty$ by Lemma 2.3. Hence 2.2 implies $\overline{U} \cap \overline{V} \neq \emptyset$, contrary to our assumption.

4.6. Lemma. *For every nonempty open set G in X there is a point $x^* \in X_s^e$ such that $ax^* \in (G)$.*

Proof. Let $E_0 = \{x \in X^e\setminus\{a\}\colon ax \in (G)\}$. We may assume $E_0 \neq \emptyset$, for otherwise X reduces to a one-point set according to a Borsuk's lemma [1] that every arc in a dendroid is a subarc of a maximal arc. By the last assertion of 3.5 we may assume that $\alpha_{1a}(x) > 0$ for each $x \in E_0$. Choose $x_0 \in E_0$ such that $\alpha_{1a}(x_0) > \frac{1}{2}\sup\{\alpha_{1a}(x)\colon x \in E_0\}$. Since $A_1(x_0, a)$ is compact (an arc), by the definition of $\alpha_{1a}(x_0)$ there is a point $\tilde{x}_0 \in A_1(x_0, a)\setminus\{x_0\}$ such that $\alpha_{1a}(x_0) = \varrho(\tilde{x}_0, x_0)$.

There exist small enough neighbourhoods U_0, V_0 of respectively x_0 and $\tilde{x}_0$ with the properties

$$a \notin U_0, \quad d(U_0, V_0) > \tfrac{1}{2}\sup\{\alpha_{1a}(x)\colon x \in E_0\}$$

and

(1) $az \in (U_0) \Rightarrow az \in (G)$ for each $z \in X$ (see Lemma 2.1).

Set

$$E_1 = P(U_0) \cap \{x \in X\colon ax \in (V_0, U_0, V_0)\} \cap X^e.$$

By the quoted Borsuk's lemma each arc ay in X is a subarc of an arc ax with $x \in X^e$, hence Lemmas 4.3 and 4.5 imply $E_1 \neq \emptyset$. By 3.5 we may assume $\alpha_{1a}(x) > 0$ for each $x \in E_1$. Choose $x_1 \in E_1$ such that $\alpha_{1a}(x_1) > \frac{1}{2}\sup\{\alpha_{1a}(x)\colon x \in E_1\}$.

As above there is a point $\tilde{x}_1 \in A_1(x_1, a)\setminus\{x_1\}$ such that $\alpha_{1a}(x_1) = \varrho(\tilde{x}_1, x_1)$. There exist small enough neighbourhoods U_1, V_1 of respectively x_1 and $\tilde{x}_1$ such that

$$a \notin U_1, \quad d(U_1, V_1) > \tfrac{1}{2}\sup\{\alpha_{1a}(x)\colon x \in E_1\},$$

$$C_{U_0} \subset \operatorname{Int} C_{U_1}$$

and

$$az \in (U_1) \Rightarrow az \in (V_0, U_0, V_0) \text{ for each } z \in X \text{ (by 2.1 and the definition of } E_1).$$

Repeating the above arguments one can construct the sets E_n, U_n, V_n and the points $x_n, \tilde{x}_n \in A_1(x_n, a)$ such that $\alpha_{1a}(x_n) = \varrho(\tilde{x}_n, x_n)$ and

(2) $$x_n \in E_n = P(U_{n-1}) \cap \{x \in X\colon ax \in (V_{n-1}, U_{n-1}, V_{n-1})\} \cap X^e,$$

(3) $a \notin U_{n-1}$, U_{n-1} is a neighbourhood of x_{n-1} and V_{n-1} is a neighbourhood of $\tilde{x}_{n-1}$,

(4) $$d(U_n, V_n) > \tfrac{1}{2}\sup\{\alpha_{1a}(x)\colon x \in E_n\},$$

(5) $$C_{U_{n-1}} \subset \operatorname{Int} C_{U_n},$$

(6) $az \in (U_n) \Rightarrow az \in (V_{n-1}, U_{n-1}, V_{n-1})$ for each $z \in X$,

for each $n \geqslant 1$. Let

(7) $$C = \bigcup_{n=0}^{\infty} C_{U_n}.$$

By (5) we obtain

(8) $a \in C$ and C is an open and arcwise connected subset of X.

By (3), (5) and (7) we infer that $C \neq X$. By the Borsuk lemma and (8) there is a point $x^* \in X^e$ such that $ax^* \not\subset C$. Hence condition (8) implies

(9) $$x^* \notin C.$$

We claim that

(10) $$ax^* \in (U_n) \quad \text{for each } n \geqslant 0.$$

In fact, otherwise $x^* \in ax^* \subset C_{U_n} \subset C$, which contradicts (9). From (6) and (10) we get

(11) $$ax^* \in (V_{n-1}, U_{n-1}, V_{n-1}) \quad \text{for each } n \geqslant 1.$$

We claim that

(12) $$x^* \in P(U_{n-1}) \quad \text{for each } n \geqslant 1.$$

In fact, $x^* \notin C_{U_n}$ by (9), hence there is a neighbourhood W of x^* disjoint with C_{U_n}. Since $C_{U_n} \subset C_W$, hence (12) follows from (5).

By (2), (11) and (12) we obtain $x^* \in E_n$ for each $n \geqslant 1$, and by (4)

(13) $$\alpha_{1a}(x^*) \leqslant 2d(U_n, V_n) \quad \text{for each } n \geqslant 1.$$

For each $n \geqslant 0$ let aw_n be the component of $\overline{ax^* \setminus U_n}$ containing a. By (3) and (10) we have $w_n \neq x^*$. Clearly, $w_n \in \overline{w_n x^* \cap U_n}$ and $aw_n \subset C_{U_n}$. By (4) we have

$\overline{U}_n \cap \overline{V}_n = \emptyset$. Also $\overline{U}_{n+1} \cap C_{U_n} = \emptyset$ by (5). These facts imply that there is a point $u_n \in U_n \cap ax^*$ such that $a \leqslant w_n < u_n < x^*$ and

(14) $$au_n \cap U_{n+1} = \emptyset,$$

(15) $$w_n u_n \cap V_n = \emptyset.$$

It is evident by (14) that

(16) $$a < u_0 < u_1 < \ldots < x^*.$$

We shall show that

(17) $$u_n u_{n+1} \cap V_n \neq \emptyset \quad \text{for each } n \geqslant 0.$$

Suppose $u_n u_{n+1} \cap V_n = \emptyset$. By (6) there are points $p, q, r \in au_{n+1}$ such that $p \in V_n$, $q \in U_n$, $r \in V_n$ and $a < p < q < r$. Since $au_{n+1} = au_n \cup u_n u_{n+1}$, by the supposition we have $r \in au_n$. Also $r \in aw_n$ by (15), because $au_n = aw_n \cup w_n u_n$. However this implies that $q \in U_n \cap C_{U_n}$, a contradiction.

Conditions (16) and (17) imply that $ax^* \in (U_0, V_0, U_1, V_1, \ldots)_\infty$. Lemma 2.2 implies that $d(U_n, V_n) \to 0$. Hence by (13) we get $\alpha_{1a}(x^*) = 0$. Applying 3.5 this gives $x^* \in X_s^e$ because $x^* \in X^e$. Moreover, by (10) and (1) we have $ax^* \in (G)$, which completes the proof of the lemma.

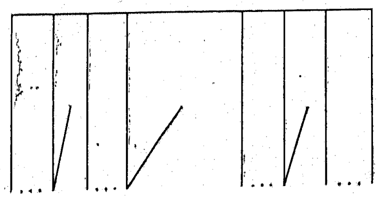
Proof of Theorem 4.1. By the preceding lemma $X_s^e \neq \emptyset$. Take a point $a \in X_s^e$. The minimal arcwise connected set spanning X_s^e in X is the union of all arcs joining a with the other points from X_s^e. Again using the lemma one easily sees that the theorem holds.

Theorem 3.5 says that X_s^e is a G_δ-set in X. One might suppose that the minimal arcwise connected set spanning X_s^e is also a G_δ-set. We give an example which shows that this is not true. Moreover, we construct a planar dendroid X for which the complement of the minimal arcwise connected set spanning X_s^e is a second (Baire) category subset of X.

4.7. Example (comp. [3]). Let C denote the ternary Cantor set on the interval $I = [0, 1]$. Let (r_n, s_n), $r_n < s_n$, $n = 1, 2, \ldots$, be the open intervals in I contiguous to C. In the plane $R \times R$ consider the set

$$X = C \times I \cup I \times \{1\} \cup \bigcup_{n=1}^{\infty} I_n,$$

where I_n is the stright segment in $R \times R$ joining $(r_n, 0)$ and $(\tfrac{1}{2}(r_n + s_n), \tfrac{1}{2})$.



Clearly, X is a (contractible) dendroid and $X_e^s = \{(\frac{1}{2}(r_n+s_n), \frac{1}{2}) : n = 1, 2, \ldots\}$. The set $G = \{(x, y) \in X \colon \frac{1}{2} < y < 1\}$ is open in X and the intersection of G with the minimal arcwise connected set spanning X_s^e (in X) is an F_σ-set in X with empty interior. Observe also that $X^e \setminus X_s^e$ is not an F_σ-set in X.

5. On the set $X^e \setminus X_s^e$. In this section we prove the following

5.1. THEOREM. *For a dendroid X the set $X^e \setminus X_s^e$ is of the first category in X.*

First we have to prove two lemmas.

5.2. LEMMA. *For each $r>0$ and for each $a \in X$ the set $\alpha_{2a}^{-1}([r, \infty))$ is nowhere dense in X.*

Proof. Suppose, to the contrary, that the interior of $\overline{\alpha_{2a}^{-1}([r, \infty))}$ is not empty. Then there are two open nonempty sets U and V such that

$$U \subset \overline{\alpha_{2a}^{-1}([r, \infty))}, \tag{1}$$

$$\overline{U} \cap \overline{V} = \emptyset, \tag{2}$$

$$x \in U \cap \alpha_{2a}^{-1}([r, \infty)) \Rightarrow A_2(x, a) \cap V \neq \emptyset. \tag{3}$$

This follows from the fact that for some $\varepsilon > 0$ and for some $z \in \operatorname{Int} \overline{\alpha_{2a}^{-1}([r, \infty))}$ we have $K(z, \varepsilon) \subset \overline{\alpha_{2a}^{-1}([r, \infty))}$ and for every $y \in K(z, \varepsilon) \cap \alpha_{2a}^{-1}([r, \infty))$ we have $A_2(y, a) \setminus K(z, 2\varepsilon) \neq \emptyset$. We construct a sequence of open sets $U_1, U_2, \ldots$ such that for each $n \geqslant 1$ we have

$$\emptyset \neq \overline{U}_n \subset U_{n-1}, \tag{4$_n$}$$

(5)$_n$ every arc joining a with U_n is of the type (U_{n-1}, V, U_{n-1}),

where $U_0 = U$. The set U_1 is constructed as follows. Let $x \in U_0 \cap \alpha_{2a}^{-1}([r, \infty))$ (see (1)). By (3) there is a point $y \in U_0$ such that $ay \in (U_0, V, U_0)$. From 2.1 we get a neighbourhood U_1 of y satisfying $(4)_1$ and $(5)_1$. Similarly we construct U_{n+1} having defined the set U_n.

By $(4)_n$ for $n = 1, 2, \ldots$ there is a point $z \in \bigcap_n \overline{U}_n$. Again by $(4)_{n+1}$ we have $z \in U_n$ for $n = 1, 2, \ldots$ By $(5)_1, \ldots, (5)_n$ we have

$$az \in (U_0, V, U_1, V, \ldots, U_{n-2}, V, U_{n-1}),$$

which implies that $az \in (U, V, \ldots, V, U)_{2n-1}$. Using 2.2, 2.3 and (2) we get a contradiction.

5.3. COROLLARY. *For each $a \in X$ the set $\alpha_{2a}^{-1}((0, \infty))$ is of the first category in X.*

5.4. LEMMA. *For each $r>0$ and for each point a of a dendroid X we have*

$$X^e \cap \alpha_{1a}^{-1}([r, \infty)) \cap \operatorname{Int} \overline{X^e \cap \alpha_{1a}^{-1}([r, \infty))} \subset \alpha_{2a}^{-1}((0, \infty)).$$

Proof. Let x be an arbitrary endpoint with $\alpha_{1a}(x) \geqslant r$ belonging to the interior of $\overline{X^e \cap \alpha_{1a}^{-1}([r, \infty))}$. Let G be a neighbourhood of x contained in $\overline{X^e \cap \alpha_{1a}^{-1}([r, \infty))}$

such that $\operatorname{diam} G < r$. Assume there is a point $y \in ax \setminus \{x\}$ such that for every neighbourhood M' of y and for every neighbourhood $N' \subset G$ of x there exists an arc joining a and N' of the type (N', M', N'). Then using 2.1 one easily checks that $\alpha_{2a}(x) > 0$. Hence in such a case the point x belongs to the right hand side of the inclusion. So suppose that for each $y \in ax \setminus \{x\}$ there exist neighbourhoods M_y of y and $N_y \subset G$ of x satisfying the condition

(1) no arc joining a and N_y is of the type (N_y, M_y, N_y).

There is a sequence $y_1, y_2, \ldots$ of points from $ax \setminus \{x\}$ such that $ax \setminus \{x\} \subset \bigcup_{i=1}^{\infty} M_{y_i}$.

We claim that for some index m the component F_m of the set $X \setminus \bigcup_{i=1}^{m} M_{y_i}$ which contains x is a subset of G. Otherwise $F_n \setminus G \neq \emptyset$ for each $n \geqslant 1$. And since $F_{n+1} \subset F_n$ the intersection $\bigcap_{n=1}^{\infty} F_n$ would be a continuum joining x and $X \setminus G$ having just the point x in common with the arc ax. Hence x would be an interior point of some arc, contrary to the choice of x.

Let

$$M = \bigcup_{i=1}^{m} M_{y_i} \quad \text{and} \quad N = \bigcap_{i=1}^{m} N_{y_i}.$$

By the construction the component F_m of $X \setminus M$ containing x is a subset of G. Since N is a neighbourhood of x there is a point $y \in ax \setminus \{x\}$ and a neighbourhood V of y such that

$$xy \subset G, \tag{2}$$

$$V \subset N \cap M_y. \tag{3}$$

We shall show that there is a neighbourhood U of x satisfying the conditions:

$$U \subset N \cap N_y, \tag{4}$$

(5) for each $z \in V$ and for each $b \in az \cap U$ the arc bz is contained in G.

Suppose there is no such U. Then for each $n \geqslant 1$ there are two points z_n, b_n and a neighbourhood U_n of x such that: $U_n \subset N \cap N_y$, $\operatorname{diam} U_n < 1/n$, $z_n \in V$, $b_n \in az_n \cap U_n$ and $b_n z_n \not\subset G$. For some index k the arc $b_k z_k$ intersects M, for otherwise there would be a continuum joining x and $X \setminus G$ outside M. This continuum would be a subset of F_m, contrary to the construction of F_m. There is an index $j \leqslant m$ such that $b_k z_k \cap M_{y_j} \neq \emptyset$. Since $z_k \in V \subset N \subset N_{y_j}$ and $b_k \in U_k \subset N \subset N_{y_j}$ the arc $az_k = ab_k \cup b_k z_k$ is of the type $(N_{y_j}, M_{y_j}, N_{y_j})$. This contradicts (1). Hence there is an U with the required properties.

Since $x \in X^e \cap \alpha_{1a}^{-1}([r, \infty))$, $xy \subset G$ by (2), and $\operatorname{diam} G < r$, there is an arc $ax' \in (U, V)$ such that $x' \in V$. Let $V' \subset V$ be a neighbourhood of x' such that every arc joining a and V' is of type (U, V) (see 2.1). We have $V' \subset N \subset G$, hence there is a point $z \in V' \cap X^e \cap \alpha_{1a}^{-1}([r, \infty))$. By the construction $az \in (U, V)$. Let $b \in az \cap U$. By (5) we have $bz \subset G$. Since $\alpha_{1a}(z) \geqslant r > \operatorname{diam} G$, by definition of $\alpha_{1a}(z)$

there is a point $c \in U$ such that $ac \in (V', U)$. By the construction of V' we have $ac \in (U, V, U)$. By (3) and (4) we see that $ac \in (N_y, M_y, N_y)$ and $c \in N_y$. This contradicts (1), and proves the lemma.

Proof of 5.1. Let $a \in X \setminus X^e$. Let $B_n = X^e \cap \alpha_{1a}^{-1}([1/n, \infty))$ for $n = 1, 2, \ldots$ The union $\bigcup_{n=1}^{\infty} B_n$ is equal to $X^e \setminus X^e_s$ by 3.5. Note that for each $n \geqslant 1$ we have

$$B_n \subset (\bar{B}_n \setminus \operatorname{Int} \bar{B}_n) \cup (B_n \cap \operatorname{Int} \bar{B}_n) .$$

The set $\bar{B}_n \setminus \operatorname{Int} \bar{B}_n$ is nowhere dense. By the above lemma $B_n \cap \operatorname{Int} \bar{B}_n \subset \alpha_{2a}^{-1}((0, \infty))$. Hence by 5.3 and the above remark B_n is of the first category in X, which proves the theorem.

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