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A note on automorphisms of semigroups and near-rings of mappings

by

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Abstract. Let $S(E)$ be a semigroup, under composition, of selfmaps of a real locally convex space E. We show that the automorphisms of $S(E)$ which are given as conjugation by an element of $U(E)$ (the group of continuous linear invertible mappings of E, with continuous inverses) are precisely the automorphisms of $S(E)$ which fix $U(E)$ (Theorem 1). The dimension of E must be greater than one and E must have the weak or Mackey topology.

The semigroup $\mathscr{L}(E)$ (all continuous linear selfmaps of E) then represents E (Corollary 2). That is, if φ from $\mathscr{L}(E)$ to $\mathscr{L}(F)$ is a semigroup isomorphism, there exists a linear homeomorphism h from E to F such that $\varphi(f) = hfh^{-1}$, for every $f \in \mathscr{L}(E)$.

If $S(E)$ forms a near-ring we show in § 2 that the near-ring automorphisms are precisely those fixing $U(E)$. In § 3 the latter are also shown to be the d-automorphisms of Yamamuro [18].

Introduction. Many authors in the past forty years have investigated the relationship between a Hausdorff locally convex space (E) over the reals (R) and an associated semigroup, group, near-ring, or ring of functions from E to itself. Nagumo, Eidelheit, Mackey, Whittaker, Magill and Yamamuro are amongst such writers (see references).

Let $\mathscr{E}$ be a class of real Hausdorff locally convex spaces and A be a category whose objects comprise all elements of $\mathscr{E}$. For any pair $E, F \in \mathscr{E}$, the morphisms $A(E, F)$ are functions from E to F with the usual composition as their product. For example, let $\mathscr{E}$ be all real Banach spaces and $A(E, F)$ be all continuous functions from E to F. Immediately $A(E) = A(E, E)$, $E \in \mathscr{E}$, forms a semigroup, so three levels of the above problem are suggested:

(R) If φ is a semigroup isomorphism (a multiplicative bijection) from $A(E)$ onto $A(F)$ for $E, F \in \mathscr{E}$, there exists an invertible $h \in A(E, F)$ such that $h^{-1} \in A(F, E)$ and

$$\varphi(f) = hfh^{-1} \quad \text{for every } f \in A(E).$$

Following Hofer [4] we say that every isomorphism is *representable*.

(C) If $A(E)$ and $A(F)$ are semigroup isomorphic, for $E, F \in \mathscr{E}$, then E and F are linearly homeomorphic. That is, $A(E)$ *characterises* E.

(M) If φ is a semigroup automorphism of $A(E)$, for $E \in \mathscr{E}$, there exists an invertible $h \in A(E)$ such that $h^{-1} \in A(E)$ and

$$\varphi(f) = hfh^{-1} \quad \text{for every } f \in A(E).$$

That is, every automorphism of $A(E)$ is inner, or following Yamamuro, $A(E)$ has the *Magill property*.

In the event that each $A(E)$, $E \in \mathscr{E}$, forms a group, near-ring or ring (where addition is defined pointwise) we may make corresponding definitions in the natural way.

Let us assume without loss of generality that each $A(E)$ forms a semigroup. Certainly, (R) implies (M). Given property (M) a notational change in all proofs to date has sufficed to show (R), while if all maps are Hadamard differentiable (R) implies (C), ([16], p. 182). However, (C) does not imply (R) in the event that each $A(E)$ forms a group, as we now show.

Let $U(H)$ be the group of all linear homeomorphisms of a Hilbert space H, with u^* the adjoint of $u \in U(H)$ ([15], p. 98). A consequence of [7], p. 251, is that $U(H)$ characterises H. But the automorphism φ of $U(H)$ given by

$$\varphi(u) = (u^{-1})^* \quad \text{for every } u \in U(H),$$

is not inner.

If we denote by ξ the scalar selfmap (a map from a set to itself) of H given by $x \to \xi x$ for every $x \in H$, $0 \neq \xi \in R$, and by M the set of all such scalar maps, then every inner automorphism of $U(H)$ will fix every element of M. But φ does not fix any $\xi \in M$.

Let X be any set and $S(X)$ any semigroup of selfmaps of X containing the constant selfmaps $I(X)$. That every automorphism φ of $S(X)$ has the form

$$\varphi(f) = hfh^{-1} \quad \text{for every } f \in S(X),$$

for some bijection h of E was proved by J. Schreier in 1937, [12]. So if we form the semigroup $U \cup I(H)$ by adjoining the constant maps, we have at once that every automorphism is given as conjugation by a bijection. Whether (M) held for this semigroup motivated Theorem 1, from which the main results (Theorems 2 and 3, and Corollary 2) emerged.

Notation. The class of all real, Hausdorff locally convex spaces is denoted by LCS. Greek symbols will be used for real numbers and Roman symbols for elements of $E \in \mathrm{LCS}$. We let $\overline{E}$ be the conjugate space of E and $\mathscr{L}(E)$ be the space of all continuous linear selfmaps of E, where necessary equipped with the topology of uniform convergence on bounded sets, while $U(E)$ will be the group of units in $\mathscr{L}(E)$.

When $x \in E$ and $\bar{x} \in \overline{E}$, $\langle x, \bar{x}\rangle$ will denote the value of $\bar{x}$ at x, while $x \otimes \bar{x} \in \mathscr{L}(E)$ will be the one-dimensional map given by

$$x \otimes \bar{x}(a) = \langle a, \bar{x}\rangle x \quad \text{for every } a \in E.$$

The constant selfmap of E whose single value is $x \in E$ is denoted by C_x, and the collection of all constant maps by $I(E)$.

The symbol for a family of selfmaps of E, $S(E)$, will often be abbreviated to S. The semigroup operation will at all times be function composition while addition will always be defined pointwise.

1. The main theorem. From now, unless otherwise stated, $E, F \in \mathrm{LCS}$ will be equipped with either the weak or the Mackey topology and will have dimension greater than one.

THEOREM 1. *Let S be a semigroup of selfmaps of E such that I, $U \subset S$ and let φ be an automorphism of S. Then φ is inner and $h \in U$ if and only if $\varphi(U) = U$.*

Proof. The necessity is straightforward. Suppose $\varphi(U) = U$. The proof that φ is conjugation by an element of U is in ten stages.

1. *There exists a bijection h of E such that $\varphi(f) = hfh^{-1}$ for every $f \in S$.*

This is an application of Schreier's result, [12]. Note that as φ fixes U, $h(0) = 0$. Since φ^{-1} fixes U and $\varphi^{-1}(f) = h^{-1}fh$, for every $f \in S$, any statement about h will hold for h^{-1}. With the non-zero scalar map ξ, and M the set of such maps as before, we show

2. *There exists a bijection $\lambda\colon R \to R$ such that $\varphi(\xi) = \lambda(\xi)$ for every $\xi \in R$.*

If $Z(U)$ denotes the centre of U, routine methods show $Z(U) = M$. But φ is an automorphism of U, so preserves M. Since $\varphi(0) = 0$ there exists a real valued function of the real variable ξ such that $\varphi(\xi) = \lambda(\xi)$ for every $\xi \in R$. That λ is one-to-one, onto and multiplicative follows from the corresponding properties for φ.

Certainly, $\lambda(1) = \varphi(1) = 1$, while since

$$(\lambda(-1))^2 = \varphi(-1)\varphi(-1) = \varphi(1) = 1$$

and $\varphi(-1) \neq \varphi(1)$, $\lambda(-1) = -1$.

Further, $\lambda(0) = \varphi(0) = 0$.

3. *Given $a, b \in E$, there exists $\mu, \varrho \in R$ such that*

$$h^{-1}(a+b) = \mu h^{-1}(a) + \varrho h^{-1}(b).$$

With $\langle x, \bar{x}\rangle \neq -1$, $\varphi(1+x\otimes\bar{x})$ is linear, so

$$\begin{aligned}(1)\qquad h[h^{-1}(a+b)+\langle h^{-1}(a+b), \bar{x}\rangle x] = h[h^{-1}(a)+\langle h^{-1}(a), \bar{x}\rangle x]+\\ +h[h^{-1}(b)+\langle h^{-1}(b), \bar{x}\rangle x]\end{aligned}$$

for arbitrary $a, b \in E$. When both $\langle h^{-1}(a), \bar{x}\rangle = 0$ and $\langle h^{-1}(b), \bar{x}\rangle = 0$ we have from (1)

$$h^{-1}(a+b) + \langle h^{-1}(a+b), \bar{x}\rangle x = h^{-1}(a+b),$$

so $\langle h^{-1}(a+b), \bar{x}\rangle = 0$, since x may be chosen to be non-zero. This means that $h^{-1}(a+b)$ lies in the subspace spanned by $h^{-1}(a)$ and $h^{-1}(b)$, so the result follows.

4. *When $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent, $\mu = \varrho$.*

Note that $h(\xi x) = \lambda(\xi)h(x)$ and $h^{-1}(\xi x) = \lambda^{-1}(\xi)h^{-1}(x)$ for any $\xi \in R$, $x \in E$. Choose $\bar{x}$ such that $\langle h^{-1}(a), \bar{x}\rangle = 1 = \langle h^{-1}(b), \bar{x}\rangle$. Then with $x = -h^{-1}(a)-h^{-1}(b)$, (1) becomes

$$h[h^{-1}(a+b)-(\mu+\varrho)\big(h^{-1}(a)+h^{-1}(b)\big)]$$
$$= h\big(-h^{-1}(b)\big)+h\big(-h^{-1}(a)\big) = -(a+b),$$

since $\lambda(-1) = -1$. Thus,

$$h^{-1}(a+b) = \frac{\mu+\varrho}{2}\big(h^{-1}(a)+h^{-1}(b)\big) = \mu h^{-1}(a)+\varrho h^{-1}(b).$$

Since $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent, $(\mu+\varrho)/2 = \mu$, or $\varrho = \mu$. Note that μ cannot be zero.

5. *h preserves linearly independent sets of elements.*

Suppose that $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent, and $\alpha a+\beta b = 0$ for some $\alpha, \beta \in R$. Then

$$\begin{aligned} 0 &= h^{-1}(\alpha a+\beta b) \\ &= \mu[h^{-1}(\alpha a)+h^{-1}(\beta b)], \quad \text{since } h^{-1}(\alpha a) \text{ and } h^{-1}(\beta b) \text{ are linearly independent,} \\ &= \mu[\lambda^{-1}(\alpha)h^{-1}(a)+\lambda^{-1}(\beta)h^{-1}(b)]. \end{aligned}$$

Thus, $\lambda^{-1}(\alpha) = \lambda^{-1}(\beta) = 0$, so $\alpha = \beta = 0$.

In a similar way we may show this result for any finite set of linearly independent elements.

6. *$h(a+b) = h(a)+h(b)$, for every pair of linearly independent elements, $a, b \in E$.*

Since a, b and $c = a+b$ are pairwise linearly independent we have

$$\begin{aligned} h(c) = h(a+b) &= \mu[h(a)+h(b)], \text{ for some } \mu \in R, \\ &= \mu[h(c-b)+h(b)] \\ &= \mu[\mu'\big(h(c)-h(b)\big)+h(b)], \quad \text{for some } \mu' \in R, \\ &= \mu\mu' h(c)+\mu(1-\mu')h(b). \end{aligned}$$

But $h(b)$ and $h(c)$ are linearly independent, so $\mu\mu' = 1$ and $\mu-\mu\mu' = 0$, whence $\mu = 1$.

7. *$\lambda(\xi+\eta) = \lambda(\xi)+\lambda(\eta)$ for any $\xi, \eta \in R$.*

We may assume $\xi, \eta \neq 0$. Choose linearly independent $a, b \in E$. Then

$$\begin{aligned} \lambda(\xi+\eta)h(a)+h(b) &= h\big((\xi+\eta)a+b\big) \\ &= h\big(\xi a+(\eta a+b)\big) \\ &= h(\xi a)+h(\eta a+b) \\ &= \lambda(\xi)h(a)+\lambda(\eta)h(a)+h(b) \\ &= [\lambda(\xi)+\lambda(\eta)]h(a)+h(b). \end{aligned}$$

So $\lambda(\xi+\eta) = \lambda(\xi)+\lambda(\eta)$.

8. *$\lambda(\xi) = \xi$ for all $\xi \in R$.*

Since λ is an additive and multiplicative bijection of the reals, it follows from [1], p. 41, Theorem 4, that λ is the identity function.

9. *h is linear.*

We have h homogeneous, and additive on linearly independent elements. Suppose a, b are linearly dependent, with $b = \alpha a$. Then

$$h(a+b) = h(a+\alpha a) = (1+\alpha)h(a) = h(a)+\alpha h(a) = h(a)+h(b).$$

10. *h is continuous.*

For this we need the conditions on the topology of E.

(i) *E has the weak topology.*

Suppose the net $\{x_a\}$ converges weakly to zero. For $\langle x, \bar{x}\rangle \neq -1$,

$$\begin{aligned} \varphi(1+x\otimes\bar{x})(x_a) &= h(1+x\otimes\bar{x})h^{-1}(x_a) \\ &= h[h^{-1}(x_a)+\langle h^{-1}(x_a), \bar{x}\rangle x] \\ &= x_a+\langle h^{-1}(x_a), \bar{x}\rangle h(x) \end{aligned}$$

converges weakly, with a, to zero. Thus $\langle h^{-1}(x_a), \bar{x}\rangle$ converges to zero, and since $\bar{x}$ may be arbitrarily chosen, h^{-1} is continuous with respect to the weak topology at zero, hence everywhere. We may show the same result for h.

(ii) *E has the Mackey topology.*

Since every map in $\mathscr{L}(E)$ is also continuous in the weak topology ([11], p. 39, Proposition 13), we may use the above and [11], p. 62, Proposition 14 to obtain the result.

Remark. The theorem is not true when E has dimension one. Let $h(x) = x^3$, for every $x \in R$. Then the automorphism φ of $M \cup I(R)$ given by

$$\varphi(f) = hfh^{-1} \quad \text{for every } f \in M \cup I(R),$$

is not inner.

COROLLARY 1. *Every automorphism of the semigroup $U \cup I(E)$ is inner.*

Since any automorphism of $U \cup I(E)$ fixes U, this is an immediate consequence of Theorem 1. Recall that $U(H)$, H a Hilbert space, does not have this property.

Let E be as in Theorem 1. The main consequence of Theorem 1 is the following:

THEOREM 2. *Every semigroup automorphism of $\mathscr{L}(E)$ is inner.*

Proof. Let φ be an automorphism of $\mathscr{L}$. We show that φ can be extended to an automorphism φ' of $\mathscr{L} \cup I$. The keystone of the proof is the construction of a bijection h of E such that

(2) $$\varphi(l) = hlh^{-1} \quad \text{for every } l \in \mathscr{L}.$$

The method, an unpublished result of S. Yamamuro, sidesteps Schreier's need for the constant mappings. Let

$$R_x = \{x \otimes \bar{x}\colon \bar{x} \in \bar{E}\} \quad \text{for } 0 \neq x \in E.$$

1. *R_x is a minimal right ideal in $\mathscr{L}$.*

For any $x \otimes \bar{x} \in R_x$ and $l \in \mathscr{L}$,

$$(x \otimes \bar{x})l = x \otimes (\bar{x}l),$$

again an element of R_x. Suppose that D is a right ideal contained in R_x. Take $x \otimes \bar{x}$ in R_x and any $x \otimes \bar{y}$ in D. Pick $y \in E$ such that $\langle y, \bar{y}\rangle = 1$. Then

$$x \otimes \bar{x} = (x \otimes \bar{y})(y \otimes \bar{x}),$$

an element of D, so $D = R_x$.

2. *If D is a minimal right ideal, $D = R_x$ for some $x \in E$.*

Take $l \in D$, $a \in E$. Since $l(a) \otimes \bar{x} = l(a \otimes \bar{x}) \in D$,

$$R_{l(a)} \subseteq D, \quad \text{so} \quad D = R_{l(a)}.$$

Each one-dimensional subspace of E gives rise to a family of identical right ideals, so we select a single member, R_x say, to represent each such family. Since φ preserves minimal right ideals,

$$\varphi(R_x) = R_y \quad \text{for some } y \in E.$$

3. *There exists a bijection h of E such that* (2) *holds.*

We define a self-map h of E by letting

$$h(x) = y \text{ if } \varphi(R_x) = R_y \quad \text{and} \quad h(ax) = \varphi(a)h(x) \text{ for } a \in R.$$

Recall that $\varphi(a) \in M$ whenever $a \in M$. It may readily be verified that h is both one-to-one and onto. We now show that (2) holds. As before we may show that sets of the form

$$L_{\bar{x}} = \{x \otimes \bar{x}\colon x \in E\}, \quad 0 \neq \bar{x} \in \bar{E},$$

are precisely the minimal left ideals in $\mathscr{L}$. So $\varphi(L_{\bar{x}}) = L_{\bar{y}}$, for some $\bar{y} \in \bar{E}$. Now for arbitrary $a = ax \in E$ as above,

$$\begin{aligned}\varphi(a \otimes \bar{x}) &= \varphi(ax \otimes \bar{x}) = \varphi(a)\varphi(x \otimes \bar{x}) \\ &= \varphi(a)h(x) \otimes \bar{y} = h(ax) \otimes \bar{y} = h(a) \otimes \bar{y}.\end{aligned}$$

Take $l \in \mathscr{L}$. Then,

$$\begin{aligned}\varphi(l)h(a) \otimes \bar{y} &= \varphi(l)\varphi(a \otimes \bar{x}) = \varphi\big(l(a \otimes \bar{x})\big) \\ &= \varphi(l(a) \otimes \bar{x}) = hl(a) \otimes \bar{y}.\end{aligned}$$

Since $\bar{y} \neq 0$, $\varphi(l)h(a) = hl(a)$, for all $a \in E$, or

$$\varphi(l) = hlh^{-1} \quad \text{for every } l \in \mathscr{L}.$$

4. *φ is inner.*

We define an automorphism φ' of $\mathscr{L} \cup I(E)$ extending φ, by

$$\varphi'(l) = \varphi(l) \text{ for } l \in \mathscr{L}, \quad \text{and} \quad \varphi'(C_x) = C_{h(x)} \text{ for } C_x \in I.$$

It is easy to check that

$$\varphi'(f) = hfh^{-1} \quad \text{for every } f \in \mathscr{L} \cup I(E).$$

Since our semigroup includes the constant maps, h is the only bijection to represent φ' in this way. Finally, as $\varphi'(U) = U$, Theorem 1 gives that $h \in U$ so φ is inner.

Remarks. In [13] Stephenson showed that a prime ring with identity and a non-zero idempotent is a unique addition ring. Such a ring has the property that every semigroup isomorphism from the ring to another ring is a ring isomorphism. We may verify that $\mathscr{L}(E)$, for E with dimension greater than one, is a unique addition ring.

For E a real Banach space with dimension greater than one, Eidelheit [3] showed that every ring automorphism and every continuous semigroup automorphism of $\mathscr{L}(E)$ was inner. In [10], Rickart, showing that $\mathscr{L}(E)$ was a unique addition ring, was able to eliminate the continuity requirement. Theorem 2 extends this result.

Mackey ([6], p. 536) has proved that the ring $\mathscr{L}(E)$, for $E \in \mathrm{LCS}$ with the weak or Mackey topology, characterises E. From Stephenson's result we have for such spaces,

Result 1. *The semigroup $\mathscr{L}(E)$ characterises E. That is, if $\mathscr{L}(E)$ and $\mathscr{L}(F)$ are isomorphic semigroups, E and F are linearly homeomorphic.*

A notational change in Schreier's Theorem and Theorems 1 and 2 enables us to derive, for E, F both having either the weak topology or the Mackey topology and dimension greater than one, the stronger result, COROLLARY 2. *Every semigroup isomorphism from $\mathscr{L}(E)$ onto $\mathscr{L}(F)$ is representable. That is, given a semigroup isomorphism $\varphi\colon \mathscr{L}(E)\to\mathscr{L}(F)$, there exists a linear homeomorphism h from E onto F such that*

$$\varphi(f) = hfh^{-1} \quad \text{for every } f\in\mathscr{L}(E).$$

2. Near-rings of mappings. Many semigroups, S, of selfmaps of a locally convex space, form near-rings. Let C be the near-ring of continuous selfmaps of E (as in Theorem 1) and suppose that N is a near-ring with $I, U\subset N\subseteq C$. Then we have,

THEOREM 3. *The following statements are equivalent:*

(1) *φ is a semigroup automorphism of N and $\varphi(U) = U$,*

(2) *φ is an inner semigroup automorphism of N and $h\in U$,*

(3) *φ is a near-ring automorphism of N.*

Proof. Statements (1) and (2) are equivalent by Theorem 1, so it will suffice to show that (2) implies (3), and (3) implies (1). The former implication is straightforward, so suppose that φ is a near-ring automorphism. Schreier's result provides a bijection h of E such that

$$\varphi(f) = hfh^{-1} \quad \text{for every } f\in N.$$

For any $x, y\in E$,

$$\varphi(C_x+C_y) = \varphi(C_x)+\varphi(C_y)$$

so

$$h(x+y) = h(x)+h(y).$$

As we have the same property for h^{-1}, for any $u\in U$ $\varphi(u)$ is continuous and additive, so $\varphi(u)\in\mathscr{L}$. Similarly, $\varphi(u^{-1}) = \varphi(u)^{-1}\in\mathscr{L}$, so $\varphi(u)\in U$, or $\varphi(U)\subseteq U$. Equally, $\varphi^{-1}(U)\subseteq U$ so $\varphi(U) = U$.

Remarks. 1. That every near-ring automorphism of such an N is inner refines [17]. Here Theorem 3 was proved for E a Banach space and $I, \mathscr{L}\subset N\subseteq C$.

2. Suppose that F is the near-ring of all selfmaps of E and h in F is discontinuous and additive. Then φ, given by

$$\varphi(f) = hfh^{-1} \quad \text{for every } f\in F,$$

is a near-ring automorphism but $h\notin U$, indicating that a containment condition is necessary.

3. *d*-automorphisms. In this section we extend the idea of a d-automorphism of a semigroup of differentiable functions, first introduced by Yamamuro in [18], and present a theorem showing them to be precisely those we have considered earlier.

A selfmap f of E is said to be *Hadamard differentiable* [2] at $a\in E$ if there exists an $l\in\mathscr{L}$ such that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}[f(a+\varepsilon x)-f(a)-l(\varepsilon x)] = 0,$$

uniformly for x in any sequentially compact subset of E. Then l is uniquely determined and denoted by $f'(a)$. Since the chain rule holds for such differentiation ([2], p. 234), the family of all such Hadamard differentiable selfmaps of E, $\mathscr{D}_H$, forms a semigroup.

For the remainder, E is as in Theorem 1, while S is a semigroup of selfmaps of E such that

$$I, U\subset S\subseteq\mathscr{D}_H.$$

We say that φ is a *d-automorphism* of S if

$$\{\varphi(f)'(a)\colon a\in E\} = \{\varphi[f'(a)]\colon a\in E\},$$

for every invertible $f\in S$ for which $f^{-1}\in S$.

THEOREM 4. *The following statements are equivalent:*

(1) *φ is an automorphism of S and $\varphi(U) = U$,*

(2) *φ is an inner automorphism of S and $h\in U$,*

(3) *φ is a d-automorphism of S.*

Proof. It suffices to show that (3) implies (1), and (2) implies (3). Suppose φ is a d-automorphism. We show that $\varphi(U) = U$.

If $u\in U$, we can find an invertible $f\in S$ such that $\varphi(f) = u$. Since $f^{-1} = \varphi^{-1}(u^{-1})$, $f^{-1}\in S$. Now

$$\{u\} = \{\varphi(f)'(a)\colon a\in E\} = \{\varphi(f'(a))\colon a\in E\}$$

so $f'(a)$ is constant. By the Mean Value Theorem for functionals [5], since $f(0) = 0$, $f\in\mathscr{L}$. Further, $\varphi(f^{-1}) = u^{-1}$ so similarly $f^{-1}\in\mathscr{L}$, whence $f\in U$.

Suppose $f\in U$. Then f is invertible, $f^{-1}\in S$ and $f = \varphi(u)$ for some u. Now

$$\{\varphi(u)'(a)\colon a\in E\} = \{\varphi(u)\},$$

so $\varphi(u)\in\mathscr{L}$. Similarly, $\varphi(u)^{-1}\in\mathscr{L}$ so $\varphi(u) = f\in U$, whence $\varphi(U) = U$.

If (2) holds and $f, f^{-1}\in S$, we have

$$\begin{aligned}\{\varphi(f)'(a)\colon a\in E\} &= \{(hfh^{-1})'(a)\colon a\in E\}\\ &= \{hf'(h^{-1}(a))h^{-1}\colon a\in E\}\\ &= \{hf'(a)h^{-1}\colon a\in E\}\\ &= \{\varphi(f'(a))\colon a\in E\},\end{aligned}$$

so φ is a d-automorphism.

Remark. If S is also a near-ring each of the above is equivalent to

(4) *φ is a near-ring automorphism of S.*

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Bounded complete Finsler structures I

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Abstract. It is proved that any C^∞ Banach manifold satisfying a stability condition admits a complete bounded Finsler structure; in particular, any C^∞ separable Hilbert manifold admits a bounded complete Riemannian structure. In finite dimensions, only compact manifolds admit complete bounded Finsler structures. A "global" definition of Finsler structures (in the sense of Palais) is also given.

The principal purpose of this note is to resolve a problem proposed some years ago by Elworthy, [2]. We have, however, appended some easy remarks on related topics. In § 1, we give a definition of the notion of "Finsler structure" in the sense of Palais [8], which, although obvious, does not seem to have appeared before, and is at least of some theoretical interest. In § 2, we discuss Elworthy's problem in finite dimensions, and in § 3 we answer the problem in infinite dimensions.

§ 1. Suppose that E is a topological vector space whose topology admits a norm. Let $\mathfrak{N}$ denote the set of norms on E which define the given topology.

(a) $\mathfrak{N}$ is a cone, in the space of real-valued functions on E with pointwise addition and multiplication. That is, if $\lambda \in \boldsymbol{R}$ and $\lambda > 0$, and $\nu \in \mathfrak{N}$, then $\lambda\nu \in \mathfrak{N}$; if $\nu_1, \nu_2 \in \mathfrak{N}$, then $\nu_1 + \nu_2 \in \mathfrak{N}$.

Choose $\nu \in \mathfrak{N}$, and define Δ_ν: $\mathfrak{N} \times \mathfrak{N} \to \boldsymbol{R}$ by the following technique. For $\nu' \in \mathfrak{N}$, let $B(\nu') = \{x \in E\colon \nu'(x) \leqslant 1\}$ be the closed unit ball with respect to ν'. Thus the correspondence $\nu' \leftrightarrow B(\nu')$ is a bijection between $\mathfrak{N}$ and the set of bounded, absolutely convex, closed neighbourhoods of 0 in E. Now let $\Delta_\nu(\nu_1, \nu_2)$ be the Hausdorff distance between $B(\nu_1)$ and $B(\nu_2)$ in the metric on E defined by ν. Formally,

$$(*) \qquad \Delta_\nu(\nu_1, \nu_2) = \max\big(a_\nu(B(\nu_1), B(\nu_2)),\ a_\nu(B(\nu_2), B(\nu_1))\big),$$

where, for any two bounded nonnull sets A, A' in E,

$$a_\nu(A, A') = \sup_{x\in A} \inf_{y\in B} \nu(x-y).$$