Pour le second facteur dans la dernière inégalité on a la limitation:

$$(3.12)\qquad \Big(\mathop{\mathsf{S}}_{\Omega_p}(\varphi_{n,p}h_n-\varphi_{n,p}^{(1)}h_n)^2\Big)^{1/2}\leqslant\frac{\sqrt{m}}{2^{N_n^{(1)}+\ldots+N_n^{(m)}}}\cdot\|h_n\|_n.$$

Enfin, moyennant (3.5), (3.11) et (3.12) on obtient

$$(\mathrm{d}_2)\qquad |\psi_p(\varphi_{n,p}x_n,\varphi_{n,p}h_n)-\psi_n(x_n,h_n)|\leqslant\frac{\sqrt{m}\,\big(L_0\|x_n\|_n+|a_0(0)|\big)}{2^{N_n^{(1)}+\ldots+N_n^{(m)}}}\|h_n\|_n.$$

Remarque (3.2). Dans le cas où les fonctions a_r dépendent de x et de $t=(t_1,\ldots,t_m)$, les limitations sont pareilles.

Bibliographie

[1] J. L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, v. 1. Paris 1968.
[2] T. Leżański, *Eine effektive Lösungsmethode nichtlinearer Gleichungen in Hilbertschen Räumen*, Math. Annalen 158 (1965), pp. 377–386.
[3] I. M. Gelfand, S. W. Fomin, *Rachunek wariacyjny*, PWN, Warszawa 1970.

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On projections and unconditional bases in direct sums of Banach spaces II

by

P. WOJTASZCZYK (Warszawa)

Abstract. We prove that an unconditional basis in $X+Y$ is a direct sum of bases in summands provided every operator from Y into X is compact.

The main result of the present paper is Theorem 2.1, which says that all unconditional bases in $X+Y$ are direct sums of bases in summands, provided every operator from Y into X is compact. The interesting thing is that we do not know a priori that X and Y have unconditional bases. This decomposition of unconditional bases in direct sums of l_p-spaces was proved in [4]. One of the main steps in the proof of Theorem 2.1 is Theorem 1.1, which generalizes Theorem 3.5 of [4] and describes complemented subspaces in certain direct sums of Banach spaces.

This paper is a direct continuation of [4]. The proofs use the same ideas as in [4].

The proof of Theorem 1.1 is an extension of the argument given in [4], so we only point out the necessary changes. The first section of this paper cannot be read independently of Section 3 of [4]. The proof of Theorem 2.1 is selfcontained. All our notations and unexplained notions are those of [4].

1. The following theorem improves Theorem 3.5 of [4].

THEOREM 1.1. *Let X and Y be Banach spaces (real or complex) such that every operator from Y into X is strictly singular. Let V be a complemented subspace of $X+Y$. Then there exists an isomorphism $\varphi\colon X+Y\xrightarrow{\text{onto}}X+Y$ such that*

$$\varphi(V)=\varphi(V)\cap X+\varphi(V)\cap Y.$$

For the proof we give only two propositions, the rest is exactly the same as in [4]. We will denote by P_X the projection from $X+Y$ onto X annihilating Y and $P_Y=I-P_X$.

PROPOSITION 1.2. *Let X, Y, V be complex Banach spaces satisfying the assumptions of Theorem 1.1. Then there exists a projection $P_1\colon X+Y\to X+Y$ such that $P_XP_1P_Y=0$ and there exists a Fredholm operator* *of index* 0, $\Phi_1\colon X+Y\to X+Y$ *such that* $\Phi_1(V)$ *is a subspace of finite codimension in* $P_1(X+Y)$.

The proof follows the proof of Proposition 3.3 of [4]. The only difference is that given P, the projection onto V, we define $Q = P - P_X P P_Y$, and so the resulting projection P_1 satisfies $P_X P_1 Y_Y = 0$.

PROPOSITION 1.3. *Let* P_1 *and* P *be projections in a Banach space* Z *and let* $PP_1(I-P) = 0$. *Then there is an isomorphism* $\psi\colon Z \xrightarrow{\text{onto}} Z$ *such that* $P\psi P_1\psi^{-1}(I-P) = 0$ *and* $(I-P)\psi P_1\psi^{-1}P = 0$.

Proof. Let the matrix form of the projection P_1 in the decomposition P, $(I-P)$ be

$$P_1 = \begin{bmatrix}\alpha, & \beta\\ 0, & \gamma\end{bmatrix};$$

thus

$$P_1 = P_1^2 = \begin{bmatrix}\alpha^2, & \alpha\beta+\beta\alpha\\ 0, & \gamma^2\end{bmatrix}$$

and this means that

$$\Pi = \begin{bmatrix}\alpha, & 0\\ 0, & \gamma\end{bmatrix} = P_1 - (I-P)P_1P$$

is a projection. Write

$$S = (I-P)P_1P = \begin{bmatrix}0, & \beta\\ 0, & 0\end{bmatrix}.$$

Obviously, $S^2 = 0$. Our goal now is to construct ψ in such a way that $\psi P_1\psi^{-1} = \Pi$.

Since $P_1 - S = (P_1-S)^2 = -P_1S - SP_1 + P_1$, we infer that $P_1S + SP_1 = S$. From this identity we get $P_1SP_1 = P_1(SP_1+P_1S)P_1 = 2P_1SP_1$, so $P_1SP_1 = 0$. Analogously, $(I-P_1)S(I-P_1) = 0$. Now we define $\tilde{S} = (I-P_1)SP_1 - P_1S(I-P_1)$ and put $\psi = I - \tilde{S}$. If we now consider the decomposition $P_1, I-P_1$, we will have

$$P_1 = \begin{bmatrix}I, & 0\\ 0, & 0\end{bmatrix}, \quad S = \begin{bmatrix}0, & \delta_1\\ \delta_2, & 0\end{bmatrix}, \quad \tilde{S} = \begin{bmatrix}0, & -\delta_1\\ \delta_2, & 0\end{bmatrix},$$

$$\psi = \begin{bmatrix}I, & \delta_1\\ -\delta_2, & I\end{bmatrix}, \quad \psi^{-1} = \begin{bmatrix}I, & -\delta_1\\ \delta_2, & I\end{bmatrix}.$$

Moreover, $\delta_1\delta_2 = \delta_2\delta_1 = 0$ because $S^2 = 0$. So it is obvious that $\tilde{S}^2 = 0$, and ψ is an isomorphism and $\psi^{-1} = I + \tilde{S}$. Thus

$$\psi P_1\psi^{-1} = \begin{bmatrix}I, & \delta_1\\ -\delta_2, & I\end{bmatrix}\begin{bmatrix}I, & 0\\ 0, & 0\end{bmatrix}\begin{bmatrix}I, & -\delta_1\\ \delta_2, & I\end{bmatrix} = \begin{bmatrix}I, & 0\\ -\delta_2, & 0\end{bmatrix}\begin{bmatrix}I, & -\delta_1\\ \delta_2, & I\end{bmatrix}$$
$$= \begin{bmatrix}I, & -\delta_1\\ -\delta_2, & \delta_2\delta_1\end{bmatrix} = P_1 - S = \Pi.$$

This completes the proof of Proposition 1.3.

The above propositions give us

PROPOSITION 1.4. *Let* X, Y, V *be complex Banach spaces satisfying the assumptions of Theorem* 1.1. *Then there is a Fredholm operator* $\Phi\colon X+Y\to X+Y$ *of index* 0 *and complemented subspaces* $X_1\subset X$ *and* $Y_1\subset Y$ *such that* $\Phi(V)$ *is a subspace of finite codimension in* X_1+Y_1.

Proof. We apply Proposition 1.3 to P_1 obtained in Proposition 1.2 and $P = P_X$. Then $X_1 = \mathrm{Im}(P_X\psi P_1\psi^{-1})$ and $Y_1 = \mathrm{Im}(P_Y\psi P_1\psi^{-1})$ and $\Phi = \psi\Phi_1$.

This proposition corresponds to Proposition 3.3 of [4] and we complete the proof of Theorem 1.1 exactly as in [4].

2. In this section we prove our main result.

THEOREM 2.1. *Let* X *and* Y *be Banach spaces such that every operator from* Y *into* X *is compact. Suppose that* $X+Y = Z$ *has an unconditional basis* (z_n). *Then one can divide the set of integers into two sets* N_1 *and* N_2 *in such a way that* $\mathrm{span}\{z_n\colon n\in N_1\}\sim X$ *and* $\mathrm{span}\{z_n\colon n\in N_2\}\sim Y$.

To find sets N_1 and N_2 we use two propositions, one being "dual" to the other. In the case where Z is reflexive they follow from each other by the duality argument. In the general case we are forced to give separate, although very similar proofs. Those proofs heavily depend on the ingenious idea of Edelstein [3] to construct a block sequence which lives "in between" X and Y.

In the following we will assume that the basis (z_n) is unconditionally monotone. We will denote always $P_X(z_n) = x_n$ and $P_Y(z_n) = y_n$. If A is a subset of integers, then $Q\{A\}$ will denote the projection in Z defined by

$$Q\{A\}\Big(\sum_{n=1}^{\infty} a_n z_n\Big) = \sum_{n\in A} a_n z_n.$$

It is a norm one projection.

In the proofs of our propositions we will pass to subsequences many times. We will omit the subscripts to simplify the notation. This should not lead to misunderstandings.

PROPOSITION 2.2. *Under the assumptions of Theorem* 2.1, *let* (n_s) *be a sequence of indices such that* $\mathrm{span}\{z_{n_s}\colon s = 1, 2, \ldots\}$ *contains an infinite-dimensional complemented subspace isomorphic to a complemented subspace of* X. *Then* $\limsup \|x_{n_s}\| \geq 0.1\|P_X\|^{-1}$.

Proof. If our conclusion is false, then we can assume $\|x_{n_s}\| < 0.1\|P_X\|^{-1}$. We will construct a sequence of blocks $g_k = \sum_{\nu_k}^{\mu_k} a_s z_{n_s}$ where $\nu_k < \mu_k < \nu_{k+1}$ such that

$$\|g_k\| = 1, \tag{1}$$

(2) $\quad 1/2 \geqslant \|P_X(g_k)\| \geqslant \|P_X(g)\| \quad$ for all $\quad g = \sum_{\nu_k}^{\mu_k} \beta_s z_{n_s},\ \|g\| = 1,$

(3) $$\|P_X(g_k)\| \geqslant 1/2 - 0.1\|P_X\|^{-1} \geqslant 0.4.$$

Having constructed ν_k, μ_k, g_k for $k = 1, 2, \ldots, N$, we put $\nu_{N+1} = \mu_N + 1$ and

$$\mu_{N+1} = \max\{r \geqslant \nu_{N+1}\colon \|P_X|\operatorname{span}\{z_{n_s}\colon \nu_{N+1} \leqslant s \leqslant r\}\| \leqslant 1/2\}.$$

and g_{N+1} is the element of $\operatorname{span}\{z_{n_s}\colon \nu_{N+1} \leqslant s \leqslant \mu_{N+1}\}$ of norm 1, where $P_X|\operatorname{span}\{z_{n_s}\colon \nu_{N+1} \leqslant s \leqslant \mu_{N+1}\}$ attains its norm.

The fact that μ_{N+1} is finite results from the following

LEMMA 2.3. *For every* N, $\|P_X|\operatorname{span}\{z_{n_s}\colon s \geqslant N\}\| \geqslant 1$.

Proof. If for some N we have $\|P_X|\operatorname{span}\{z_{n_s}\colon s > N\}\| < 1$, then $\|P_X Q\{n_s\colon s > N\}\| < 1$ and this implies that $I - P_X Q\{n_s\colon s > N\}$ is an isomorphism of Z. But

$$(I - P_X Q\{n_s\colon s > N\})(\operatorname{span}\{z_{n_s}\colon s > N\}) = P_Y(\operatorname{span}\{z_{n_s}\colon s > N\}),$$

and so $\operatorname{span}\{z_{n_s}\colon s > N\}$ is isomorphic to a complemented subspace of Y, which contradicts our assumptions.

The fact that (1), (2) and (3) are satisfied follows easily from the construction.

Let us write

$$P_X(g_k) = a_k + b_k, \quad \text{where} \quad a_k = Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k\}(P_X(g_k)),$$

$$P_Y(g_k) = \tilde{a}_k - b_k, \quad \text{where} \quad \tilde{a}_k = Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k\}(P_Y(g_k)).$$

Observe that

$$1 = \frac{\|P_X(g_k)\|}{\|P_X(g_k)\|} = \frac{\|P_X(a_k) + P_X(b_k)\|}{\|a_k + b_k\|} \leqslant \frac{\|P_X(a_k)\| + \|P_X(b_k)\|}{\|a_k + b_k\|}$$
$$\leqslant \frac{\|P_X(a_k)\|}{\|a_k\|} + \frac{\|P_X(b_k)\|}{\|P_X(g_k)\|} \leqslant \frac{1}{2} + \frac{\|P_X(b_k)\|}{\|P_X(g_k)\|},$$

and this implies $\|P_X(b_k)\| \geqslant 1/2\,\|P_X(g_k)\| > 0.2$.

Our goal now is to get a contradiction by constructing a non-compact operator from Y into X. Consider $P_Y(g_k)$. Passing to a subsequence, we can assume that $P_Y(g_k)$ is weakly Cauchy. Indeed, if $P_Y(g_k)$ has no weakly Cauchy subsequence, then by a deep theorem of Rosenthal [7] (see [9] in the complex case) we can find a subsequence spanning l_1, and so by C7 of [2] this implies that Y has a complemented subspace isomorphic to l_1, which easily produces a non-compact operator from Y into X. For a fixed j, the sequence $Q\{n_s\colon \nu_j \leqslant s \leqslant \mu_j\}(P_Y(g_k))$ is a norm convergent sequence, because $P_Y(g_k)$ is weakly Cauchy.

Let us write

$$u_j = \lim_k Q\{n_s\colon \nu_j \leqslant s \leqslant \mu_j\}(P_Y(g_k)).$$

Observe that for $|\varepsilon_j| = 1$ and $k = 1, 2, \ldots$ we have $\|\sum_{j=1}^{k} \varepsilon_j u_j\| \leqslant \|P_Y\|$. Now we will pass to a subsequence to ensure

(4) $$\|Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k\}(b_r) + u_k\| \leqslant (0.1)^{k+r} \quad \text{for} \quad k < r,$$

(5) $$\|Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k\}(b_r)\| \leqslant (0.1)^{k+r} \quad \text{for} \quad r < k.$$

We are able to ensure those conditions because for $k_1 \neq k_2$

$$Q\{n_s\colon \nu_{k_1} \leqslant s \leqslant \mu_{k_1}\}(b_{k_2} + P_Y(g_{k_2})) = 0.$$

Observe that (4) and (5) imply

(*) $$\Big\|Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k,\ k = 1, 2, \ldots\}\Big(b_r - \sum_{j=1}^{r-1} u_j\Big)\Big\| \leqslant (0.1)^{r-1}.$$

In particular, $Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k,\ k = 1, 2, \ldots\}(b_r)$ is weakly Cauchy since so is $\sum_{j=1}^{r-1} u_j$. Also

$$(I - Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k,\ k = 1, 2, \ldots\})(b_r)$$
$$= -(I - Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k;\ k = 1, 2, \ldots\})(P_Y(g_r))$$

is weakly Cauchy, and so b_k is weakly Cauchy. This implies that $\tilde{a}_k$ is weakly Cauchy, and since $\tilde{a}_k$ is unconditionally basic, $\tilde{a}_k$ is weakly null. This enables us to pass to a subsequence to ensure that:

There exists a sequence of indices N_k such that for every k

(6) $$\Big\|Q\{n\colon n < N_k\}\Big(P_X\Big(\sum_{j=1}^{k-1} u_j\Big)\Big) - P_X\Big(\sum_{j=1}^{k-1} u_j\Big)\Big\| \leqslant 2^{-k}\Big\|P_X\Big(\sum_{j=1}^{k-1} u_j\Big)\Big\|$$

and

$$\|Q\{n\colon N_k < n \leqslant N_{k+1}\}(P_X(\tilde{a}_k)) - P_X(\tilde{a}_k)\| \leqslant 2^{-k}\|P_X(\tilde{a}_k)\|.$$

Recall that $P_X(\tilde{a}_k) = P_X(b_k)$ and has a norm greater than 0.2. Now having conditions (1)–(6) satisfied, we define $Q = Q\{n_s\colon \nu_k \leqslant s \leqslant \mu_k,\ k = 1, 2, \ldots\}$ and define a non-compact operator from Y into X by $P_X Q P_Y$. To see that it is a non-compact operator consider

$$P_X Q P_Y(g_k) = P_X Q(\tilde{a}_k - b_k) = P_X Q(\tilde{a}_k) - P_X Q(b_k) = P_X(\tilde{a}_k) - P_X Q(b_k)$$
$$= P_X(\tilde{a}_k) + P_X\Big(\sum_{j=1}^{k-1} u_j\Big) + v_k,$$

where $\|v_k\| \leqslant (0.1)^{k-1}\|P_X\|$. The last equality is by (*). Condition (6) implies that this sequence has no norm convergent subsequences. This contradiction concludes the proof of the proposition.

Remark. All complications in the above proof vanish if Z is reflexive. Then we have g_k weakly null and we easily pass to a subsequence satisfying (1), (2), (3) and

(4′) The sequence $P_Y(g_k)$ is almost disjoint with respect to (z_n).

If $P_Y(g_k)$ were exactly disjoint, then $P_X Q P_Y(g_k) = P_X Q(\tilde{a}_k - b_k) = P_X(\tilde{a}_k) = P_X(b_k)$ would have norm $\geqslant 0.2$. In the general case an easy approximation is necessary.

Our next proposition deals with the dual situation. The main difficulty here is that the biorthogonal functionals (z_n^*) need not span the whole space Z^*.

PROPOSITION 2.4. *Under the assumptions of Theorem 2.1 let (n_s) be a sequence of indices such that* $\operatorname{span}\{z_{n_s}\}$ *contains an infinite-dimensional complemented subspace isomorphic to a complemented subspace of Y. Then* $\limsup \|P_Y^*(z_{n_s}^*)\| \geqslant 0.1\,\|P_X\|^{-1}$.

Proof. Suppose the conclusion is false. As in the proof of Proposition 2.2, we construct a sequence of blocks $g_k^* = \sum_{\nu_k}^{\mu_k} a_s z_{n_s}^*$, where $\nu_k < \mu_k < \nu_{k+1}$ such that

$$\|g_k^*\| = 1, \tag{7}$$

$$1/2 \geqslant \|P_Y^*(g_k^*)\| \geqslant \|P_Y^*(g^*)\| \quad \text{for all} \quad g^* = \sum_{\nu_k}^{\mu_k} \beta_s z_{n_s}^*,\ \|g^*\| = 1, \tag{8}$$

$$\|P_Y^* g_k^*\| \geqslant 1/2 - 0.1\,\|P_X\|^{-1} \geqslant 0.4. \tag{9}$$

The construction is made possible by

LEMMA 2.5. $\|P_Y^*|\operatorname{span}\{z_{n_s}^*\colon s \geqslant N\}\| \geqslant 1$ *for all* N.

Proof. If for some N, $\|P_Y^*|\operatorname{span}\{z_{n_s}^*\colon s \geqslant N\}\| < 1$, then we have $P_Y^* Q^*\{n_s\colon s \geqslant N\} < 1$. Observe that we use the fact that $Q^*\{n_s\colon s \geqslant N\}$ is w^*-continuous and the unit ball of $\operatorname{span}\{z_{n_s}^*\colon s \geqslant N\}$ is w^*-dense in the unit ball of $\operatorname{Im} Q^*\{n_s\colon s \geqslant N\}$. But this gives $\|Q\{n_s\colon s \geqslant N\} P_Y\| < 1$, and so $\Phi = I - Q\{n_s\colon s \geqslant N\} P_Y$ is an isomorphism of Z. Since $\Phi|X = I|X$ and $\Phi(Y) \subset \operatorname{span}\{z_n\colon n \neq n_s, s \geqslant N\}$, we conclude that $\operatorname{span}\{z_{n_s}\colon s \geqslant N\}$ is isomorphic to a complemented subspace of X. This contradiction finishes the proof of the Lemma.

Write

$$P_X^*(g_k^*) = a_k^* + b_k^*, \quad \text{where} \quad a_k^* = Q^*\{n_s\colon \nu_k \leqslant s \leqslant \mu_k\}(P_X^*(g_k^*)),$$
$$P_Y^*(g_k^*) = \tilde{a}_k^* - b_k^*, \quad \text{where} \quad \tilde{a}_k^* = Q^*\{n_s\colon \nu_k \leqslant s \leqslant \mu_k\}(P_Y^*(g_k^*)).$$

By the same computations as in Proposition 2.2, we get $\|P_Y^* b_k^*\| > 0.2$ and $P_Y^* a_k^* = -P_Y^* b_k^*$. Observe that since $g_k^* \xrightarrow[w^*]{} 0$ and $a_k^* \xrightarrow[w^*]{} 0$, we have $P_X^*(g_k^*) \xrightarrow[w^*]{} 0$ and $b_k^* \xrightarrow[w^*]{} 0$.

LEMMA 2.6. *Given an arbitrary natural number N, an element $b^* \in Z^*$ and $\varepsilon > 0$, there is a natural number r such that for every set of natural numbers A with $\inf A > r$ we have*

$$\|Q^*\{n\colon n \leqslant N\} P_Y^* Q^*\{A\}(b^*)\| \leqslant \varepsilon\,\|b^*\|.$$

Proof. If the claim is false, then there exist $\varepsilon > 0$, an integer N, an element $b^* \in Z^*$ and a sequence of disjoint finite subsets of integers A_n such that

$$\|Q^*\{n\colon n \leqslant N\} P_Y^* Q^*\{A_n\}(b^*)\| \geqslant \varepsilon\,\|b^*\| \quad \text{for} \quad n = 1, 2, \ldots$$

But it is clear that $Q^*\{A_n\}(b^*)$ is an unconditional basic sequence equivalent to the unit vector basis in c_0; so $Q^*\{n\colon n \leqslant N\} P_Y^*|\operatorname{span}\{Q^*\{A_n\}(b^*)\}$ is a non-compact operator with a finite-dimensional range, which is absurd. This contradiction finishes the proof.

To get the final contradiction we find three increasing sequences of natural numbers k_r, l_r, and v_r such that

$$\|Q^*\{n\colon n < l_r\} P_Y^* a_{k_r}^*\| > 0.2, \tag{10}$$

$$\|Q^*\{n\colon n < l_r\} P_Y^* Q^*\{n_s\colon \nu_{k_i} \leqslant s \leqslant \mu_{k_i} \text{ for } i = 1, 2, \ldots, r-1\}(b_{k_r}^*)\| \leqslant 0.05, \tag{11}$$

(12) $\|Q^*\{n\colon n < l_r\} P_Y^* Q^*\{A\}(b_{k_r}^*)\| \leqslant 0.05$ for all sets of integers A with $\inf A > v_r$,

$$\nu_{k_{r+1}} \geqslant v_r. \tag{13}$$

We start the induction with $k_1 = 1$ and l_1 big enough to ensure (10). (This is possible since $\|Q^*\{n\colon n < l\}(b^*)\| \xrightarrow[l]{} \|b^*\|$.) Condition (11) is satisfied vacuously. We choose v_1, using Lemma 2.6.

Suppose we have k_r, l_r, and v_r and we want to find k_{r+1}, l_{r+1} and v_{r+1}. First we choose k_{r+1} big enough to have

$$\nu_{k_{r+1}} \geqslant v_r$$

and

$$\|Q^*\{n_s\colon \nu_{k_i} \leqslant s \leqslant \mu_{k_i},\ i = 1, 2, \ldots, r\}(b_{k_{r+1}}^*)\| \leqslant 0.05\,\|P_Y^*\|^{-1}.$$

This is possible since $b_k^* \xrightarrow[w^*]{} 0$ and $Q^*\{n_s\colon \nu_{k_i} \leqslant s \leqslant \mu_{k_i},\ i = 1, 2, \ldots, r\}$ is a finite-dimensional w^*-continuous operator. This choice of k_{r+1} ensures (13) and (11) for arbitrary l_{r+1}. Now we choose l_{r+1} to satisfy (10). To those k_{r+1} and l_{r+1} we find v_{r+1} by Lemma 2.6 to satisfy (12).

Let us write $A_0 = \bigcup_{r=1}^{\infty} \{n_s\colon \nu_{k_r} \leqslant s \leqslant \mu_{k_r}\}$ and consider $P_Y^* Q^*\{A_0\} P_X^*$. By our assumptions this operator is a w^*-continuous, norm-compact operator, and so it transforms sequences w^*-convergent to zero into sequences norm convergent to zero. But

$$\begin{aligned} P_Y^*Q^*\{A_0\}P_X^*(g_{k_r}^*) &= P_Y^*Q^*\{A_0\}(a_{k_r}^*+b_{k_r}^*) = P_Y^*(a_k^*)+P_Y^*Q^*\{A_0\}(b_{k_r}^*) \\ &= P_Y^*(a_{k_r}^*)+P_Y^*Q^*\{A_0\cap\{1,2,\ldots,\mu_{k_{r-1}}\}\}(b_{k_r}^*)+ \\ &\qquad +P_Y^*Q^*\{A_0\cap\{n:\ n\geqslant \nu_{k_{r+1}}\}\}(b_{k_r}^*). \end{aligned}$$

Conditions (10)–(13) imply that

$$\|Q^*\{n:\ n<l_r\}P_Y^*Q^*\{A_0\}P_X^*(g_{k_r}^*)\|\geqslant 0.1,$$

and so also

$$\|P_Y^*Q^*\{A_0\}P_X^*(g_{k_r}^*)\|\geqslant 0.1.$$

This contradiction finishes the proof of the proposition.

Using the above propositions, we can start the proof of Theorem 2.1.

Proof of Theorem 2.1. Let us write

$$N_1 = \{n:\ \|x_n\|\geqslant 0.05\,\|P_X\|^{-1}\},$$
$$N_2 = \{n:\ \|x_n\|< 0.05\,\|P_X\|^{-1}\}.$$

Proposition 2.2 and Theorem 1.1 imply that span$\{z_n:\ n\in N_2\}$ is isomorphic to a complemented subspace of Y. Now we want to prove that span$\{z_n:\ n\in N_1\}$ does not contain any infinite-dimensional complemented subspace isomorphic to a complemented subspace of Y. Suppose that it contains such a subspace. Then Proposition 2.4 gives us a subsequence $(z_{n_s})\subset N_1$ such that $\|P_Y^*(z_{n_s}^*)\|\geqslant 0.1\,\|P_X\|^{-1}$. But $P_Y^*(z_{n_s}^*)\xrightarrow[w^*]{} 0$; so by the theorem of Johnson and Rosenthal [6] we can pass to a subsequence such that there are $y_{n_s}\in Y$, $\|y_{n_s}\|\leqslant M$ and $P_Y^*(z_{n_s}^*)(y_{n_k})=\delta_{s,k}$. Because Y does not contain l_1 (cf. the proof of Proposition 2.2), we can assume that y_{n_s} are weakly null. But now the operator T: $Y\to X$ defined by $T(y)=P_X\big(\sum_s z_{n_s}^*(y)z_{n_s}\big)$ is a non-compact operator because $\liminf\|T(y_{n_s})\|>0$. So Theorem 1.1 gives us an isomorphism φ: $X+Y\to X+Y$ such that $\varphi(\mathrm{span}\{z_n:\ n\in N_1\}) = X_1+V$ where X_1 is a subspace of X of finite codimension and V is a finite-dimensional space.

To ensure span$\{z_n:\ n\in N_1\}\sim X$ and span$\{z_n:\ n\in N_2\}\sim Y$ we can shift a finite number of integers from N_1 to N_2 or vice versa, if necessary.

3. Concluding remarks.

a. Our Theorem 2.1 can be considered as a statement about certain purely atomic Banach lattices. We do not know if a theorem like Theorem 2.1 holds for some non-atomic Banach lattices. In particular, let us mention the problem asked in [1]. Let X be a Banach lattice, isomorphic as a Banach space to $L_1(0,1)+L_2(0,1)$. Do there exist two disjoint bands in X, X_1 and X_2, such that $X=X_1+X_2$ and such that X_1 is as a Banach space isomorphic to $L_1(0,1)$ and X_2 is as a Banach space isomorphic to $L_2(0,1)$. Recall that the results of [1] show that in this case X_1 is order-isomorphic to the L_1-space and X_2 is order isomorphic to the L_2-space.

b. We do not know if in Theorem 2.1 we can replace the condition that "all operators from Y into X are compact" by the weaker condition that "all operators from Y into X are strictly singular". Observe, however, that we have to assume something about X and Y for the conclusion of Theorem 2.1 to be true. For example, in $L_p(0,1)$, $1<p\neq 2<\infty$ the Haar system is an unconditional basis and $L_p(0,1)\sim l_2+L_p(0,1)$, but the Haar system has no subsequence spanning l_2 (cf. [5]).

c. In a sense Theorem 1.1, Theorem 2.1 and Remark a deal with a special cases of the following general problem: Suppose we have a bounded Boolean algebra of projections $\mathscr{B}$ on the space Z, and a projection P: $Z\to Z$. Under what conditions does there exist an isomorphism φ: $Z\xrightarrow[\text{onto}]{} Z$ such that the algebra $\varphi\mathscr{B}\varphi^{-1}$ commutes with P? The easiest unknown special case of this problem seems to be to extend Theorem 2.1 to unconditional finite-dimensional decompositions.

d. Using the assumption that $X+Y$ has an unconditional basis we can avoid the use of Rosenthal's theorem [7] in the proof of Theorem 2.1.

References

[1] Yu. A. Abramovič, P. Wojtaszczyk, *On the uniqueness of order in spaces l_p and $L_p(0,1)$, $1<p<\infty$*, Mat. Zamietki 18 (1975), pp. 313–325 (in Russian).

[2] C. Bessaga, A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), pp. 151–164.

[3] I. S. Edelstein, *On complemented subspaces and unconditional bases in l_p+l_2*, Teor. Funkcii, Functional Anal. i Prilozen. 10 (1970), pp. 132–143 (in Russian).

[4] —, P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces*, Studia Math. 56 (1976), pp. 263–276.

[5] J. L. B. Gamlen, R. J. Gaudet, *On subsequences of the Haar system in $L_p(0,1)$, $(1<p<\infty)$*, Israel J. Math. 15 (1973), pp. 404–413.

[6] W. B. Johnson, H. P. Rosenthal, *On w^*-basic sequences and their applications to the study of Banach spaces*, Studia Math. 43 (1972), pp. 77–92.

[7] H. P. Rosenthal, *A characterisation of Banach spaces containing l_1*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), pp. 2411–2413.

[8] P. Wojtaszczyk, *On complemented subspaces and unconditional bases in l_p+l_q*, Studia Math. 47 (1973), pp. 197–206.

[9] L. Dor, *On sequences spanning a complex l_1 spaces*, Proc. Amer. Math. Soc. 47 (1975), pp. 515–516.

INSTYTUT MATEMATYCZNY
POLSKA AKADEMIA NAUK
INSTITUT OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

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