[2], for instance. The proof of Theorem (a) can now be followed word for word, except that formula (1) requires alteration. If g_M, g_N are the Riemannian structures on M and N, respectively (both C^∞ and complete; g_N is bounded), and if f is a C^∞ function on N which is everywhere strictly positive and has infimum zero, define for $i = 1, 2$, $\xi_i \in T_x M$, $\eta_i \in T_y N$,

$$g(x, y)\big((\xi_1, \eta_1), (\xi_2, \eta_2)\big) = \big(f(y)\big)^2 g_M(x)(\xi_1, \xi_2) + g_N(\eta_1, \eta_2).$$

The previous proof now shows that g is a complete bounded C^∞ Riemannian structure on $M \times N$, which may be transferred by the given diffeomorphism to M.

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Interpolation of weighted L_p-spaces

by

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Abstract. We characterize the interpolation spaces with respect to couples of weighted L_p-spaces. This is done in terms of the K-functional of Peetre. The main tool is a generalization of the theorem of Hardy, Littlewood, and Polya on doubly stochastic matrices.

Contents

0. Introduction. We are concerned with the following problem: Let $\bar{A} = \{A_0, A_1\} = \{L_{p_0a_0}, L_{p_1a_1}\}$, $\bar{B} = \{B_0, B_1\} = \{L_{p_0b_0}, L_{p_1b_1}\}$, $0 < p_0, p_1 \leqslant \infty$, be two couples of weighted L_p-spaces assigned to some, not necessarily the same, measure spaces. Then characterize *all* interpolation spaces with respect to $\bar{A}, \bar{B}$, i.e. all spaces A, B obeying

$$(0.1) \qquad T\colon A_\mu \to B_\mu \ (\mu = 0, 1) \quad \text{implies} \quad T\colon A \to B, \quad \|T\|_{\mathcal{L}(A;B)} \leqslant c \max_{\mu=0,1} \|T\|_{\mathcal{L}(A_\mu;B_\mu)}$$

(operator norms). A both necessary and sufficient condition is found to be

$$(0.2) \qquad K(t, g; \bar{B}) \leqslant K(t, f; \bar{A}),\ t > 0, \quad \text{implies} \quad \|g\|_B \leqslant C\|f\|_A$$

("K-monotonicity"), where K is the Peetre functional

$$(0.3) \qquad K(t, f; \bar{A}) = \inf_{f=f_0+f_1} (\|f_0\|_{A_0} + t\|f_1\|_{A_1}).$$

Careing also about the constants c and C, we find that (0.2) implies (0.1) with $c = C$ ("exactness"), while in the other direction (0.1) only implies (0.2) with $C \leqslant c\gamma_{p_0p_1}$, $1 \leqslant \gamma_{p_0p_1} \leqslant 2$. Using instead a modified K, we obtain the equality $C = c$ in the latter direction (but in general not in the former). If $0 < p_0, p_1 \leqslant 1$, however, we get a complete characterization even in the exact sense.

Our results were announced in [30]. They generalize those of Mitjagin [19], Calderón [6], and Cotlar [7] for $\{L_1, L_\infty\}$; Lorentz–Shimogaki [17] for $\{L_p, L_\infty\}$ and $\{L_1, L_p\}$, $1 \leqslant p \leqslant \infty$; Sedaev–Semenov [25], and Sedaev [26] for $\{L_{pa_0}, L_{pa_1}\}$, $1 \leqslant p \leqslant \infty$. Recently, and independently of us, Cwikel [8] has treated the general Banach case $\{L_{p_0a_0}, L_{p_1a_1}\}$, $1 \leqslant p_0, p_1 \leqslant \infty$. His results are about the same as ours, although obtained by different methods.

The plan of the paper is as follows. In Section 1 we give some preliminaries on interpolation theory. In Section 2, also for background purposes, we briefly recapitulate the classical results and methods for $\{L_1, L_\infty\}$. In Section 3 we introduce and study the modified K-functionals, mentioned above. In Section 4 we establish our main results, having as a consequence the equivalence between (0.1) and (0.2). In Section 5 we investigate the relation between the constants c and C. In Section 6 we illustrate how our methods apply to the interpolation of Lorentz spaces. In the Appendix, finally, we derive a matrix lemma which plays a major role throughout the paper. This lemma generalizes a classical result of Hardy, Littlewood, and Polya on doubly stochastic matrices.

1. General background. As a general source for the theory of interpolation we refer to [4]. Here we recapitulate a few notions only.

Let A_0 and A_1 be two (quasi-) Banach spaces, both continuously imbedded in some Hausdorff topological vector space $\mathscr{A}$. They then constitute what is called a *(quasi-) Banach couple*, denoted by $\bar{A} = \{A_0, A_1\}$. (For a quasi-Banach space only holds a quasi-triangle inequality: $\|f+g\| \leqslant c(\|f\|+\|g\|)$, $c \geqslant 1$.) To $\bar{A}$ are associated the (quasi-) Banach spaces $\Sigma(\bar{A})$ and $\Delta(\bar{A})$, defined by the (quasi-) norms

$$\|f\|_{\Sigma(\bar{A})} = \inf_{f=f_0+f_1} (\|f_0\|_{A_0} + \|f_1\|_{A_1}),$$

$$\|f\|_{\Delta(\bar{A})} = \max(\|f\|_{A_0}, \|f\|_{A_1}).$$

Given two spaces A and B, we denote by $\mathscr{L}(A; B)$ the set of continuous linear operators $T\colon A \to B$, provided with the operator (quasi-) norm

$$\|T\|_{\mathscr{L}(A;B)} = \sup \|Tf\|_B/\|f\|_A.$$

Analogously, if $\bar{A}$ and $\bar{B}$ are two (quasi-) Banach couples we denote by $\mathscr{L}(\bar{A}; \bar{B})$ the set of operators $T\colon \Sigma(\bar{A}) \to \Sigma(\bar{B})$ such that $T \in \mathscr{L}(A_\mu, B_\mu)$, $\mu = 0, 1$. Here and in the sequel, by abuse of notation, we do not distinguish between an operator and its restrictions to various subspaces. A (quasi-) norm on $\mathscr{L}(\bar{A}; \bar{B})$ is defined by

$$\|T\|_{\mathscr{L}(\bar{A};\bar{B})} = \max_{\mu=0,1} \|T\|_{\mathscr{L}(A_\mu, B_\mu)}.$$

Denote by $\mathscr{L}_c(A; B)$ and $\mathscr{L}_c(\bar{A}; \bar{B})$ the respective balls of radius c, i.e.

$$T \in \mathscr{L}_c(A; B) \quad \text{iff} \quad T \in \mathscr{L}(A; B), \quad \|T\|_{\mathscr{L}(A;B)} \leqslant c,$$

$$T \in \mathscr{L}_c(\bar{A}; \bar{B}) \quad \text{iff} \quad T \in \mathscr{L}(\bar{A}; \bar{B}), \quad \|T\|_{\mathscr{L}(\bar{A};\bar{B})} \leqslant c.$$

If $A = B$, $\bar{A} = \bar{B}$, we simply write $\mathscr{L}(A)$, $\mathscr{L}(\bar{A})$, $\mathscr{L}_c(A)$, $\mathscr{L}_c(\bar{A})$.

Now consider spaces A, B such that, with continuous imbeddings,

$$\Delta(\bar{A}) \subset A \subset \Sigma(\bar{A}), \qquad \Delta(\bar{B}) \subset B \subset \Sigma(\bar{B}). \tag{1.1}$$

We say that A, B are *interpolation spaces* with respect to $\bar{A}, \bar{B}$ if

$$T \in \mathscr{L}(\bar{A}; \bar{B}) \quad \text{implies} \quad T \in \mathscr{L}(A; B), \quad \|T\|_{\mathscr{L}(A;B)} \leqslant c\|T\|_{\mathscr{L}(\bar{A};\bar{B})}$$

for some constant c, independent of T. By homogeneity this is equivalent to

$$\mathscr{L}_1(\bar{A}; \bar{B}) \subset \mathscr{L}_c(A; B) \tag{Int}$$

for some c. (In fact, by the closed graph thoerem, (Int) is equivalent to the seemingly less restrictive condition $\mathscr{L}(\bar{A}; \bar{B}) \subset \mathscr{L}(A; B)$.) When on places we want to emphasize a particular value of c, this is done by referring to the above inclusion as (c-Int). The case $c = 1$ is important in many applications. A, B are then called *exact interpolation spaces* with respect to $\bar{A}, \bar{B}$, obeying thus

$$\mathscr{L}_1(\bar{A}; \bar{B}) \subset \mathscr{L}_1(A; B). \tag{Ex Int}$$

If, in particular, $\bar{A} = \bar{B}$, $A = B$, we simply say that A is an (exact) interpolation space with respect to $\bar{A}$. This case will be referred to as the *diagonal case.*

Given $\bar{A}, \bar{B}$, the problem of characterizing the interpolation spaces A, B thus can be treated on two different levels, a non-exact and an exact one. In the former case one does not distinguish between spaces having equivalent norms. However, it can be shown ([1], p. 74) that to A, B obeying (Int) there exist equivalent spaces A', B' obeying (Ex Int). Hence, whichever level one works on, it suffices to consider the condition (Ex Int).

As was announced in the Introduction, the solution of the problem can be expressed in terms of the K-functional (0.3). To this end we define a quasi-order (relative to $\bar{A}, \bar{B}$)

$$g \leqslant f[K] \quad \text{iff} \quad K(t, g; \bar{B}) \leqslant K(t, f; \bar{A}), \quad t > 0. \tag{1.2}$$

Two spaces A, B obeying (1.1) then are said to be *K-monotonic* with respect to $\bar{A}, \bar{B}$ if

(1.3) $$f \in A,\ g \leqslant f[K] \quad \text{implies} \quad g \in B,\ \|g\|_B \leqslant C\|f\|_A$$

for some C, independent of f, g. Wanting to stress a particular value of C, they are said to be *$(C; K)$-monotonic*. If $C = 1$, i.e.

(1.4) $$f \in A,\ g \leqslant f[K] \quad \text{implies} \quad g \in B,\ \|g\|_B \leqslant \|f\|_A,$$

we speak about *exact K-monotonicity*.

The applicability of these notions depends on

LEMMA 1.1. *If $g = Tf$ with $T \in \mathscr{L}_1(\bar{A}; \bar{B})$, then $g \leqslant f[K]$.*

Proof. The assumption $T \in \mathscr{L}_1(\bar{A}; \bar{B})$ yields

$$K(t, g; \bar{B}) = K(t, Tf; \bar{B}) \leqslant \inf_{f=f_0+f_1} (\|Tf_0\|_{B_0} + t\|Tf_1\|_{B_1}) \leqslant \inf_{f=f_0+f_1} (\|f_0\|_{A_0} + t\|f_1\|_{A_1}) = K(t, f; \bar{A}),$$

i.e. $g \leqslant f[K]$. ∎

We now immediately get

THEOREM 1.1. *(ExInt) is a consequence of exact K-monotonicity.*

Proof. Let $T \in \mathscr{L}_1(\bar{A}; \bar{B})$. Then, by Lemma 1.1, $Tf \leqslant f[K]$ for every $f \in \Sigma(\bar{A})$. Hence, if A, B are exactly K-monotonic, $\|Tf\|_B \leqslant \|f\|_A$, i.e. $\|T\|_{\mathscr{L}(A;B)} \leqslant 1$. ∎

A natural question is to what extent and for which couples the converse of this theorem holds true. To describe that situation, we say that $\bar{A}, \bar{B}$ are *K-adequate* if the conditions (Int) and K-monotonicity are equivalent. If moreover (ExInt) and exact K-monotonicity are equivalent, they are said to be *exactly K-adequate*. In the diagonal case $\bar{A} = \bar{B}, A = B$, we simply say that $\bar{A}$ is (exactly) K-adequate. (In [8] the term "Calderón couple" was used in the same sense.)

Not every couple is K-adequate. Counterexamples were given in [25] (even a finite-dimensional one) and in [8] ($\{L_1, W_1^1\}$, where W_1^1 is a Sobolev space). What we intend to prove is that every two couples of weighted L_p-spaces are K-adequate, provided they refer to the same p's.

Using this fact, quite a general class of K-adequate couples was discovered by Cwikel [9]. In fact, for arbitrary $\bar{A} = \{A_0, A_1\}$, every couple $\{\bar{A}_{\theta_0 q_0}, \bar{A}_{\theta_1 q_1}\}$, $0 < \theta_0, \theta_1 < 1$, $1 \leqslant q_0, q_1 \leqslant \infty$, is K-adequate. Here, as usual, $\bar{A}_{\theta q}$ stands for the space defined by

$$\|f\|_{\bar{A}_{\theta q}} = \left(\int (t^{-\theta} K(t, f))^q \frac{dt}{t}\right)^{1/q}.$$

This is an exact interpolation space, by virtue of Theorem 1.1. For special examples of K-adequate couples, see Remark 6.1 below.

2. Review of the case $\{L_1, L_\infty\}$. Let us briefly recapitulate the results and methods in the case $\{L_1, L_\infty\}$. We restrict ourselves to the diagonal case and to functions defined on $\boldsymbol{R}$. The exact interpolation spaces then can be characterized in the following two, seemingly different, ways:

THEOREM M (Mitjagin [19]). *Let $\mathscr{G}$ be the set of operators of the form*

$$Tf(x) = \varepsilon(x) f(\gamma(x)),$$

where γ is a measure preserving bijection on $\boldsymbol{R}$ and $|\varepsilon(x)| = 1$. Then A is an exact interpolation space if and only if

$$\mathscr{G} \subset \mathscr{L}_1(A).$$

THEOREM C (Calderón [6], Cotlar [7]). *A is an exact interpolation space if and only if*

$$f \in A,\quad \int_0^t g^*(s)\,ds \leqslant \int_0^t f^*(s)\,ds \text{ for } t > 0 \quad \text{implies} \quad g \in A,\ \|g\|_A \leqslant \|f\|_A.$$

(Here f^* stands for the non-increasing rearrangement of $|f|$, i.e. $f^*\colon R_+ \to R_+$ is non-increasing and $\operatorname{meas}\{x \mid f^*(x) > t\} = \operatorname{meas}\{x \mid |f(x)| > t\}$, $t > 0$.)

It is well known that

$$K(t, f; \{L_1, L_\infty\}) = \int_0^t f^*(s)\,ds.$$

Hence Theorem C may be restated as

COROLLARY. *$\{L_1, L_\infty\}$ is exactly K-adequate.*

Searching for a link between these theorems, by approximation with simple functions one reduces to the n-dimensional case $\bar{A} = \{l_1^{(n)}, l_\infty^{(n)}\}$. We note that a matrix $T = (t_{ij})$ belongs to $\mathscr{L}_1(\bar{A})$ if and only if

(2.1) $$\|T\|_{l_1^{(n)}} = \max_j \sum_i |t_{ij}| \leqslant 1, \qquad \|T\|_{l_\infty^{(n)}} = \max_i \sum_j |t_{ij}| \leqslant 1.$$

As an important subset we recognize the *doubly stochastic matrices* $\mathscr{D}$, defined by

$$t_{ij} \geqslant 0, \qquad \sum_i t_{ij} = \sum_j t_{ij} = 1 \qquad (i, j = 1, \ldots, n).$$

$\mathscr{D}$ in turn contains as a subset the permutation matrices $\mathscr{P}$.

There exists an extensive literature about doubly stochastic matrices, cf. the survey [18]. The two major theorems are

THEOREM B (Birkhoff). *$\mathscr{D}$ = the convex hull of $\mathscr{P}$.*

THEOREM HLP (Hardy, Littlewood, Pólya). *Let $f = (f_1, \ldots, f_n)$,*

$g = (g_1, \ldots, g_n)$ *with* $f_i, g_i \geqslant 0$ $(i = 1, \ldots, n)$. *Then*

$$g = Tf \quad \textit{with} \quad T \in \mathscr{D} \quad \textit{iff} \quad \sum_1^k g_i^* \leqslant \sum_1^k f_i^* \quad (k = 1, \ldots, n-1),$$
$$\sum_1^n g_i^* = \sum_1^n f_i^*.$$

These theorems are closely related to the interpolation theorems above. In fact, since $\mathscr{L}_1(\bar{A})$ is convex, in verifying (Ex Int) it suffices to consider its set $\mathscr{G}$ of extreme points. But by means of Theorem B it is easily verified that $\mathscr{G} = \varepsilon\mathscr{P}$ with $\varepsilon = (\pm 1 \ldots \pm 1)$ (n-row matrix). In this way we obtain the finite-dimensional analogue of Theorem M.

On the other hand, turning to Theorem C, we rewrite (Ex Int) as

$$g = Tf \text{ with } T \in \mathscr{L}_1(\bar{A}) \quad \text{implies } \|g\|_A \leqslant \|f\|_A. \tag{2.2}$$

It suffices to consider f, g and T with non-negative elements. It is readily seen that any such matrix $T \in \mathscr{L}_1(\bar{A})$ is submatrix of a $\mathscr{D}$-matrix of order $2n$. Then by means of the $2n$-dimensional version of Theorem HLP it follows that in (2.2) the condition "$g = Tf$ with $T \in \mathscr{L}_1(\bar{A})$" is equivalent to "$\sum_1^k g_i^* \leqslant \sum_1^k f_i^*$ $(k = 1, \ldots, n)$". This proves Theorem C in the finite-dimensional case.

In treating the general case $\{L_{p_0a_0}, L_{p_1a_1}\}$ (where a_0 and a_1 are weight functions) we choose the latter of the above two approaches. Then in particular we need a result of HLP-type, i.e. necessary and sufficient conditions for "$g = Tf$ with $T \in \mathscr{L}_1$". A general necessary condition, coinciding with that of Theorem HLP if $\bar{A} = \{l_1^{(n)}, l_\infty^{(n)}\}$, was formulated in Lemma 1.1. A sufficient condition, less trivial, will be given in Lemma 4.2 below. Here the proof is founded on a matrix lemma of the Appendix. As a matter of fact the latter one generalizes Theorem HLP too, but in quite other directions.

3. The functionals $K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$. Now consider couples of the form

$$\bar{A} = \{L_{p_0a_0}(a, \mathscr{R}, X), L_{p_1a_1}(a, \mathscr{R}, X)\} = \{L_{p_0a_0}, L_{p_1a_1}\}$$

where $0 < p_0, p_1 < \infty$ and $(a, \mathscr{R}, X)$ is an arbitrary measure space. Here by L_{pa} is meant the space defined by the (quasi-)norm

$$\|f\|_{pa} = \Big(\int_X |fa|^p da\Big)^{1/p}.$$

As is well known, hereby also the seemingly more general case

$$\bar{A} = \{L_{p_0}(a_0, \mathscr{R}, X), L_{p_1}(a_1, \mathscr{R}, X)\}$$

is covered. In fact, since a_0 and a_1 both are absolutely continuous with respect to $a_0 + a_1 = a$, by the Radon–Nikodym theorem there exist a_0 and a_1 such that $da_0 = a_0^{p_0} da$, $da_1 = a_1^{p_1} da$.

To make certain that $\bar{A}$ is a (quasi-)Banach couple, we claim that $a_\mu > 0$ $(\mu = 0,1)$, or, equivalently, that the measures a_0 and a_1 are absolutely continuous with respect to each other. However, after minor modifications also the semi-normed case $a_\mu \geqslant 0$ $(\mu = 0, 1)$ can be included in our treatment.

For couples like $\bar{A}$ it is advantageous besides K to work with

$$K_{\bar{p}}(t, f) = K_{\bar{p}}(t, f; \bar{A}) = \inf_{f = f_0 + f_1} \big(\|f_0\|_{p_0a_0}^{p_0} + t\|f_1\|_{p_1a_1}^{p_1}\big), \qquad \bar{p} = (p_0, p_1),$$

cf. [22] where this functional was denoted by L. Its usefulness depends on the formula

$$K_{\bar{p}}(t, f) = \int_X \inf_{f(x) = f_0(x) + f_1(x)} \big(|f_0(x)a_0(x)|^{p_0} + t|f_1(x)a_1(x)|^{p_1}\big) da. \tag{3.1}$$

A more explicit formula (without "inf") is derived in Lemma 3.4 below. We shall also use another functional, similar to $K_{\bar{p}}$,

$$\mathscr{K}_{\bar{p}}(t, f) = \mathscr{K}_{\bar{p}}(t, f; \bar{A}) = \int_X \min\big(|f(x)a_0(x)|^{p_0}, t|f(x)a_1(x)|^{p_1}\big) da. \tag{3.2}$$

LEMMA 3.1. *$K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$ are equivalent. More precisely,*

$$\varkappa_{\bar{p}} \mathscr{K}_{\bar{p}}(t, f) \leqslant K_{\bar{p}}(t, f) \leqslant \mathscr{K}_{\bar{p}}(t, f), \qquad t > 0,$$

where $\varkappa_{\bar{p}} = \inf_{x + y = 1} (x^{p_0} + y^{p_1})$. *If* $0 < p_0, p_1 \leqslant 1$, *then* $K_{\bar{p}} = \mathscr{K}_{\bar{p}}$.

Proof. It suffices to consider pointwise the scalar-valued integrands in (3.1) and (3.2). The right inequality then is obvious. The left one follows from

$$f_0^{p_0} + tf_1^{p_1} = \Big(\frac{f_0}{f}\Big)^{p_0} f^{p_0} + t\Big(\frac{f_1}{f}\Big)^{p_1} f^{p_1} \geqslant \bigg(\Big(\frac{f_0}{f}\Big)^{p_0} + \Big(\frac{f_1}{f}\Big)^{p_1}\bigg) \min(f^{p_0}, tf^{p_1}),$$

i.e.

$$\inf_{f = f_0 + f_1} (f_0^{p_0} + tf_1^{p_1}) \geqslant \varkappa_{\bar{p}} \min(f^{p_0}, tf^{p_1}).$$

Since $\varkappa_{\bar{p}} = 1$ if $0 < p_0, p_1 \leqslant 1$, $K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$ coincide in that case. ■

Now let

$$\bar{B} = \{L_{p_0b_0}(\beta, \mathscr{S}, Y), L_{p_1b_1}(\beta, \mathscr{S}, Y)\} = \{L_{p_0b_0}, L_{p_1b_1}\}$$

be another (quasi-) Banach couple, assigned to the same $\bar{p}$. In analogy with (1.2), using $K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$ instead of K, we define the quasi-orders $g \leqslant f[K_{\bar{p}}]$ and $g \leqslant f[\mathscr{K}_{\bar{p}}]$. In an obvious fashion, replacing in (1.3) K by $K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$, respectively, we obtain the definitions of $K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$-

monotonicity, with the different prefices. In analogy with the K-case we also define the notions of (exact) $K_{\bar p}$ and $\mathscr{K}_{\bar p}$-*adequateness.*

LEMMA 3.2. *Generally holds:*

(i) *The concepts of* $K, K_{\bar p}$ *and* $\mathscr{K}_{\bar p}$-*monotonicity coincide.*

In particular cases moreover hold:

(ii) *If* $1 \leqslant p_0, p_1 < \infty$, *then exact* K *and exact* $K_{\bar p}$-*monotonicity coincide.*

(iii) *If* $0 < p_0, p_1 \leqslant 1$, *then exact* $K_{\bar p}$ *and exact* $\mathscr{K}_{\bar p}$-*monotonicity coincide.*

This lemma is an immediate consequence of

LEMMA 3.3. *Generally holds:*

(i) *The following statements are equivalent:*

$$g \leqslant cf[K] \quad \text{for some } c > 0, \tag{3.3}$$

$$g \leqslant cf[K_{\bar p}] \quad \text{for some } c > 0, \tag{3.4}$$

$$g \leqslant cf[\mathscr{K}_{\bar p}\} \quad \text{for some } c > 0. \tag{3.5}$$

In particular cases moreover hold:

(ii) *If* $1 \leqslant p_0, p_1 < \infty$, *then* $g \leqslant f[K]$ *iff* $g \leqslant f[K_{\bar p}]$.

(iii) *If* $0 < p_0, p_1 \leqslant 1$, *then* $g \leqslant f[K_{\bar p}]$ *iff* $g \leqslant f[\mathscr{K}_{\bar p}]$.

Proof. Part (iii) and the equivalence between (3.4) and (3.5) are immediate consequences of Lemma 3.1. What remains are the statements about K and $K_{\bar p}$. To verify them, we use the functional

$$E(s) = E(s, f) = \inf_{\|f_0\|_{p_0 a_0} \leqslant s} \|f - f_0\|_{p_1 a_1}.$$

There is a close connection between K and E, cf. [24], notably Proposition 4.4. In fact,

$$K(t) = K(t, f) = \inf_s \big(s + tE(s)\big), \tag{3.6}$$

$$K_{\bar p}(t) = K_{\bar p}(t, f) = \inf_s \big(s^{p_0} + tE^{p_1}(s)\big) = \inf_s \big(s + tE^{p_1}(s^{1/p_0})\big). \tag{3.7}$$

Put

$$E_{\bar p}(s) = E^{p_1}(s^{1/p_0}).$$

The formulas (3.6) and (3.7) then state that K and $K_{\bar p}$ are Legendre transforms of E and $E_{\bar p}$, respectively.

Beginning with (ii), we assume that $1 \leqslant p_0, p_1 < \infty$. Then, as is readily verified, $E(s)$ is convex and decreasing. This is in fact the case with $E_{\bar p}$ too. To verify that, we use in order the concavity of $x \mapsto x^{1/p_0}$ and the decreasingness of E, the convexity of E, the convexity of $x \mapsto x^{p_1}$, thus obtaining, with $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} E_{\bar p}(\lambda_1 s_1 + \lambda_2 s_2) &= E^{p_1}\big((\lambda_1 s_1 + \lambda_2 s_2)^{1/p_0}\big) \leqslant E^{p_1}(\lambda_1 s_1^{1/p_0} + \lambda_2 s_2^{1/p_0}) \\ &\leqslant \big(\lambda_1 E(s_1^{1/p_0}) + \lambda_2 E(s_2^{1/p_0})\big)^{p_1} \leqslant \lambda_1 E^{p_1}(s_1^{1/p_0}) + \lambda_2 E^{p_1}(s_2^{1/p_0}) \\ &= \lambda_1 \big(E_{\bar p}(s_1)\big) + \lambda_2 E_{\bar p}(s_2). \end{aligned}$$

This proves the convexity. Under these circumstances, making an inverse Legendre transformation in (3.6) and (3.7), we get

$$E(s) = \sup_t \left(\frac{K(t)}{t} - \frac{s}{t}\right), \tag{3.8}$$

$$E_{\bar p}(s) = \sup_t \left(\frac{K_{\bar p}(t)}{t} - \frac{s}{t}\right). \tag{3.9}$$

Now by virtue of (3.6)–(3.9) we have the chain of equivalences

$$\begin{aligned} &K(t, g) \leqslant K(t, f),\ t > 0 \ \Leftrightarrow\ E(s, g) \leqslant E(s, f),\ s > 0 \\ &\Leftrightarrow\ E_{\bar p}(s, g) \leqslant E_{\bar p}(s, f),\ s > 0\ \Leftrightarrow\ K_{\bar p}(t, g) \leqslant K_{\bar p}(t, f),\ t > 0. \end{aligned}$$

This proves (ii).

Turning to the general case $0 < p_0, p_1 < \infty$, E and $E_{\bar p}$ need not be convex. Consequently, the inverse transformation applied to (3.6) and (3.7) does not, in general, give back E and $E_{\bar p}$. Instead we get their greatest convex minorants E^* and $E^*_{\bar p}$. However, taking into account that

$$E^*(s) \leqslant E(s) \leqslant 2E^*(s/2),$$

$$E^*_{\bar p}(s) \leqslant E_{\bar p}(s) \leqslant 2E^*_{\bar p}(s/2),$$

by an argument similar to that above the equivalence between (3.3) and (3.4) can be proved. We omit the details. ∎

Remark 3.1. The definitions of $K, K_{\bar p}$ and $E_{\bar p}$ have significance for general quasi-Banach couples. With no changes the proof above applies to this general situation. Part (ii) of the lemma then generalizes a result of [14] for the case $p_0 = p_1$.

The next section will motivate a closer study of the relationship between $K_{\bar p}$ and $\mathscr{K}_{\bar p}$-monotonicity. Such investigations are then carried out in Section 5. They are based on the following integral representation of $K_{\bar p}$.

LEMMA 3.4. *Let* $1 \leqslant p_0, p_1 < \infty$. *Put*

$$\Delta_{\bar p}(\sigma) = \frac{p_1}{p_0} \cdot \frac{(1 - \sigma^{1/p_0})^{p_1 - 1}}{\sigma^{1 - 1/p_0}}. \tag{3.10}$$

Then

$$\inf_{\sigma+\tau=a}(\sigma^{p_0}+t\tau^{p_1}) = \int_0^1 \min\big(p_0 a^{p_0}\sigma^{p_0-1},\ tp_1 a^{p_1}(1-\sigma)^{p_1-1}\big)\,d\sigma = \int_0^1 \min\big(a^{p_0},\ t\Delta_{\bar p}(\sigma)a^{p_1}\big)\,d\sigma, \tag{3.11}$$

$$K_{\bar p}(t,f) = \int_0^1 \mathscr{K}_{\bar p}\big(t\Delta_{\bar p}(\sigma), f\big)\,d\sigma. \tag{3.12}$$

Proof. Here (3.12) is a consequence of (3.11) and (3.1). In fact,

$$\begin{aligned} K_{\bar p}(t,f) &= \int_X \inf_{f(x)=f_0(x)+f_1(x)} (|f_0 a_0|^{p_0}+t|f_1 a_1|^{p_1})\,d\alpha \\ &= \int_X \Big(\int_0^1 \min\big(|fa_0|^{p_0},\ t\Delta_{\bar p}(\sigma)|fa_1|^{p_1}\big)\,d\sigma\Big)\,d\alpha \\ &= \int_0^1 \Big(\int_X \min\big(|fa_0|^{p_0},\ t\Delta_{\bar p}(\sigma)|fa_1|^{p_1}\,d\alpha\Big)\,d\sigma \\ &= \int_0^1 \mathscr{K}_{\bar p}\big(t\Delta_{\bar p}(\sigma), f\big)\,d\sigma. \end{aligned}$$

In verifying (3.11), without restrictions we may assume $a = 1$. Put

$$k(s) = \inf_{\sigma+\tau=1} (\sigma^{p_0}+s\tau^{p_1}) = \inf_{0\leqslant\sigma\leqslant 1}\big(\sigma^{p_0}+s(1-\sigma)^{p_1}\big). \tag{3.13}$$

Being an infimum of positive concave functions on $\boldsymbol{R}_+$, vanishing at the origin, k itself has these properties. Moreover $k(s) < 1$, so that $k(s)/s \to 0$, $s\to\infty$. By partial integrations one then verifies the formula

$$k(t) = -\int_0^\infty \min(s,t)\,dk'(s), \tag{3.14}$$

cf. formula (A.5) of the Appendix and the reference given there.

We now restrict ourselves to the case $p_0, p_1 > 1$. The remaining cases need some minor modifications, omitted here. To get an expression for k' in (3.14), first consider k. A derivation yields that the infimum in (3.13) is attained for σ obeying

$$p_0\sigma^{p_0-1} = sp_1(1-\sigma)^{p_1-1}. \tag{3.15}$$

This formula defines a bijection between $0 < s < \infty$ and $0 < \sigma < 1$. Expressing k and k' in terms of σ, we get

$$k(s) = \sigma^{p_0}+s(1-\sigma)^{p_1},$$

$$k'(s) = \frac{dk}{d\sigma}\Big/\frac{ds}{d\sigma} = \frac{p_0\sigma^{p_0-1}-sp_1(1-\sigma)^{p_1-1}+\dfrac{ds}{d\sigma}(1-\sigma)^{p_1}}{ds/d\sigma} = (1-\sigma)^{p_1}.$$

Thus, making the change of variables (3.15) in (3.14), we obtain

$$\begin{aligned} k(t) &= -\int_0^1 \min\Big(\frac{p_0\sigma^{p_0-1}}{p_1(1-\sigma)^{p_1-1}},\ t\Big)\,d(1-\sigma)^{p_1} \\ &= \int_0^1 \min\big(p_0\sigma^{p_0-1},\ tp_1(1-\sigma)^{p_1-1}\big)\,d\sigma, \end{aligned}$$

i.e. the first equality in (3.11). Making another change of variables, we get the second one

$$\begin{aligned} k(t) &= \int_0^1 \min\Big(1,\ t\,\frac{p_1(1-\sigma)^{p_1-1}}{p_0\sigma^{p_0-1}}\Big)\,d\sigma^{p_0} \\ &= \int_0^1 \min\big(1,\ t\Delta_{\bar p}(\sigma)\big)\,d\sigma. \ \blacksquare \end{aligned}$$

4. The general case $\{L_{p_0a_0}, L_{p_1a_1}\}$, $0 < p_0, p_1 < \infty$.

As in the preceding section, when not otherwise stated, let

$$\bar A = \{L_{p_0a_0}(\alpha,\mathscr{R},X),\ L_{p_1a_1}(\alpha,\mathscr{R},X)\} = \{L_{p_0a_0}, L_{p_1a_1}\},$$
$$\bar B = \{L_{p_0b_0}(\beta,\mathscr{S},Y),\ L_{p_1b_1}(\beta,\mathscr{S},Y)\} = \{L_{p_0b_0}, L_{p_1b_1}\}.$$

A sufficient condition for A, B to be interpolation spaces with respect to these couples was given in Theorem 1.1 above. However, for comparison with the necessary condition to be derived below, it is convenient to restate it by means of $K_{\bar p}$ instead of K.

THEOREM 4.1. (ExInt) *is a consequence of exact $K_{\bar p}$-monotonicity.*

The proof is analogous to that of Theorem 1.1, depending on

LEMMA 4.1. *If $g = Tf$ with $T\in\mathscr{L}_1(\bar A;\bar B)$, then $g\leqslant f[K_{\bar p}]$.*

Below a necessary condition is derived in two (overlapping) cases:
(i) $1\leqslant p_0, p_1 < \infty$ with arbitrary measure spaces,
(ii) $0 < p_0, p_1 < \infty$ with the following restrictions imposed on the measures:

(M.1) *$(\beta,\mathscr{S},Y)$ is non-atomic,*

(M.2) *For every $D\in\mathscr{R}$, $E\in\mathscr{S}$ with $\alpha(D),\beta(E) < \infty$, the normalized measure spaces $(\alpha/\alpha(D),\mathscr{R}_D,D)$ and $(\beta/\beta(E),\mathscr{S}_E,E)$ are isomorphic.*

Here the latter condition means that there exists a measurable bijection $\pi\colon D\to E$ such that for every measurable set $D'\subset D$,

$$\frac{\alpha(D')}{\alpha(D)} = \frac{\beta(\pi D')}{\beta(E)}.$$

We note that if $X = Y = \boldsymbol{R}$, $\alpha = \beta =$ Lebesgue measure, (M.1) and (M.2) are both obeyed. Consequently they also are whenever X and Y are isomorphic to $\boldsymbol{R}$ (or some subinterval of $\boldsymbol{R}$). This in turn is guaranteed by their being separable, non-atomic and σ-finite (cf. e.g. [12], p. 173). Also note that if X is discrete, then (M.2) is automatically fulfilled.

The case $p = \infty$ is commented upon in Remark 5.1 below.

We are now prepared to state the main result:

THEOREM 4.2. *Under the hypotheses of* (i) *or* (ii), (ExInt) *implies exact $\mathscr{K}_{\bar{p}}$-monotonicity.*

Theorems 4.1 and 4.2 now yield, by virtue of Lemma 3.2.

COROLLARY 4.1. *$\bar{A}, \bar{B}$ are K-adequate.*

COROLLARY 4.2. *If $0 < p_0, p_1 \leqslant 1$ and* (M.1), (M.2) *are obeyed, then $\bar{A}, \bar{B}$ are exactly $\mathscr{K}_{\bar{p}}$-adequate.*

Theorem 4.2 is a consequence of the following lemma. When combined with Lemma 4.1 we get the generalization of Theorem HLP inquired for at the end of Section 2, cf. also Remark 4.1 below.

LEMMA 4.2. *Under the hypotheses of* (i) *or* (ii), *if $g \leqslant f[\mathscr{K}_{\bar{p}}]$ then to every $\varrho > 1$ there exists $T \in \mathscr{L}_\varrho(\bar{A}; \bar{B})$ such that $Tf = g$.*

Proof of Theorem 4.2. Let A, B obey (ExInt) and $g \leqslant f[\mathscr{K}_{\bar{p}}]$. For $\varrho > 1$, let $T = T_\varrho$ be given by the lemma. Then

$$\|g\|_B = \|T_\varrho f\|_B \leqslant \varrho \|f\|_A.$$

Since this holds for any $\varrho > 1$, $\|g\|_B \leqslant \|f\|_A$. Hence A, B are exactly $\mathscr{K}_{\bar{p}}$-monotonic. ■

Before proving Lemma 4.2 we introduce some more notations. Let χ_D denote the characteristic function of the set D. For arbitrary functions f we write $f_D = f\chi_D$. We consider functions $f = \sum_{i\in I} f_i \chi_{F_i}$, $I \subset \boldsymbol{Z}$, with the property that the values $\{f_i\}$ can be renumbered in increasing order. Such functions are called *elementary* or, if I is finite, *simple*.

Proof of Lemma 4.1. The disposition is as follows: In Step 1 we interpret the condition $g \leqslant f[\mathscr{K}_{\bar{p}}]$ for certain elementary functions f, g, a_μ, b_μ ($\mu = 0, 1$). In Steps 2(i) and 2(ii), for such functions we construct in the respective cases the operator T. Here we even get $T \in \mathscr{L}_1(\bar{A}; \bar{B})$. In Step 3 is performed the passage from elementary to arbitrary functions. Steps 1 and 3 are common for (i) and (ii).

It suffices to consider the case $f > 0$, $g > 0$.

Step 1. Suppose that f, g, a_μ, b_μ ($\mu = 0, 1$) all are elementary. Hence

$$f = \sum_{i\in I} f_i \chi_{F_i} = \sum_{i\in I} f_{F_i}, \qquad g = \sum_{j\in J} g_j \chi_{G_j} = \sum_{j\in J} g_{G_j}.$$

Without restrictions we may assume that a_0, a_1 and b_0, b_1 are constant on F_i ($i \in I$) and G_j ($j \in J$), respectively. For the $\mathscr{K}_{\bar{p}}$-functional we then have

$$\begin{aligned}\mathscr{K}_{\bar{p}}(t, f; \bar{A}) &= \int \min(|fa_0|^{p_0}, t|fa_1|^{p_1})\, d\alpha \\ &= \sum_i \int \min\big((fa_0)^{p_0}_{F_i}, t(fa_1)^{p_1}_{F_i}\big)\, d\alpha \\ &= \sum_i \min\Big(\int (fa_0)^{p_0}_{F_i} d\alpha, t\int (fa_1)^{p_1}_{F_i} d\alpha\Big) \\ &= \sum_i \min(\|f_{F_i}\|^{p_0}_{p_0 a_0}, t\|f_{F_i}\|^{p_1}_{p_1 a_1}),\end{aligned}$$

analogously for $\mathscr{K}_{\bar{p}}(t, g; \bar{B})$. The condition $g \leqslant f[\mathscr{K}_{\bar{p}}]$ thus means

$$\sum_j \min(\|g_{G_j}\|^{p_0}_{p_0 b_0}, t\|g_{G_j}\|^{p_1}_{p_1 b_1}) \leqslant \sum_i \min(\|f_{F_i}\|^{p_0}_{p_0 a_0}, t\|f_{F_i}\|^{p_1}_{p_1 a_1}), \qquad t > 0.$$

Now suppose that each set of plane vectors $(\|f_{F_i}\|^{p_0}_{p_0 a_0}, \|f_{F_i}\|^{p_1}_{p_1 a_1})$, $i \in I$, and $(\|g_{G_j}\|^{p_0}_{p_0 b_0}, \|g_{G_j}\|^{p_1}_{p_1 b_1})$, $j \in J$, can be arranged in non-decreasing order with respect to the order relation $\underline{\varepsilon}$ of the Appendix. Then Lemma A.2 applies. It provides us with numbers $\theta_{ij} \geqslant 0$ such that

$$\sum_{i\in I} \theta_{ij} = 1 \quad (j \in J), \tag{4.1}$$

$$\sum_{j\in J} \theta_{ij} \|g_{G_j}\|^{p_\mu}_{p_\mu b_\mu} \leqslant \|f_{F_i}\|^{p_\mu}_{p_\mu a_\mu} \quad (\mu = 0, 1,\ i \in I). \tag{4.2}$$

These relations shall now be used in constructing the operators.

Step 2(i). Under the hypothesis (i), for $\varphi \in \sum(\bar{A})$ we define

$$T\varphi = \sum_{i,j} \theta_{ij} \Big(\frac{1}{\alpha(F_i)} \int_{F_i} \frac{\varphi}{f}\, d\alpha\Big) g_{G_j}.$$

Then obviously, by (4.1), $Tf = g$. We prove that $T \in \mathscr{L}_1(\bar{A}; \bar{B})$. To begin with, (4.1) and the convexity of $x \mapsto x^{p_\mu}$ ($\mu = 0, 1$) yield

$$\begin{aligned}\|T\varphi\|^{p_\mu}_{p_\mu b_\mu} &= \sum_j \Big|\sum_i \theta_{ij} \Big(\frac{1}{\alpha(F_i)} \int_{F_i} \frac{\varphi}{f}\, d\alpha\Big)\Big|^{p_\mu} \|g_{G_j}\|^{p_\mu}_{p_\mu b_\mu} \\ &\leqslant \sum_j \sum_i \theta_{ij} \Big(\frac{1}{\alpha(F_i)} \int_{F_i} \Big|\frac{\varphi}{f}\Big|\, d\alpha\Big)^{p_\mu} \|g_{G_j}\|^{p_\mu}_{p_\mu b_\mu}.\end{aligned} \tag{4.3}$$

But here, again by the convexity and the fact that f and a_μ ($\mu = 0, 1$) are constant on each F_i ($i \in I$),

$$\begin{aligned}\Big(\frac{1}{\alpha(F_i)} \int_{F_i} \Big|\frac{\varphi}{f}\Big|\, d\alpha\Big)^{p_\mu} &\leqslant \frac{1}{\alpha(F_i)} \int_{F_i} \Big|\frac{\varphi}{f}\Big|^{p_\mu} d\alpha \\ &= \frac{1}{\alpha(F_i)} \int_{F_i} \Big|\frac{\varphi a_\mu}{f a_\mu}\Big|^{p_\mu} d\alpha = \frac{1}{\alpha(F_i)(fa_\mu)^{p_\mu}_{F_i}} \int_{F_i} |\varphi a_\mu|^{p_\mu} d\alpha \\ &= \|\varphi_{F_i}\|^{p_\mu}_{p_\mu a_\mu} / \|f_{F_i}\|^{p_\mu}_{p_\mu a_\mu} \quad (\mu = 0, 1).\end{aligned}$$

Inserting this in (4.3), we get, by virtue of (4.2),

$$\|T\varphi\|_{p_\mu b_\mu}^{p_\mu} \leqslant \sum_i \|\varphi_{F_i}\|_{p_\mu a_\mu}^{p_\mu} \sum_j \theta_{ij} \|g_{G_j}\|_{p_\mu b_\mu}^{p_\mu} / \|f_{F_i}\|_{p_\mu a_\mu}^{p_\mu}$$
$$\leqslant \sum_i \|\varphi_{F_i}\|_{p_\mu a_\mu}^{p_\mu} = \|\varphi\|_{p_\mu a_\mu}^{p_\mu} \quad (\mu = 0, 1).$$

Hence $T \in \mathscr{L}_1(\bar{A}; \bar{B})$.

Step 2(ii). We make use of the following fact: Let T be a mapping defined for all characteristic functions χ_D, $D \in \mathscr{R}$, with values in $\sum(\bar{B})$, such that

(4.4) $$T\chi_{D_1 \cup D_2} = T\chi_{D_1} + T\chi_{D_2} \quad \text{if} \quad D_1 \cap D_2 = \emptyset,$$

(4.5) $$\operatorname{supp} T\chi_{D_1} \cap \operatorname{supp} T\chi_{D_2} = \emptyset \quad \text{if} \quad D_1 \cap D_2 = \emptyset,$$

(4.6) $$\|T\chi_D\|_{p_\mu b_\mu} \leqslant \|\chi_D\|_{p_\mu a_\mu} \quad (\mu = 0, 1).$$

Then, extending T by linearity to simple functions and thereafter by continuity to $\sum(\bar{A})$, we obtain an operator $T \in \mathscr{L}_1(\bar{A}; \bar{B})$. Thus, to verify the lemma it suffices to construct T obeying (4.4)–(4.6) and, in addition, $Tf = g$. (Also note that T constructed this way gets the property $\operatorname{supp} T\varphi \cap \cap \operatorname{supp} T\psi = \emptyset$ if $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$.)

We begin by cutting the sets G_j ($j \in J$) into pieces. By virtue of (M.1) and (4.1) there exist sets G_{ij} such that, disjointly,

$$G_j = \bigcup_i G_{ij} \quad \text{with } \beta(G_{ij}) = \theta_{ij}\beta(G_j) \ (i \in I,\ j \in J)$$

(cf. [13], p. 174). Since g and b_μ ($\mu = 0, 1$) are constant on G_{ij}, it follows that

$$\theta_{ij} \int_{G_j} |gb_\mu|^{p_\mu} d\beta = \int_{G_{ij}} |gb_\mu|^{p_\mu} d\beta,$$

i.e.

$$\theta_{ij} \|g_{G_j}\|_{p_\mu b_\mu}^{p_\mu} = \|g_{G_{ij}}\|_{p_\mu b_\mu}^{p_\mu} \quad (\mu = 0, 1,\ i \in I,\ j \in J).$$

Hence, by (4.2),

(4.7) $$\sum_j \|g_{G_{ij}}\|_{p_\mu b_\mu}^{p_\mu} \leqslant \|f_{F_i}\|_{p_\mu a_\mu}^{p_\mu} \quad (\mu = 0, 1,\ i \in I).$$

Now put

$$\Gamma_i = \bigcup_j G_{ij} \quad (i \in I).$$

These sets are pairwise disjoint. Thus, defining T so that

$$\operatorname{supp} T\chi_D \subset \Gamma_i \quad \text{if} \quad D \subset F_i,$$

it suffices to verify (4.4)–(4.6) for $D \subset F_i$ with $i \in I$ fixed. Since f is constant on F_i, we may use f_D instead of χ_D.

To this end, for fixed i, let $\pi_{ij}\colon F_i \to G_{ij}$ be chosen in accordance with (M.2). For $D \subset F_i$, put

$$q_D = \frac{\alpha(D)}{\alpha(F_i)} = \frac{\beta(\pi_{ij}D)}{\beta(G_{ij})} \quad (j \in J).$$

Then

(4.8) $$\|f_D\|_{p_\mu a_\mu}^{p_\mu} = q_D \|f_{F_i}\|_{p_\mu a_\mu}^{p_\mu}, \quad \|g_{\pi_{ij}D}\|_{p_\mu b_\mu}^{p_\mu} = q_D \|g_{G_{ij}}\|_{p_\mu b_\mu}^{p_\mu}.$$

Now define (cf. Fig. 4.1)

$$Tf_D = \sum_j g_{\pi_{ij}D}.$$

Then (4.4) and (4.5) are satisfied. To verify (4.6), by virtue of (4.7) and (4.8) we have

$$\|Tf_D\|_{p_\mu b_\mu}^{p_\mu} = \sum_j \|g_{\pi_{ij}D}\|_{p_\mu b_\mu}^{p_\mu} = q_D \sum_j \|g_{G_{ij}}\|_{p_\mu b_\mu}^{p_\mu}$$
$$\leqslant q_D \|f_{F_i}\|_{p_\mu a_\mu}^{p_\mu} = \|f_D\|_{p_\mu a_\mu}^{p_\mu} \quad (\mu = 0, 1).$$

Finally, since $Tf_{F_i} = g_{\Gamma_i}$, we also get $Tf = g$. This proves the assertion.

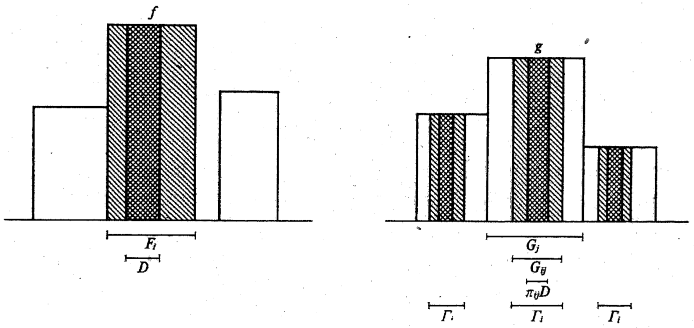

Fig. 4.1

Step 3. Now consider the general case, dropping thus the assumption the functions to be elementary. In order to reduce to that case, for $\varrho > 1$ and positive functions φ we define the discretizing operator $\varphi \mapsto \varphi^\varrho$ by

$$\varphi^\varrho(x) = \varrho^k \quad \text{if} \quad \varrho^k \leqslant \varphi(x) < \varrho^{k+1} \quad (k \in \mathbf{Z}).$$

Then

(4.9) $$\varphi^\varrho \leqslant \varphi \leqslant \varrho\varphi^\varrho.$$

For technical reasons we also have to work with ϱ_μ defined by

$$\varrho_\mu = \varrho^{(p_1-p_0)/p_\mu} \quad \text{if} \quad p_1 > p_0, \qquad \varrho_\mu = \varrho \quad \text{if} \quad p_1 = p_0 \qquad (\mu = 0,1).$$

Put

$$\tilde\varrho = \max(\varrho, \varrho_0, \varrho_1).$$

Let

$$\bar A^\varrho = \{L_{p_0 a_0^{\varrho_0}}, L_{p_1 a_1^{\varrho_1}}\}, \qquad \bar B^\varrho = \{L_{p_0 b_0^{\varrho_0}}, L_{p_1 b_1^{\varrho_1}}\}.$$

By (4.9) and the hypothesis $g \leqslant f[\mathcal{K}_{\bar p}]$, we have

$$\mathcal{K}_{\bar p}(t, g^\varrho; \bar B^\varrho) \leqslant \mathcal{K}_{\bar p}(t, g; \bar B) \leqslant \mathcal{K}_{\bar p}(t, f; \bar A) \leqslant \mathcal{K}_{\bar p}(t, \tilde\varrho^2 f^\varrho; \bar A^\varrho), \qquad t > 0,$$

i.e. $g^\varrho \leqslant \tilde\varrho^2 f^\varrho[\mathcal{K}_{\bar p}]$ with respect to $\bar A^\varrho, \bar B^\varrho$. Here all functions involved are elementary. To use the first two parts of the proof we also have to verify the monotonicity assumption made in the course of Step 1. To this end, considering f, let F_i $(i \in I)$ be common sets of constancy for f^ϱ and $a_0^{\varrho_0}$, $a_1^{\varrho_1}$. Then, for some integers k, k_0, k_1, the inclination coefficients (cf. the Appendix) of the vectors $(\|f^\varrho_{F_i}\|^{p_0}_{p_0 a_0^{\varrho_0}}, \|f^\varrho_{F_i}\|^{p_1}_{p_1 a_1^{\varrho_1}})$ are of the form

$$\|f^\varrho_{F_i}\|^{p_1}_{p_1 a_1^{\varrho_1}} / \|f^\varrho_{F_i}\|^{p_0}_{p_0 a_0^{\varrho_0}} = \varrho^{p_1 k} \varrho_1^{p_1 k_1} / \varrho^{p_0 k} \varrho_0^{p_0 k_0} = \varrho^{(p_1 - p_0)k} \varrho_0^{-p_0 k_0} \varrho_1^{p_1 k_1}.$$

Hence, by the definition of ϱ_μ, they are integral powers of $\varrho^{p_1-p_0}$ if $p_1 > p_0$, ϱ^p if $p_1 = p_0 = p$. The vectors $(\|f^\varrho_{F_i}\|^{p_0}_{p_0 a_0^{\varrho_0}}, \|f^\varrho_{F_i}\|^{p_1}_{p_1 a_1^{\varrho_1}})$ thus can be numbered in non-decreasing order. Doing the same with g, by Step 2 there exists $\tilde T \in \mathcal{L}_1(\bar A^\varrho; \bar B^\varrho)$ such that

$$\tilde\varrho^2 \tilde T f^\varrho = g^\varrho.$$

Regarding $\tilde T$ as a map $\bar A \to \bar B$, in view of (4.9) we have $\|\tilde T\|_{\mathcal{L}(\bar A; \bar B)} \leqslant \tilde\varrho$. In order to get a T with $Tf = g$, $\tilde T$ has to be somewhat modified. To this end, let

$$U\varphi = \varphi f^\varrho / f \quad \text{for} \quad \varphi \in \sum(\bar A), \qquad V\psi = \psi g / g^\varrho \quad \text{for} \quad \psi \in \sum(\bar B).$$

Then, again by (4.9), $\|U\|_{\mathcal{L}(\bar A)} \leqslant 1$, $\|V\|_{\mathcal{L}(\bar B)} \leqslant \tilde\varrho$. Thus, defining T by

$$T = \tilde\varrho^2 V \tilde T U,$$

we obtain an operator with the properties wanted: $\|T\|_{\mathcal{L}(\bar A; \bar B)} \leqslant \tilde\varrho^4$, $Tf = g$. ■

Remark 4.1. In certain cases the lemma is true even with $\varrho = 1$. In Step 2 of the proof above, this was seen for particular elementary functions f, g, a_μ, b_μ $(\mu = 0, 1)$. If $1 \leqslant p_0, p_1 < \infty$, it is true also without this assumption. This can be verified in the following way, using a slight modification of the corresponding argument of [6], Lemma 2, for $\{L_1, L_\infty\}$, cf. also [8].

Let Λ be a linear functional on l_∞, of norm one, such that $\Lambda((c_n)_n) = \lim_{n\to\infty} c_n$ if this limit exists. Let T_n be the operator obtained from Lemma 4.2 if $\varrho = 1 + 1/n$. For $\varphi \in \sum(\bar A)$ one verifies that

$$\lambda_\varphi(E) = \Lambda\left(\frac{1}{1+1/n} \int_E T_n \varphi \, d\beta\right)$$

defines a measure on Y, absolutely continuous with respect to β. Let $T\varphi = d\lambda_\varphi / d\beta$, the Radon–Nikodym derivative. Then

$$\int_E T\varphi \, d\beta = \Lambda\left(\frac{1}{1+1/n} \int_E T_n \varphi \, d\beta\right). \tag{4.10}$$

In particular, with $\varphi = f$,

$$\int_E Tf \, d\beta = \Lambda\left(\frac{1}{1+1/n} \int_E T_n f \, d\beta\right) = \Lambda\left(\frac{1}{1+1/n} \int_E g \, d\beta\right) = \int_E g \, d\beta.$$

Hence $Tf = g$. To prove that $T \in \mathcal{L}_1(\bar A; \bar B)$, consider simple functions $\psi \in (L_{p_\mu b_\mu})' = L_{p'_\mu b_\mu^{-1}}$, where $1/p_\mu + 1/p'_\mu = 1$ $(\mu = 0, 1)$. Then by virtue of (4.10)

$$\int T\varphi \cdot \psi \, d\beta = \Lambda\left(\frac{1}{1+1/n} \int T_n \varphi \cdot \psi \, d\beta\right).$$

Hence

$$\left|\int T\varphi \cdot \psi \, d\beta\right| \leqslant \sup_n \frac{1}{1+1/n} \left|\int T_n \varphi \cdot \psi \, d\beta\right|$$

$$\leqslant \sup_n \frac{1}{1+1/n} \|T_n \varphi\|_{p_\mu b_\mu} \|\psi\|_{p'_\mu b_\mu^{-1}} \leqslant \|\varphi\|_{p_\mu b_\mu} \|\psi\|_{p'_\mu b_\mu^{-1}} \qquad (\mu = 0, 1).$$

It then follows that $T \in \mathcal{L}_1(\bar A; \bar B)$, which concludes the proof.

We do not know whether Lemma 4.2 admits a generalization to $\varrho = 1$ also in the quasi-Banach case (ii).

Example 4.1. Let $H: \mathbf{R}_+^2 \to \mathbf{R}_+$ be homogeneous, i.e. $H(\lambda x^0, \lambda x^1) = \lambda H(x^0, x^1)$, $\lambda > 0$. We say that H is an *exact $\bar p$-interpolation function* if, for arbitrary $\bar A = \{L_{p_0 a_0}, L_{p_1 a_1}\}$, $\bar B = \{L_{p_0 b_0}, L_{p_1 b_1}\}$,

(4.11) $g = Tf$ with $T \in \mathcal{L}_1(\bar A; \bar B)$ implies

$$\int H(|g b_0|^{p_0}, |g b_1|^{p_1}) \, d\beta \leqslant \int H(|f a_0|^{p_0}, |f a_1|^{p_1}) \, d\alpha.$$

We try to characterize such functions H. There also is the analogous non-exact problem, with a constant c inserted in (4.11). In that case the interpolation functions are characterized by being equivalent to the exact ones. Since the functional

$$f \mapsto \int H(|fa_0|^{p_0}, |fa_1|^{p_1})\,d\alpha \tag{4.12}$$

only accidentally defines a quasi-norm, in general in (4.11) we do not deal with interpolation spaces. However, if $p_0 = p_1 = p$ (the most interesting case), (4.12) reads

$$f \mapsto \int |f|^p H(a_0^p, a_1^p)\,d\alpha,$$

i.e. it is the pth power of a weighted L_p-norm. This case has been extensively studied, cf. notably [12], [20], and [21]. In the case $p_0 \neq p_1$, claiming on H some kind of convexity property, by means of (4.12) it is possible to define a quasi-normed space O_H (a "weighted Orlicz space") by

$$\|f\|_{O_H} = \inf\left\{\varepsilon \,\middle|\, \int H\left(\left|\frac{fa_0}{\varepsilon}\right|^{p_0}, \left|\frac{fa_1}{\varepsilon}\right|^{p_1}\right)d\alpha \leqslant 1\right\}.$$

In this way we thus obtain exact interpolation spaces. For the case $a_\mu, b_\mu \equiv 1$ ($\mu = 0, 1$), see [22].

Returning to (4.11), by the homogeneity of H we may write $H(x^0, x^1) = x^1 h(x^0/x^1)$ with $h(x) = H(x, 1)$. Hence there is a one-to-one correspondence between the exact $\bar{p}$-interpolation functions and a class, denoted by $\mathscr{I}_{\bar{p}}$, of non-negative functions h on $\boldsymbol{R}_+$. As in the Appendix, let $\mathscr{C}_+$ be the set of non-negative concave functions on $\boldsymbol{R}_+$. For $1 \leqslant p_0, p_1 < \infty$, we define a class $\mathscr{I}_{\bar{p}}^K$, consisting of functions of the form

$$h(x) = \int \Delta_{\bar{p}}(\sigma)\varphi\big(x/\Delta_{\bar{p}}(\sigma)\big)\,d\sigma,$$

where $\varphi \in \mathscr{C}_+$, $\varphi(x) = o\big(\max(1, t)\big)$ as $t \to 0$ or ∞, and $\Delta_{\bar{p}}$ is defined by (3.10). Note that here, by (A.5),

$$\varphi(x) = \int \min(x, t)\,dw(t) \tag{4.13}$$

with a positive measure dw.

We are now able to prove

$$\mathscr{I}_{\bar{p}}^K \subset \mathscr{I}_{\bar{p}} \subset \mathscr{C}_+. \tag{4.14}$$

In the cases $p_0 = p_1$ and $p_0 \neq p_1$, $a_\mu, b_\mu \equiv 1$ ($\mu = 0, 1$), these inclusions are well known, cf. the references given above. (One verifies that if $p_0 = p_1$ our class $\mathscr{I}_{\bar{p}}^K$ coincides with that of, e.g., [12].) It will be apparent from the proof below, using also Lemma 3.2(iii), that if $0 < p_0, p_1 \leqslant 1$, then $\mathscr{I}_{\bar{p}} = \mathscr{C}_+$. If $p_0 = p_1$, we thus obtain a result of [21]. By means of Example 5.1 below one easily constructs an example showing that if $1 \leqslant p_0, p_1 < \infty$ the right inclusion in (4.14) is strict. Concerning the left one, in [11] was proved that $\mathscr{I}_{(2,2)}^K = \mathscr{I}_{(2,2)}$. It is not known whether there holds equality for other values of $\bar{p}$.

To verify the right inclusion in (4.14), let H obey (4.11). Then by Lemma 4.2

$$g \leqslant f[\mathscr{K}_{\bar{p}}] \quad \text{implies} \quad \int H(|gb_0|^{p_0}, |gb_1|^{p_1})\,d\beta \leqslant \int H(|fa_0|^{p_0}, |fa_1|^{p_1})\,d\alpha.$$

Applying this condition to simple functions, we get

$$\sum \min(x_j^0, tx_j^1) \leqslant \sum \min(y_i^0, ty_i^1),\ t > 0, \quad \text{implies} \quad \sum H(x_j^0, x_j^1) \leqslant \sum H(y_i^0, y_i^1).$$

That the function h, corresponding to H, belongs to $\mathscr{C}_+$ now is a consequence of Lemma A.3(i).

To verify the left inclusion in (4.14), by Lemma 4.1 a sufficient condition for (4.11) is

$$g \leqslant f[K_{\bar{p}}] \quad \text{implies} \quad \int H(|gb_0|^{p_0}, |gb_1|^{p_1})\,d\beta \leqslant \int H(|fa_0|^{p_0}, |fa_1|^{p_1})\,d\alpha.$$

One instance when such an implication holds true is

$$K_{\bar{p}}(t, g) \leqslant K_{\bar{p}}(t, f),\ t > 0, \quad \text{implies} \quad \int K_{\bar{p}}(t, g)\,dw(t) \leqslant \int K_{\bar{p}}(t, f)\,dw(t), \tag{4.15}$$

where dw is a positive measure. But here, by Lemma 3.4,

$$\begin{aligned}\int K_{\bar{p}}(t, f)\,dw(t) &= \iint \mathscr{K}_{\bar{p}}\big(t\Delta_{\bar{p}}(\sigma), f\big)\,d\sigma\,dw(t)\\ &= \iiint \min\big(|fa_0|^{p_0}, t\Delta_{\bar{p}}(\sigma)\,|fa_1|^{p_1}\big)\,d\sigma\,dw(t)\,d\alpha\\ &= \int H(|fa_0|^{p_0}, |fa_1|^{p_1})\,d\alpha,\end{aligned}$$

where, with φ given by (4.12),

$$\begin{aligned}H(x, 1) = h(x) &= \iint \min\big(x, t\Delta_{\bar{p}}(\sigma)\big)\,d\sigma\,dw(t)\\ &= \int\Big(\Delta_{\bar{p}}(\sigma)\int \min\big(x/\Delta_{\bar{p}}(\sigma), t\big)\,dw(t)\Big)\,d\sigma = \int \Delta_{\bar{p}}(\sigma)\varphi\big(x/\Delta_{\bar{p}}(\sigma)\big)\,d\sigma.\end{aligned}$$

Hence $h \in \mathscr{I}_{\bar{p}}^K$. Treating in the same way $\int K_{\bar{p}}(t, g)\,dw(t)$, we conclude that if $h \in \mathscr{I}_{\bar{p}}^K$ then (4.15) and, consequently, (4.11) are valid. This proves the assertion.

EXAMPLE 4.2. Let $X = Y = \boldsymbol{R}_+$, provided with the Lebesgue measure, and let $\bar{A} = \{L_{p_0}, L_{p_1}\}$, $0 < p_0, p_1 < \infty$. Consider the dilation operator σ_s defined by

$$\sigma_s f(x) = f(x/s), \quad s > 0.$$

Then

$$\mathscr{K}_{\bar{p}}(t, \sigma_s f) = \int \min\left(|f(x/s)|^{p_0}, t|f(x/s)|^{p_1}\right) dx$$
$$= \int \min\left(|s^{1/p_0} f(x)|^{p_0}, t|s^{1/p_1} f(x)|^{p_1}\right) dx.$$

If follows that

$$\sigma_s f \leqslant \max(s^{1/p_0}, s^{1/p_1}) f[\mathscr{K}_{\bar{p}}].$$

Hence, by virtue of Theorem 4.2, for any c-interpolation space A holds

$$\|\sigma_s f\|_A \leqslant c \max(s^{1/p_0}, s^{1/p_1}) \|f\|_A. \tag{4.16}$$

This should be compared with the results for interpolation of weak type operators, i.e. operators T: $\{L_{p_0,1}, L_{p_1,1}\} \to \{L_{p_0,\infty}, L_{p_1,\infty}\}$, where $\|f\|_{L_{p,q}} = \left(\int (t^{1/p} f^*(t))^q dt/t\right)^{1/q}$. Following Boyd [5], we define

$$u(s) = \sup_f \|\sigma_s f\|_A / \|f\|_A,$$
$$a_A = \lim_{s\to\infty} \log u(s)/\log s,$$
$$\beta_A = \lim_{s\to 0} \log u(s)/\log s.$$

Then by (4.16) a necessary condition for A to be a (strong) interpolation space is

$$1/p_1 \leqslant \beta_A \leqslant a_A \leqslant 1/p_0, \quad \text{where} \quad 0 < p_0 < p_1 \leqslant \infty.$$

(Here the information about c is lost.) In the Banach case $1 \leqslant p_0 < p_1 \leqslant \infty$, under additional, rather restrictive, assumptions on A, (4.17) (or (4.16)) is also sufficient for A to be an interpolation space, cf. [27] with the addendum made in [15]. For weak interpolation, on the other hand, by [5] a both necessary and sufficient condition is, generally,

$$1/p_1 < \beta_A < a_A < 1/p_0 \quad \text{if} \quad 1 \leqslant p_0 < p_1 \leqslant \infty.$$

5. On the gap between the necessary and the sufficient conditions. In Section 4 was proved that exact $K_{\bar{p}}$-monotonicity is sufficient and exact $\mathscr{K}_{\bar{p}}$-monotonicity necessary for the condition (ExInt) to be satisfied. We now investigate quantitatively the relationship between the two kinds of monotonicity. When not otherwise stated, let $\bar{A}$ and $\bar{B}$ denote the same couples as in Section 4. However, we now restrict ourselves to $1 \leqslant p_0, p_1 < \infty$. Since $\bar{p} = (1, 1)$ was fully covered by Corollary 4.2, that case is excluded too.

It is natural first to compare the quasi-orders $g \leqslant f[K_{\bar{p}}]$ and $g \leqslant f[\mathscr{K}_{\bar{p}}]$. One result in this direction can be derived from Lemma 3.1, with the

deviation expressed in terms of $\varkappa_{\bar{p}}$. However, this yields too rough an estimate. In fact, since $\varkappa_{(p_0,p_1)} \to 0$, $p_1 \to \infty$, measured this way the gap between the necessary and sufficient conditions would increase unboundedly as $p \to \infty$. In the limit case $\bar{p} = (1, \infty)$, however, we have reason to expect Theorem C of Section 2, at least to within norm equivalences.

The following lemma settles these objections. In order to obtain there the best constant, not just an estimate, we suppose that the measure spaces are not purely atomic with finitely many atoms and that if $p_0 = p_1$ the weight functions are non-equivalent.

Lemma 5.1. *Under the above assumptions we have*

$$g \leqslant f[\mathscr{K}_{\bar{p}}] \quad \textit{implies} \quad g \leqslant f[K_{\bar{p}}], \tag{5.1}$$

$$g \leqslant f[K_{\bar{p}}] \quad \textit{implies} \quad g \leqslant \gamma_{\bar{p}} f[\mathscr{K}_{\bar{p}}], \tag{5.2}$$

where $\gamma_{\bar{p}}$, the smallest constant possible, is determined by

$$\inf_{x+y=\gamma_{\bar{p}}} (x^{p_0} + y^{p_1}) = 1. \tag{5.3}$$

Here, generally speaking,

$$1 < \gamma_{\bar{p}} < 2. \tag{5.4}$$

Proof. The main tool is Lemma 3.4, expressing $K_{\bar{p}}$ as a certain mean value of $\mathscr{K}_{\bar{p}}$-functions. As a first consequence, we immediately obtain (5.1).

Less trivial is (5.2). There the best constant $\gamma_{\bar{p}}$ can be expressed as

$$\gamma_{\bar{p}} = \sup_{g \leqslant f[K_{\bar{p}}]} c_{fg} \quad \text{where} \quad c_{fg} = \inf\{c \mid g \leqslant cf[\mathscr{K}_{\bar{p}}]\}. \tag{5.5}$$

After approximation it suffices to consider simple functions f, g, a_μ, b_μ $(\mu = 0, 1)$. $\mathscr{K}_{\bar{p}}(t, f)$ and $\mathscr{K}_{\bar{p}}(t, g)$ are then piecewise linear. By the definition of c_{fg}, for some $\tau > 0$ we have

$$\mathscr{K}_{\bar{p}}(\tau, g) = \mathscr{K}_{\bar{p}}(\tau, c_{fg} f). \tag{5.6}$$

Here

$$\mathscr{K}_{\bar{p}}(\tau, c_{fg} f) = c_{fg}^{p_0} \mathscr{K}_{\bar{p}}(\tau c_{fg}^{p_1 - p_0}, f). \tag{5.7}$$

Let $p_0 \geqslant p_1$. Consider the points

$$P\colon \left(\tau c_{fg}^{p_1-p_0}, \mathscr{K}_{\bar{p}}(\tau c_{fg}^{p_1-p_0}, f)\right), \quad Q\colon \left(\tau, \mathscr{K}_{\bar{p}}(\tau, g)\right).$$

Defining the multiplication $\#$: $\boldsymbol{R}_+ \times \boldsymbol{R}_+^2 \to \boldsymbol{R}_+^2$ by

$$c \# (x, y) = (c^{p_1 - p_0} x, c^{-p_0} y),$$

(5.6) and (5.7) express that

$$c_{fg} \# Q = P.$$

Q is, or may be chosen as, a corner of the $\mathscr{K}_{\bar{p}}(t, g)$-polygone. If $c_{fg} \geqslant 1$, which we may assume, the relative localization of P and Q is indicated by Figure 5.1.

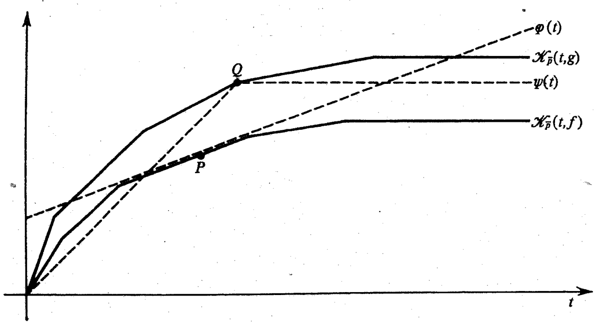

Fig. 5.1

Now let φ and ψ be functions defined by the graphs in Figure 5.1. Since

$$\psi(t) \leqslant \mathscr{K}_{\bar{p}}(t, g), \qquad \mathscr{K}_{\bar{p}}(t, f) \leqslant \varphi(t),$$

we have, in view of Lemma 3.4,

$$\int \psi\big(t\Delta(\sigma)\big)\,d\sigma \leqslant K_{\bar{p}}(t, g), \qquad K_{\bar{p}}(t, f) \leqslant \int \varphi\big(t\Delta(\sigma)\big)\,d\sigma,$$

where we have put $\Delta = \Delta_{\bar{p}}$. If $g \leqslant f[K_{\bar{p}}]$, it follows that

$$\int \psi\big(t\Delta(\sigma)\big)\,d\sigma \leqslant \int \varphi\big(t\Delta(\sigma)\big)\,d\sigma.$$

The problem of finding the supremum in (5.5) now can be reduced to an optimization problem for functions of φ and ψ-type. To this end, to every point $P \in \mathbf{R}_+^2$ we assign the set Φ_P of positive, increasing linear functions having graphs containing P, and the unique function $\psi = \psi_P = \min(a, bt)$ having its corner at P. We enunciate that

(5.8) $\quad \gamma_{\bar{p}} = \sup c, \quad$ where $c \# Q = P$ and

$$\int \psi_Q\big(t\Delta(\sigma)\big)\,d\sigma \leqslant \int \varphi\big(t\Delta(\sigma)\big)\,d\sigma \quad \text{for some } \varphi \in \Phi_P.$$

In fact, comparing (5.5) and (5.8), by the considerations above every $c = c_{fg}$ in (5.5) also appears in (5.8). Hence $\gamma_{\bar{p}} \leqslant \sup c$, where c obeys the conditions in (5.8). (Actually this estimate is all we need in the sequel.) On the other hand, since $\psi(t) = \mathscr{K}_{\bar{p}}(t, g)$ for some g and $\varphi(t)$ may be approximated by $\mathscr{K}_{\bar{p}}(t, f)$-functions, there is an inequality in the reverse direction too. (It is here the additional assumptions we made on the measures and the weight functions are needed.) We omit the details on this point.

We have to make the conditions on c in (5.8) more explicit. Without restrictions, let $Q = (1, 1)$, i.e. $\psi(t) = \min(1, t)$. Then $P = (c^{p_1-p_0}, c^{-p_0})$. Writing $\varphi(t) = \xi^{p_0} + t\eta^{p_1}$, $P \in \operatorname{graph}\varphi$ means that $\xi^{p_0} + c^{p_1-p_0}\eta^{p_1} = c^{-p_0}$, i.e.

(5.9) $$(c\xi)^{p_0} + (c\eta)^{p_1} = 1.$$

Turning to the integral inequality in (5.8), we handle the two members separately. Since $\int_0^1 \Delta(\sigma)\,d\sigma = 1$, for φ holds

$$\int_0^1 \varphi\big(t\Delta(\sigma)\big)\,d\sigma = \int_0^1 \big(\xi^{p_0} + t\Delta(\sigma)\eta^{p_1}\big)\,d\sigma = \xi^{p_0} + t\eta^{p_1}.$$

For ψ we get, using the decreasingness of Δ and defining $x = x(t)$ by $t\Delta(x) = 1$,

$$\int_0^1 \psi\big(t\Delta(\sigma)\big)\,d\sigma = \int_0^1 \min\big(1, t\Delta(\sigma)\big)\,d\sigma = x + t\int_x^1 \Delta(\sigma)\,d\sigma.$$

If $1 = p_1 < p_0$ this formula needs in fact a minor modification, omitted here, originating from the fact that $\Delta(\sigma)$ then attains no values smaller than $1/p_0$. The inequality in (5.8) now can be rewritten as

$$x + t\int_x^1 \Delta(\sigma)\,d\sigma \leqslant \xi^{p_0} + t\eta^{p_1},$$

or, since $t = 1/\Delta(x)$,

(5.10) $$x\Delta(x) + \int_x^1 \Delta(\sigma)\,d\sigma \leqslant \xi^{p_0}\Delta(x) + \eta^{p_1}.$$

In (5.8), for the extremal choice of φ, for some t there holds equality between the integrals. Hence, for some x this must be the case in (5.10) too. A derivation yields $x = \xi^{p_0}$. Inserting this in (5.10), we get

$$\int_{\xi^{p_0}}^1 \Delta(\sigma)\,d\sigma = \eta^{p_1}.$$

But here

$$\int_{\xi^{p_0}}^{1} \Delta(\sigma)\,d\sigma = \int_{\xi^{p_0}}^{1} \frac{p_1}{p_0}\,\frac{(1-\sigma^{1/p_0})^{p_1-1}}{\sigma^{1-1/p_0}}\,d\sigma = \int_{\xi^{p_0}}^{1} p_1\,(1-\sigma^{1/p_0})^{p_1-1}\,d\sigma^{1/p_0}$$
$$= \int_{\xi}^{1} p_1(1-\sigma)^{p_1-1}\,d\sigma = (1-\xi)^{p_1}.$$

Thus, for the extremal choice of φ we have

$$1-\xi = \eta. \tag{5.11}$$

In conclusion, the conditions on φ in (5.8) are equivalent to (5.9) and (5.11). Hence

$$\gamma_{\bar{p}} = \sup\{\varrho \,|\, (\varrho\xi)^{p_0}+(\varrho\eta)^{p_1} = 1 \text{ with } \xi+\eta = 1\}. \tag{5.8′}$$

But this is the same as

$$\inf_{\xi+\eta=1} (\gamma_{\bar{p}}\,\xi)^{p_0}+(\gamma_{\bar{p}}\eta)^{p_1} = 1,$$

which in turn is the same as (5.3).

What remains is (5.4). Obviously, $\gamma_{\bar{p}} > 1$ (remember the assumption $\bar{p} \neq (1,1)$). On the other hand, in view of (3.11) we have

$$1 = \int_0^1 \min\left(p_0\gamma_{\bar{p}}^{p_0}\sigma^{p_0-1}, p_1\gamma_{\bar{p}}^{p_1}(1-\sigma)^{p_1-1}\right)d\sigma$$
$$= \int_0^a p_0\gamma_{\bar{p}}^{p_0}\sigma^{p_0-1}\,d\sigma + \int_a^1 p_1\gamma_{\bar{p}}^{p_1}(1-\sigma)^{p_1-1}\,d\sigma$$

for some a, $0 < a < 1$. If $a \geqslant 1/2$ we get, cancelling the second and estimating the first integral,

$$1 > \int_0^a p_0\gamma_{\bar{p}}^{p_0}\sigma^{p_0-1}\,d\sigma \geqslant \int_0^{1/2} p_0\gamma_{\bar{p}}^{p_0}\sigma^{p_0-1}\,d\sigma = (\gamma_{\bar{p}}/2)^{p_0},$$

i.e. $\gamma_{\bar{p}} < 2$. Treating in the same way the case $a < 1/2$, we end up with (5.4). ■

It is now possible to formulate partial converses of the theorems and lemmata of Section 4. Thus, combining Lemma 5.1 with Lemma 4.2 and Lemma 4.1 respectively, we get

LEMMA 5.2. *If $g \leqslant f[K_{\bar{p}}]$ or, equivalently, $g \leqslant f[K]$, then there exists $T \in \mathscr{L}_{\gamma_{\bar{p}}}(\bar{A}; \bar{B})$ such that $Tf = g$.*

LEMMA 5.3. *If $T \in \mathscr{L}_1(\bar{A}; \bar{B})$, then $Tf \leqslant \gamma_{\bar{p}} f[\mathscr{K}_{\bar{p}}]$.*

As immediate consequences we obtain

THEOREM 5.1. (Ex Int) *implies $(\gamma_{\bar{p}}; K_{\bar{p}})$-monotonicity and, equivalently, $(\gamma_{\bar{p}}; K)$-monotonicity.*

THEOREM 5.2. *$(\gamma_{\bar{p}}$–Int) is a consequence of exact $\mathscr{K}_{\bar{p}}$-monotonicity.*

Note that here we do not assert $\gamma_{\bar{p}}$ to be the best constant. However, in Lemma 5.3 it actually is. This is seen in Example 5.3 below. On the contrary, in Lemma 5.2 and Theorem 5.1 it is not (in general). This is indicated in Remark 5.1, referring to the limit case $p = \infty$. That in these instances $\gamma_{\bar{p}}$ cannot, in general, be replaced by the constant 1 is shown by Examples 5.1 and 5.2. There the case $\bar{p} = (1, p)$ is considered. In particular, we conclude that, generally $\bar{A}$, $\bar{B}$ are *not* exactly K-adequate.

Remark 5.1. By Lemma 5.1 the deviation between the $\mathscr{K}_{\bar{p}}$ and the $K_{\bar{p}}$ (or K) quasi-orders is measured by $\gamma_{\bar{p}}$. For p-values close to 1, $\gamma_{\bar{p}}$ is close to 1 too. This is not surprising, since $\mathscr{K}_{\bar{p}}$ is modelled after the K-functional for $\{L_{1a_0}, L_{1a_1}\}$ (where $\mathscr{K}_{\bar{p}} = K_{\bar{p}} = K$). Considering the optimal partitions of f involved in calculating $K_{\bar{p}}$ and $\mathscr{K}_{\bar{p}}$, it seems equally natural that $\gamma_{\bar{p}}$ increases with p_0, p_1. One readily verifies that the extremal value 2 is approached in the limit p_0 and/or $p_1 = \infty$.

Trying to characterize the exact interpolation spaces in this limit case, a sufficient condition is as usual given by Theorem 1.1. What necessity concerns, $\mathscr{K}_{\bar{p}}$, thus also Theorem 4.2, loose their sense. However, considering instead Theorem 5.2, by a continuity argument it can be extended to the present case. Thereby, as was remarked above, $\gamma_{\bar{p}} = 2$. But if in particular $p_0 = 1$, we know from Theorem C that the best constant is 1. We thus conclude that the constant $\gamma_{\bar{p}}$ in Lemma 5.2 and Theorem 5.1 is not, in general, the best one. For the couple $\{L_p, L_\infty\}$, $1 < p < \infty$, in [17] was proved analogues of Lemma 5.2 and Theorem 5.1 with constants estimated by $2^{1/p'}$, $1/p+1/p' = 1$. Concerning this value, cf. also [2], [3].

Remark 5.2. If $\bar{p} = (p, p)$, $\gamma_{\bar{p}} = 2^{1/p'}$. For this case, in Theorem 5.2 we thus obtained the same estimate as in [26].

Showing that the constant 1 does not do in Lemma 5.3 and Theorem 5.2, we need the following partial converse of Lemma 4.2.

LEMMA 5.4. *Let a_μ, b_μ $(\mu = 0, 1)$, f and Tf be non-negative simple functions, where $T \in \mathscr{L}_1(\bar{A}; \bar{B})$. Suppose that T is non-negative (i.e. $T\varphi \geqslant 0$ if $\varphi \geqslant 0$) and $\|Tf\|_{p_\mu b_\mu} = \|f\|_{p_\mu a_\mu}$ $(\mu = 0, 1)$. Then $Tf \leqslant f[\mathscr{K}_{\bar{p}}]$.*

Proof. To fix the ideas, let $a_\mu = b_\mu \equiv 1$ $(\mu = 0, 1)$ (and hence $p_0 \neq p_1$). The modification needed in the remaining cases just consists of the insertion of certain constants.

Let $f = \sum_1^m f_i \chi_{F_i}$, $g = Tf = \sum_1^n g_j \chi_{G_j}$. Without restrictions we may assume that T maps the space spanned by $\{\chi_{F_i}\}_1^m$ onto that spanned by

$\{\chi_{G_j}\}_1^n$. Hence T is determined by a matrix (t_{ji}) in the sense that

$$g_j = \sum_{i=1}^{m} t_{ji} f_i \quad (j = 1, \ldots, n). \tag{5.12}$$

By the positivity of T, $t_{ji} \geqslant 0$.

Put $\alpha(F_i) = \alpha_i$, $\beta(G_j) = \beta_j$. By assumption we have

$$\sup \|Tx\|_{p_\mu} = \|g\|_{p_\mu} \quad \text{where} \quad \|x\|_{p_\mu} = \|f\|_{p_\mu} \quad (\mu = 0, 1),$$

or equivalently

$$\sup \sum_j \Big| \sum_i t_{ji} x_i \Big|^{p_\mu} \beta_j = \|g\|_{p_\mu}^{p_\mu} \quad \text{where} \quad \sum_i |x_i|^{p_\mu} \alpha_i = \|f\|_{p_\mu}^{p_\mu} \quad (\mu = 0, 1).$$

Since $\|T\|_{\mathscr{L}(\bar{A}, \bar{B})} = 1$, $\|f\|_{p_\mu} = \|g\|_{p_\mu}$ $(\mu = 0, 1)$, in both instances the supremum is attained for $x = f$. Hence, using Lagrangian multipliers λ_μ we get

$$\sum_j \Big(\sum_i t_{ji} f_i \Big)^{p_\mu - 1} t_{ji} \beta_j = \lambda_\mu f_i^{p_\mu - 1} \alpha_i \quad (i = 1, \ldots, n, \ \mu = 0, 1). \tag{5.13}$$

On multiplying both sides by f_i and summing after i, by means of (5.12) we obtain

$$\sum_j g_j^{p_\mu} \beta_j = \lambda_\mu \sum_i f_i^{p_\mu} \alpha_i \quad (\mu = 0, 1).$$

Thus, since $\|g\|_{p_\mu} = \|f\|_{p_\mu}$, $\lambda_\mu = 1$ $(\mu = 0, 1)$. Put

$$\theta_{ij} = t_{ji} f_i / g_j.$$

The equations (5.12) and (5.13) then are equivalent to

$$\sum_{i=1}^{m} \theta_{ij} = 1 \quad (j = 1, \ldots, n),$$

$$\sum_{j=1}^{n} \theta_{ij} g_j^{p_\mu} \beta_j = f_i^{p_\mu} \alpha_i \quad (i = 1, \ldots, m, \ \mu = 0, 1).$$

Since moreover $\theta_{ij} \geqslant 0$, application of Lemma A.2 yields the assertion $g \leqslant f[\mathscr{K}_{\bar{p}}]$. ■

Example 5.1. Let $\bar{p} = (1, p)$, $a_\mu, b_\mu \equiv 1$ $(\mu = 0, 1)$. We construct f and g with $g \leqslant f[K_{\bar{p}}]$ such that $Tf \neq g$ for all $T \in \mathscr{L}_1(\bar{A}; \bar{B})$. To this end we note that by the proof of Lemma 5.1 there exist simple functions $f = \sum_1^2 f_{F_i}$, $g = \sum_1^3 g_{G_j}$, having $\mathscr{K}_{\bar{p}}$-functionals as in Figure 5.2, such that

$$g \leqslant f[K_{\bar{p}}], \quad g \not\leqslant f[\mathscr{K}_{\bar{p}}].$$

From the behaviours at 0 and ∞ it follows that $\|f\|_{p_\mu} = \|g\|_{p_\mu}$ $(\mu = 0, 1)$. Without restrictions we may assume $\|f\|_1 = \|g\|_1 = 1$. Let $\mathscr{F}$ and $\mathscr{G}$ be the

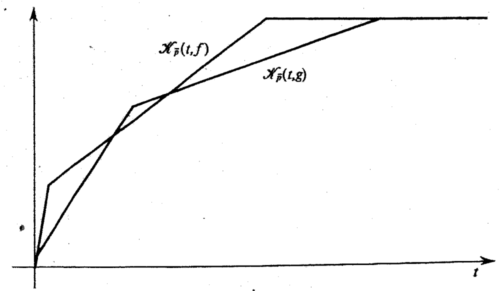

Fig 5.2

spaces generated by $\{\chi_{F_i}\}_1^2$ and $\{\chi_{G_j}\}_1^3$, respectively. Let $\mathscr{F}_1$ be the subspace of $\mathscr{F}$ defined by $\varphi \geqslant 0$, $\int \varphi \, d\alpha = 1$.

Now suppose that $T \in \mathscr{L}_1(\bar{A}; \bar{B})$, $Tf = g$ and, without restrictions, $T\colon \mathscr{F} \to \mathscr{G}$. The functional on L_1

$$\varphi \mapsto \int T\varphi \, d\beta$$

then has a norm not exceeding 1. Since it attains the value 1 on $\mathscr{F}_1$ at the interior point f, it is identically 1 on $\mathscr{F}_1$. But then $T\varphi \geqslant 0$ for $\varphi \in \mathscr{F}_1$, since otherwise

$$\int |T\varphi| \, d\beta > \int T\varphi \, d\beta = 1,$$

contrary to the hypothesis. Hence T must be non-negative. But this is impossible by virtue of Lemma 5.4. We conclude that there does not exist an operator $T \in \mathscr{L}_1(\bar{A}; \bar{B})$ such that $Tf = g$.

Example 5.2. By means of the preceding example, we now construct an exact interpolation space with respect to $\{L_1, L_p\}$ (or generally $\{L_{1a_0}, L_{pa_1}\}$), $1 < p < \infty$, which is not exactly K-monotonic. In fact, with f and g as in Example 5.1, let A be defined by (cf. [1], p. 99)

$$\|h\|_A = \inf_{h = Tf} \|T\|_{\mathscr{L}(\bar{A})}.$$

It is readily verified that A is an exact interpolation space with respect to $\bar{A}$. Obviously, $\|f\|_A \leqslant 1$. But expressed in terms of A, the content of the preceding example is that $\|g\|_A > 1$. Hence $\|g\|_A > \|f\|_A$ despite $g \leqslant f[K]$, which proves that A is not exactly K-monotonic.

Example 5.3. We show that $\gamma_{\bar{p}}$ in Lemma 5.3 cannot be replaced by any smaller constant. Referring to the proof of Lemma 5.1, let $\psi(t) = \min(1, t)$ and $\varphi(t) = \xi^{p_0} + t\eta^{p_1}$, where ξ and η are the optimal ones in (5.8′). Let $g = \chi_G$ with $\beta(G) = 1$ and let $f^n = f_{F_0}^n + f_{F_1}^n$ be a 2-valued simple function with

$$\|f^n_{F_0}\|_{p_0\alpha_0} = \xi, \quad \|f^n_{F_0}\|_{p_0\alpha_1} = n, \quad \|f^n_{F_1}\|_{p_0\alpha_0} = n, \quad \|f^n_{F_1}\|_{p_1\alpha_1} = \eta.$$

Then $\mathscr{K}_{\bar p}(t, g) = \psi(t)$ and $\mathscr{K}_{\bar p}(t, f^n) \nearrow \varphi(t)$, $n \to \infty$, cf. Figure 5.3.

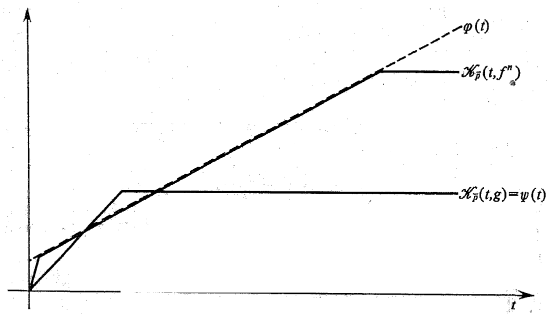

Fig. 5.3

Define T^n by

$$T^n f^n_{F_0} = \xi g, \quad T^n f^n_{F_1} = \eta g.$$

Then $T^n f^n = g$ and, as is readily seen, $T^n \in \mathscr{L}_{1+\varrho(n)}(\bar A; \bar B)$ with $\varrho(n) \to 0$, $n \to \infty$. However, by the proof of Lemma 5.1, $T^n f^n \leqslant \gamma_{\bar p} f^n [\mathscr{K}_{\bar p}]$, uniformly in n, where the constant $\gamma_{\bar p}$ cannot be improved. Taking into account the homogeneity, this proves the assertion.

6. A remark on the Lorentz case. We finally show how Lemma A.2 applies to yield a short proof of the results of Lorentz-Shimogaki [16] for interpolation of Lorentz spaces. (For Corollary 6.1 below, another simplified proof was given in [29]).

Let φ be a positive, decreasing function on $\boldsymbol{R}_+$ and let $\Phi(t) = \int_0^t \varphi(x)\,dx$. Define the Lorentz space $\Lambda(\varphi)$ by means of the norm

$$\|f\|_{\Lambda(\varphi)} = \int_0^\infty f^* \varphi\, dx = \int_0^\infty f^*\, d\Phi.$$

In particular, if $f^* = \chi_{(0,x)}$ we have $\|f\|_{\Lambda(\varphi)} = \Phi(x)$.

The following theorem and its corollary are in essence equivalent to Theorem 4 of [16].

THEOREM 6.1. *Let* $\bar A = \{\Lambda(\varphi_0), \Lambda(\varphi_1)\}$, $\bar B = \{\Lambda(\psi_0), \Lambda(\psi_1)\}$. *Then* $A = \Lambda(\varphi)$ *and* $B = \Lambda(\psi)$ *obey* (ExInt) *if and only if they are exactly* K-*monotonic.*

Proof. That K-monotonicity is sufficient for (ExInt) follows as usual from Theorem 1.1. What remains is the necessity.

It is well known that (cf. [28])

$$K(t, f; \bar A) = \int f^*(s)\, d\big(\min(\Phi_0(s), t\Phi_1(s))\big),$$

$$K(t, g; \bar B) = \int g^*(s)\, d\big(\min(\Psi_0(s), t\Psi_1(s))\big).$$

As in the proofs of Lemma 4.2 and Theorem 4.2, the problem can be reduced to the case of non-negative elementary functions f, g, i.e.

$$f = \sum_{i\in I} f_i \chi_{F_i} \quad \text{with} \quad f_{i-1} < f_i, \qquad g = \sum_{j\in J} g_j \chi_{G_j} \quad \text{with} \quad g_{j-1} < g_j.$$

Put

$$f'_i = f_i - f_{i-1}, \quad F'_i = \bigcup_{k\geqslant i} F_k, \quad g'_j = g_j - g_{j-1}, \quad G'_j = \bigcup_{k\geqslant j} G_j.$$

Then

$$f = \sum_i f'_i \chi_{F'_i}, \qquad g = \sum_j g'_j \chi_{G'_j}.$$

Putting $\operatorname{meas}(F'_i) = \alpha_i$, $\operatorname{meas}(G'_j) = \beta_j$, the condition $g \leqslant f[K]$ means

$$\sum_j g'_j \min\big(\Psi_0(\beta_j), t\Psi_1(\beta_j)\big) \leqslant \sum_i f'_i \min\big(\Phi_0(\alpha_i), t\Phi_1(\alpha_i)\big), \qquad t > 0.$$

But then, by Lemma A.2, there exists a matrix $\Theta = (\theta_{ij})$ such that

$$\sum_{i\in I} \theta_{ij} = 1 \quad (j \in J), \tag{6.1}$$

$$\sum_{j\in J} \theta_{ij} g'_j \Psi_\mu(\beta_j) \leqslant f'_i \Phi_\mu(\alpha_i) \quad (i \in I,\ \mu = 0, 1). \tag{6.2}$$

Now define operators T_i by

$$T_i \chi_{F'_i} = \frac{1}{f'_i} \sum_j \theta_{ij} g'_j \chi_{G'_j}$$

and, generally, for locally integrable functions h by

$$T_i h = \Big(\frac{1}{\alpha_i} \int_{F'_i} h\, dx\Big) T_i \chi_{F'_i}.$$

Then, by (6.2),

$$\|T_i \chi_{F'_i}\|_{\Lambda(\psi_\mu)} = \frac{1}{f'_i} \sum_j \theta_{ij} g'_j \Psi_\mu(\beta_j)$$
$$\leqslant \Phi_\mu(\alpha_i) \ \big(= \|\chi_{F'_i}\|_{\Lambda(\varphi_\mu)}\big) \quad (\mu = 0, 1).$$

Making two integrations by parts and using the fact that $\Phi_\mu(x)/x$ decreases with x, we get

$$\|T_i h\|_{\Lambda(\varphi_\mu)} = \left|\frac{1}{a_i}\int_{F_i'} h\,dx\right| \|T_i\chi_{F_i'}\|_{\Lambda(\varphi_\mu)} \leqslant \frac{\Phi_\mu(a_i)}{a_i}\int_{F_i'}|h|\,dx \leqslant \frac{\Phi_\mu(a_i)}{a_i}\int_0^{a_1} h^* dx$$

$$= \frac{\Phi_\mu(a_i)}{a_i}\Big(a_i h^*(a_i) + \int_0^{a_i} x\,d\big(-h^*(x)\big)\Big)$$

$$= \Phi_\mu(a_i)h^*(a_i) + \int_0^{a_i}\frac{\Phi_\mu(a_i)}{a_i}x\,d\big(-h^*(x)\big)$$

$$\leqslant \Phi_\mu(a_i)h^*(a_i) + \int_0^{a_i}\Phi_\mu(x)\,d\big(-h^*(x)\big)$$

$$= \int_0^{a_i} h^*(x)\,d\Phi_\mu(x) = \|h\|_{\Lambda(\varphi_\mu)} \quad (\mu = 0, 1).$$

In other words, $T_i \in \mathscr{L}_1(\bar{A}; \bar{B})$. Assuming that $\Lambda(\varphi), \Lambda(\psi)$ obey (ExInt), we get

$$\|T_i h\|_{\Lambda(\varphi)} \leqslant \|h\|_{\Lambda(\varphi)} \quad \text{if} \quad h \in \Lambda(\varphi).$$

Thus, for $h = f_i'\chi_{F_i'}$,

$$\sum_j \theta_{ij} g_j' \Psi(\beta_j) \leqslant f_i'\Phi(a_i) \qquad (i \in I).$$

A summation after i now yields, by virtue of (6.1),

$$\|g\|_{\Lambda(\psi)} = \sum_j g_j'\Psi(\beta_j) = \sum_i\sum_j \theta_{ij} g_j'\Psi(\beta_j) \leqslant \sum_i f_i'\Phi(a_i) = \|f\|_{\Lambda(\varphi)},$$

i.e. $\|g\|_{\Lambda(\psi)} \leqslant \|f\|_{\Lambda(\varphi)}$. This concludes the proof. ■

In [16] also the following condition on $\Lambda(\varphi), \Lambda(\psi)$ was dealt with

(L.S) $$\frac{\Psi(y)}{\Phi(x)} \leqslant \max\left(\frac{\Psi_0(y)}{\Phi_0(x)}, \frac{\Psi_1(y)}{\Phi_1(x)}\right) \quad \text{for all } x, y > 0.$$

We also introduce the slightly less restrictive

(L.S′) $$\frac{\Psi(y)}{\Phi(x)} \leqslant \frac{\Psi_0(y)}{\Phi_0(x)} + \frac{\Psi_1(y)}{\Phi_1(x)} \quad \text{for all } x, y > 0.$$

COROLLARY 6.1. *Let $\bar{A}$, $\bar{B}$ and A, B be the same as in Theorem 6.1. Then*

(i) (ExInt) *implies* (L.S),

(ii) (L.S′) *implies* (2.Int).

Proof. Note that, by the above theorem, (ExInt) and (2.Int) are equivalent to exact K- and $(2;K)$-monotonicity, respectively. In the proof to follow we argue throughout in terms of K-monotonicity.

(i) Let

$$g = \chi_{(0,y)}, \qquad f = \max\left(\frac{\Psi_0(y)}{\Phi_0(x)}, \frac{\Psi_1(y)}{\Phi_1(x)}\right)\chi_{(0,x)}.$$

One readily verifies that $g \leqslant f[K]$. Assuming exact K-monotonicity, we obtain $\|g\|_{\Lambda(\psi)} \leqslant \|f\|_{\Lambda(\varphi)}$. But this is exactly the same as (L.S).

(ii) Suppose that (L.S′) is valid and that $g \leqslant f[K]$. We then have relations (6.1) and (6.2). On dividing (6.2) by $\Phi_\mu(a_i)$ and adding the two inequalities ($\mu = 0, 1$), by means of (L.S′) we get

$$2f_i' \geqslant \sum_j \theta_{ij} g_j'\left(\frac{\Psi_0(\beta_j)}{\Phi_0(a_i)} + \frac{\Psi_1(\beta_j)}{\Phi_1(a_i)}\right) \geqslant \sum_j \theta_{ij} g_j' \frac{\Psi(\beta_j)}{\Phi(a_i)}.$$

Hence

$$\sum_j \theta_{ij} g_j' \Psi(\beta_j) \leqslant 2f_i'\Phi(a_i).$$

In view of (6.1), a summation after i now yields

$$\sum_j g_j'\Psi(\beta_j) \leqslant 2\sum_i f_i'\Phi(a_i),$$

i.e. $\|g\|_{\Lambda(\psi)} \leqslant 2\|f\|_{\Lambda(\varphi)}$. This proves the assertion. ■

Remark 6.1. Concerning the family of Lorentz spaces $L_{p,q}$ defined in Remark 4.2, any couple $\{L_{p_0,q_0}, L_{p_1,q_1}\}$ of such spaces is K-adequate, $1 \leqslant p_\mu, q_\mu \leqslant \infty$, $\mu = 0, 1$. This is a consequence of the result of [9], cited at the end of Section 1, and the fact that $L_{p,q} = (L_1, L_\infty)_{1-1/p,q}$. (That the couple $\{L_{p_0,\infty}, L_{p_1,\infty}\}$ is K-adequate, in fact exactly, was first proved in [23].) In particular, the couple $\{L_{p_0,1}, L_{p_1,1}\} = \{\Lambda(t^{1/p_0-1}), \Lambda(t^{1/p_1-1})\}$ is K-adequate. Hence *every* interpolation space with respect to this couple is K-monotonic, although not necessarily in the exact sense. Theorem 6.1 thus may be considered as a sharpening of this result for interpolation spaces which themselves are Lorentz spaces.

For a related result, showing that $\bar{A} = \{\Lambda(\varphi), L_\infty\}$, $\bar{B} = \{L_1, L_\infty\}$ are exactly K-adequate, see [10].

Appendix. Some matrix lemmata. Let $\boldsymbol{R}_+^2$ be the positive (vector-) quadrant, i.e. $\bar{x} = (x^0, x^1) \in \boldsymbol{R}_+^2$ iff $x^0, x^1 > 0$. On $\boldsymbol{R}_+^2$ are defined order relations (inclination) by

$$\bar{x} \in \bar{y} \quad \text{iff} \quad x^1/x^0 < y^1/y^0,$$
$$\bar{x} \subseteq \bar{y} \quad \text{iff} \quad x^1/x^0 \leqslant y^1/y^0.$$

We consider sequences (possibly non-finite) $X = (\bar{x}_j)_{j\in J} \subset \boldsymbol{R}_+^2$, $J \subset \boldsymbol{Z}$, such that $\bar{x}_j \subseteq \bar{x}_{j+1}$ and

(A.1) $$\sum_{j\geqslant 0} x_j^0 < \infty, \qquad \sum_{j<0} x_j^1 < \infty.$$

If $Y = (\bar{y}_i)_{i\in I}$ is another such sequence and $\Theta = (\theta_{ij})_{i\in I, j\in J}$ a matrix, we agree to write

$$\Theta X = Y \quad \text{iff} \quad \sum_{j\in J} \theta_{ij}\bar{x}_j = \bar{y}_i \quad (i \in I),$$
$$\Theta X \leqslant Y \quad \text{iff} \quad \sum_{j\in J} \theta_{ij}\bar{x}_j \leqslant \bar{y}_i \quad (i \in I).$$

Here $\bar{x} \leqslant \bar{y}$ stands for $x^0 \leqslant y^0$, $x^1 \leqslant y^1$. Let $\mathscr{S}$ denote the set of matrices $\Theta = (\theta_{ij})_{i\in I, j\in J}$ such that

$$\theta_{ij} \geqslant 0, \quad \sum_{i\in I} \theta_{ij} = 1 \quad (i \in I,\ j \in J),$$

and such that the non-zero elements are distributed in accordance with the figure:

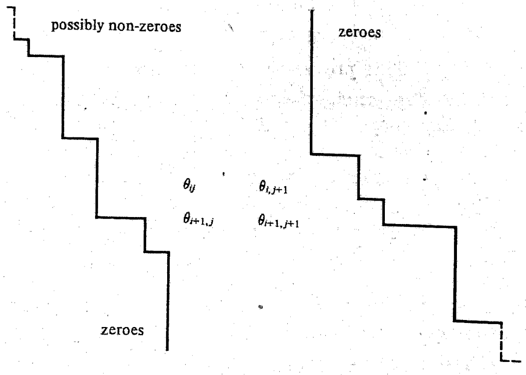

Fig. A.1

Thus, in particular, each row contains only finitely many non-zero elements, and for all but finitely many j there exists a number $i = i(j)$ such that $\theta_{ij} = 0$ for $i > i(j)$ or $i < i(j)$.

To start with we assume that I and J are finite. Then $\mathscr{S}$ is the set of all (finite) stochastic matrices. To every $X = (\bar{x}_j)_{j\in J}$ we associate the set (of vectors)

$$\omega_X = \Big\{\sum_{j\in J} \varepsilon_j \bar{x}_j \mid 0 \leqslant \varepsilon_j \leqslant 1,\ j \in J\Big\}.$$

Using the point P as origin, let $\omega_{X,P}$ be its affine representative. $\omega_{X,P}$ then is a convex parallelotope, cf. Figure A.2. By $\gamma_{X,P}$ and $\bar{\gamma}_{X,P}$ we denote the boundary polygones, and also the functions on $\boldsymbol{R}$ having them as graphs.

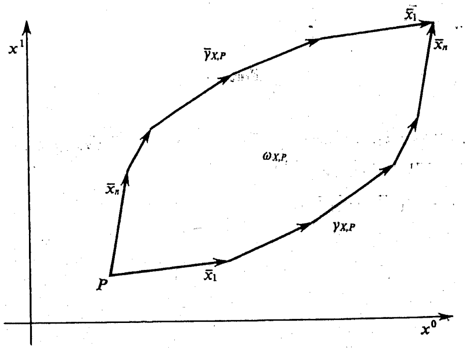

Fig. A.2

Lemma A.1. *Let I, J be finite. Then the following conditions are equivalent*

(i) $Y = \Theta X$ *with* $\Theta \in \mathscr{S}$,

(ii) $\omega_Y \subset \omega_X$, $\sum_{j\in J} \bar{x}_j = \sum_{i\in I} \bar{y}_i$,

(iii) $\gamma_{X,P} \leqslant \gamma_{Y,P}$.

Proof. The equivalence between (ii) and (iii) is obvious. As a consequence of (i), $\sum_{i\in I'} \bar{y}_i \in \omega_X$ for every $I' \subset I$. Hence $\omega_Y \subset \omega_X$. Since obviously $\sum \bar{x}_j = \sum \bar{y}_i$, we have verified that (i) implies (ii).

We now prove that (iii) implies (i). Thus let $\gamma_{X,P} \leqslant \gamma_{Y,P}$. To construct Θ we use induction over the number of $\bar{x}$-vectors n. For $n = 1$ and 2, the statement is obvious. Suppose it is true for $n-1$. Let $\bar{z}$ be chosen in accordance with Figure A.3 (along the "tangent" of $\gamma_{Y,P}$ through the corner $P + \bar{x}_1$ of $\gamma_{X,P}$).

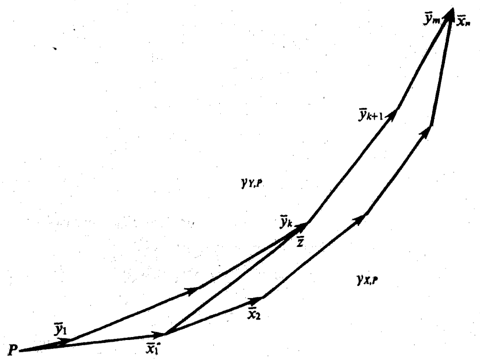

Fig. A.3

Since

$$\gamma_{(\bar x_1,\bar z),P} \leqslant \gamma_{(\bar y_1,\dots,\bar y_k),P}, \qquad \gamma_{(\bar x_2,\dots,\bar x_n),P+\bar x_1} \leqslant \gamma_{(\bar z,\bar y_{k+1},\dots,\bar y_m),P+\bar x_1},$$

the induction hypothesis yields

$$\bar y_1 = \theta_{11}\bar x_1 + \lambda_1 \bar z,$$
$$\dots\dots\dots$$
$$\bar y_k = \theta_{k1}\bar x_1 + \lambda_k \bar z \quad \text{with} \quad \sum_{i=1}^{k}\theta_{i1} = 1, \ \sum_{i=1}^{k}\lambda_i = 1,$$

and

$$\bar y_{k+1} = \theta_{k+1,2}\bar x_2 + \dots + \theta_{k+1,n}\bar x_n,$$
$$\dots\dots\dots\dots\dots$$
$$\bar y_m = \theta_{m2}\bar x_2 + \dots + \theta_{mn}\bar x_n,$$
$$\bar z = \mu_2\bar x_2 + \dots + \mu_n \bar x_n \quad \text{with} \quad \mu_j + \sum_{i=k+1}^{m}\theta_{ij} = 1 \ (j = 2,\dots,n).$$

Inserting the expression for $\bar z$ into the first k equations, we obtain a scheme of coefficients having the properties stated. ■

As a corollary we get the theorem of Hardy, Littlewood and Pólya, cited in Section 2. In fact, considering the particular vectors $\bar x_i = (x_i, 1)$, $\bar y_i = (y_i, 1)$ $(i = 1, \dots, n)$, the equations for the second components in "$Y = \Theta X$ with $\Theta \in \mathscr{S}$" say that every row sum of Θ is equal to 1. Hence (i) of Lemma A.1 is equivalent to "$y = \Theta x$ with $\Theta \in \mathscr{D}$", where $x = (x_1 \dots x_n)^t$, $y = (y_1 \dots y_n)^t$ and $\mathscr{D}$ stands for the doubly stochastic matrices. On the other hand, by virtue of Figure A.4, the condition (iii) is equivalent to "$\sum_1^k y_i^* \leqslant \sum_1^k x_i^*$ for $k = 1, \dots, n-1$, $\sum_1^n y_i^* = \sum_1^n x_i^*$". Thus Lemma A.1 reduces to Theorem HLP in this case.

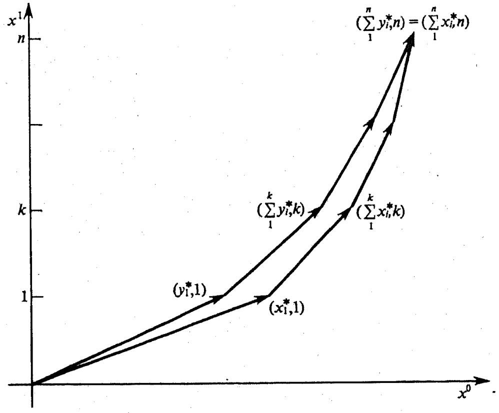

Fig. A.4

Now consider the general (i.e. not necessarily finite) case. To $X = (\bar x_j)_{j\in J}$ we assign the piecewise linear function γ_X defined by having a graph with vertices at the points (cf. Figure A.5)

$$P_m\colon \Big(-\sum_{j\geqslant m} x_j^0, \ \sum_{j<m} x_j^1\Big).$$

If $\sum_{j\in J} x_j^0$ is finite, let $\gamma_X(t) = 0$ for $t < -\sum_{j\in J} x_j^0$. (That the coordinates of P_m are finite is ensured by the assumption (A.1)). γ_X thus is located in the second quadrant and has the coordinate axes as (possibly attained) asymptotes.

Our main result reads

LEMMA A.2. *The following statements are equivalent*:

(i) $\Theta X \leqslant Y$ *with* $\Theta \in \mathscr{S}$,

(ii) $\gamma_X \leqslant \gamma_Y$,

(iii) $\sum_{j\in J}\min(x_j^0, t x_j^1) \leqslant \sum_{i\in I}\min(y_i^0, t y_i^1)$, $t > 0$.

If $\sum \bar x_j = \sum \bar y_i$, *then* (ii) *and* (iii) *are equivalent to*

(i)′ $\Theta X = Y$ *with* $\Theta \in \mathscr{S}$.

(Here by $\sum \bar x_j = \sum \bar y_i$ is meant that $\sum x_j^0 = \sum y_i^0$, $\sum x_j^1 = \sum y_i^1$, where in each instance either both sides are finite and equal or both infinite.)

Proof. (i)′ ⇔ (ii). If I, J are finite, this is exactly what is expressed by Lemma A.1. In that case the following lengthy argument thus should be excluded.

Dealing with the non-finite case, to fix the ideas let $I = J = \mathbf{Z}$. Beginning with the implication (ii) $\Rightarrow$ (i)′, let $\gamma_X \leqslant \gamma_Y$, $\sum \bar{x}_j = \sum \bar{y}_i$. The situation is illustrated in Figure A.5, where $\sum\limits_{j\geqslant 0} x_j^1$ is divergent, $\sum\limits_{j<0} x_j^0$ convergent. Without restrictions we may assume that $\bar{x}_j \in \bar{x}_{j+1}$, $\bar{y}_i \in \bar{y}_{i+1}$ strictly. Drawing from the vertices of γ_X tangents of γ_Y, the vectors $\bar{u}_k$ are constructed, $k = \pm 1, \pm 2, \ldots$ In this way the region D bordered by γ_X and γ_Y is decomposed into subregions D_k, $k \in \mathbf{Z}$. Let $X_k = (\bar{x}_j)_{j\in J_k}$ and $Y_k = (\bar{y}_i)_{i\in I_k}$ be the subsets of X and Y, associated with D_k.

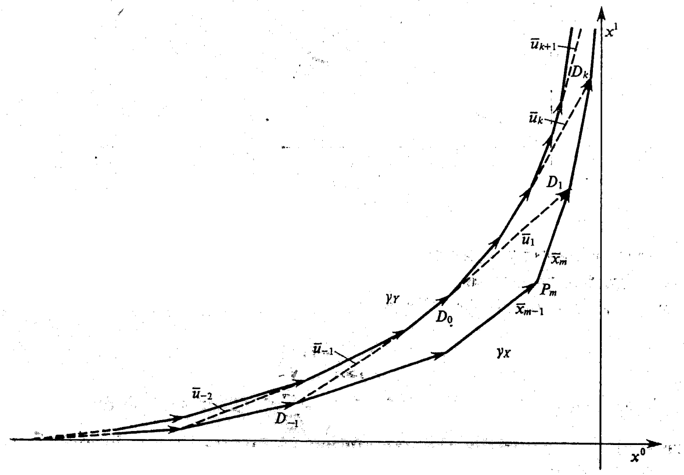

Fig. A.5

Now to every D_k, Lemma A.1 is applicable. It provides us with finite $\mathscr{S}$-matrices Θ_k such that

$$\text{(A.2)}\qquad \begin{aligned} (\bar{u}_{k+1}, Y_k) &= \Theta_k(\bar{u}_k, X_k), \quad k \geqslant 1,\\ (\bar{u}_{-1}, Y_0, \bar{u}_1) &= \Theta_0 X_0,\\ (\bar{u}_{k-1}, Y_k) &= \Theta_k(\bar{u}_k, X_k), \quad k \leqslant -1. \end{aligned}$$

On eliminating successively the $\bar{u}_k$'s, $k = 1, 2, \ldots$, $k = -1, -2, \ldots$, we find that

$$\text{if} \quad i \in I_k \quad \text{then} \quad \bar{y}_i = \sum_{j\in \bigcup\limits_0^k J_\mu} \theta_{ij}\bar{x}_j \quad (k \in \mathbf{Z}).$$

Hence $Y = \Theta X$ with Θ having the shape of Figure A.1.

What remains to prove is that all column sums equal 1. Let us talk about θ_{ij} as "the $\bar{x}_j$-coefficient for $\bar{y}_i$". In the analogous way we describe the coefficients of the equations (A.2). Beginning with $j \in J_k$, $k \geqslant 1$, put

$$\begin{aligned} a_k &= \text{the } \bar{x}_j\text{-coefficient for } \bar{u}_{k+1},\\ a_\mu &= \text{the } \bar{u}_\mu\text{-coefficient for } \bar{u}_{\mu+1} \text{ if } \mu > k. \end{aligned}$$

By their construction the elements θ_{ij} of the jth column can be interpreted in the following way:

$$\begin{array}{lll} \text{if} & i \in I_\mu,\ \mu < k: & \theta_{ij} = 0,\\ \text{if} & i \in I_k: & \theta_{ij} = \text{the } \bar{x}_j\text{-coefficient for } \bar{y}_i,\\ \text{if} & i \in I_{k+1}: & \theta_{ij} = a_k\cdot(\text{the } \bar{u}_{k+1}\text{-coefficient for } \bar{y}_i),\\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}\\ \text{if} & i \in I_\mu: & \theta_{ij} = a_k a_{k+1}\ldots a_{\mu-1}\cdot(\text{the } \bar{u}_\mu\text{-coefficient for } \bar{y}_i). \end{array}$$

Put

$$\begin{aligned} \beta_k &= \text{the sum of the } \bar{x}_j\text{-coefficients for } \bar{y}_i, \quad i \in I_k,\\ \beta_\mu &= \text{the sum of the } \bar{u}_\mu\text{-coefficients for } \bar{y}_i, \quad i \in I_\mu \text{ if } \mu > k. \end{aligned}$$

Since $a_\mu + \beta_\mu$ is a column sum in Θ_μ, we have

$$\text{(A.3)}\qquad a_\mu + \beta_\mu = 1, \quad \mu \geqslant k.$$

The problem is to determine the sum of the series

$$\beta_k + a_k\beta_{k+1} + a_k a_{k+1}\beta_{k+2} + \ldots$$

In view of (A.3) its partial sums can be written

$$s_n = \beta_k + \ldots + a_k \ldots a_{n-1}\beta_n = 1 - a_k \ldots a_n.$$

But here the product $a_k \ldots a_n$ has the geometrical significance of the $\bar{x}_j$-coefficient for $\bar{u}_{n+1}$ when in the iterative process above $\bar{u}_{n+1}$ is expressed as a linear combination of $\bar{x}_j$, $j \in \bigcup\limits_{\mu=k}^n J_\mu$. This coefficient thus is majorized by the length of the projection of $\bar{u}_{n+1}$ onto $\bar{x}_j$ along the $\bar{x}_{j_n}$-direction, where j_n is the greatest integer of J_n, cf. Figure A.6. But this length obviously tends to zero. Hence $s_n \to 1$, $n \to \infty$, which proves the assertion if $j \in J_k$, $k \geqslant 1$. In the same way one treats the case $k \leqslant -1$. Thereafter, by a combination of these two cases one settles the case $k = 0$ (where the sum has to be taken over all integers). By this we have proved the implication (ii) $\Rightarrow$ (i)′.

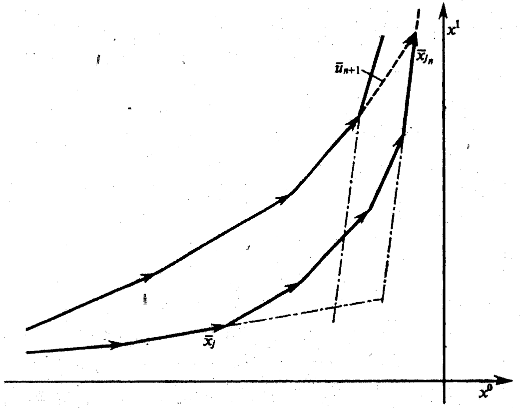

Fig. A.6

To verify (i)′ ⇒ (ii), let $Y = \Theta X$ with $\Theta \in \mathscr{S}$. Consider an arbitrary corner of γ_Y:

$$Q_k\colon \Big(-\sum_{i\geqslant k} y_i^0, \sum_{i<k} y_i^1\Big).$$

We prove it is situated above γ_X. By definition (of $\mathscr{S}$) there exist m_k and n_k, $m_k < n_k$, such that

$$\theta_{ij} = 0 \quad \text{for} \quad i<k,\ j\geqslant n_k,$$
$$\theta_{ij} = 0 \quad \text{for} \quad i\geqslant k,\ j< m_k.$$

Then

$$\sum_{i<k}\theta_{ij} = 1 \quad \text{if} \quad j<m_k,$$
$$\sum_{i<k}\theta_{ij} = a_j \quad \text{with} \quad 0\leqslant a_j\leqslant 1 \quad \text{if} \quad m_k\leqslant j<n_k,$$
$$\sum_{i<k}\theta_{ij} = 0 \quad \text{if} \quad j\geqslant n_k.$$

Taking into account also that all column sums equal 1, for the coordinates of Q_k we get

$$-\sum_{i\geqslant k} y_i^0 = -\sum_{i\geqslant k}\sum_j \theta_{ij}x_j^0 = -\sum_j x_j^0\sum_{i\geqslant k}\theta_{ij}$$
$$= -\sum_{j\geqslant n_k} x_j^0 - \sum_{n_k>j\geqslant m_k}(1-a_j)x_j^0 = -\sum_{j\geqslant m_k} x_j^0 + \sum_{n_k>j\geqslant m_k} a_j x_j^0,$$
$$\sum_{i<k} y_i^1 = \sum_{i<k}\sum_j \theta_{ij}x_j^1 = \sum_j x_j^1\sum_{i<k}\theta_{ij} = \sum_{j<m_k} x_j^1 + \sum_{m_k\leqslant j<n_k} a_j x_j^1.$$

In other words, $Q_k \in \omega_{X_k, P_{m_k}}$, with $X_k = (\bar{x}_j)_{m_k\leqslant j<n_k}$, $P_{m_k} = \big(-\sum_{j\geqslant m_k} x_j^0, \sum_{j<m_k} x_j^1\big)$. Hence all vertices of γ_Y lie above γ_X, which proves the assertion.

(i) ⇔ (ii). Let $\gamma_X \leqslant \gamma_Y$ (but not necessarily $\sum \bar{x}_j = \sum \bar{y}_i$). Choose the points R_0, R_1 in Figure A.7 so that the segment R_0R_1 intersects γ_Y. From these points we draw the tangents of γ_Y. Also connect them with γ_X. Let $\Xi = (\bar{\xi}_j)_{j\in J}$ and $H = (\bar{\eta}_i)_{i\in I}$ be defined by the figure. Then $\gamma_\Xi \leqslant \gamma_H$, $\sum\bar{\xi}_j = \sum\bar{\eta}_i$ and thus, by the equivalence just proved, $H = \Theta\Xi$ with $\Theta \in \mathscr{S}$. Since $\bar{\eta}_i \leqslant \bar{y}_i$, $\bar{x}_j \leqslant \bar{\xi}_j$ $(i\in I, j\in J)$, it follows that $\Theta X \leqslant Y$ with $\Theta \in \mathscr{S}$.

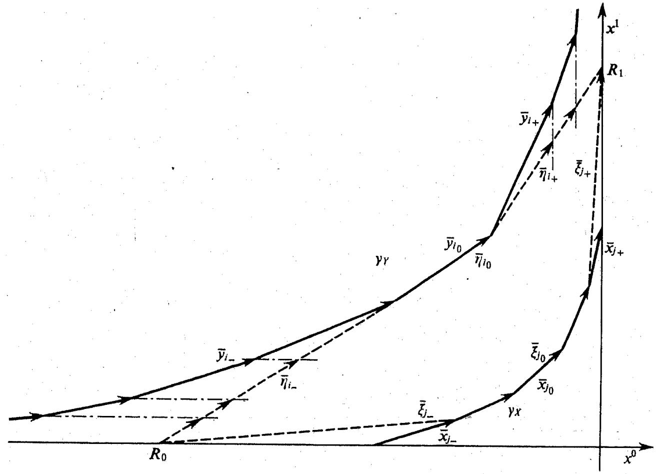

Fig. A.7

Conversely, let $Y' = \Theta X \leqslant Y$ with $\Theta \in \mathscr{S}$. Then, by the equivalence between (i)′ and (ii), $\gamma_X \leqslant \gamma_{Y'}$. One easily verifies that $\gamma_{Y'} \leqslant \gamma_Y$. Hence $\gamma_X \leqslant \gamma_Y$, which concludes this part of the proof.

(ii) ⇔ (iii). Here we use the Legendre transform

$$\mathscr{L}\gamma(t) = \inf_{s<0}\big(t\gamma(s) - s\big).$$

Since γ_X and γ_Y are convex, there holds the equivalence

$$\gamma_X \leqslant \gamma_Y \Leftrightarrow \mathscr{L}\gamma_X \leqslant \mathscr{L}\gamma_Y.$$

Thus, proving that

$$\mathscr{L}\gamma_X(t) = \sum_{j\in J}\min(x_j^0, tx_j^1), \tag{A.4}$$

the assertion will follow.

To this end, referring to Figure A.8, where we have drawn the tangent of γ_X in the $(t,1)$-direction, $\mathcal{L}\gamma_X(t)$ can be represented as the distance $|OM|$. Let the point of tangency be P_k: $(-\sum_{j\geqslant k} x_j^0, \sum_{j<k} x_j^1)$. By the figure we have

$$\mathcal{L}\gamma_X(t) = |OM| = |ON|+|NM| = \sum_{j\geqslant k} x_j^0 + t\sum_{j<k} x_j^1.$$

But here

if $j<k$, then $\bar{x}_j \underline{\varepsilon} (t,1)$, i.e. $tx_j^1 \leqslant x_j^0$, $tx_j^1 = \min(x_j^0, tx_j^1)$,

if $j\geqslant k$, then $(t,1)\,\underline{\varepsilon}\,\bar{x}_j$, i.e. $x_j^0 \leqslant tx_j^1$, $x_j^0 = \min(x_j^0, tx_j^1)$.

This proves (A.4).

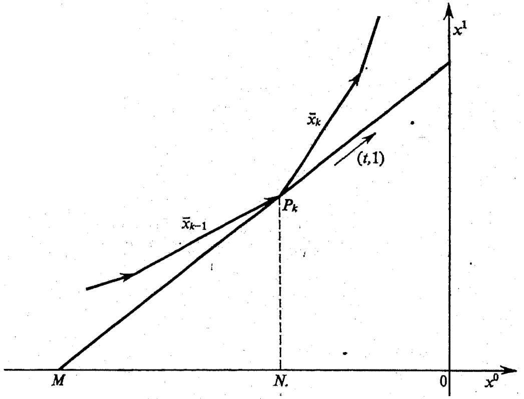

Fig. A.8

Let us denote by $\mathscr{C}_+$ the set of positive concave functions on $\boldsymbol{R}_+$. It is well known (cf. e.g. [4]) that every such function admits the representation

(A.5) $\quad \varphi(x) = \alpha+\beta x+\int_0^\infty \min(x,t)\,dw(t), \quad dw$ positive measure.

(Here, in fact, $w = -\varphi'$, $0\leqslant\alpha = \lim_{x\to 0}\varphi(x)$, $0\leqslant\beta = \lim_{x\to 0}\varphi(x)/x$.) Writing $\gamma_X\leqslant\gamma_Y$ for any one of the three equivalent conditions (i), (ii), (iii) of Lemma A.2, the following lemma generalizes a result of Pólya on doubly stochastic matrices (cf. [18]).

Lemma A.3. *Let I, J be finite. Consider the condition*

(A.6) $$\sum_{j\in J} x_j^1\varphi(x_j^0/x_j^1) \leqslant \sum_{i\in I} y_i^1\varphi(y_i^0/y_i^1).$$

Then

(i) (A.6) *is a consequence of $\gamma_X\leqslant\gamma_Y$ if and only if $\varphi\in\mathscr{C}_+$,*

(ii) (A.6) *is valid for every $\varphi\in\mathscr{C}_+$ if and only if $\gamma_X\leqslant\gamma_Y$.*

Proof. The "if" parts of (i) and (ii) coincide. To verify them, let X, Y obey (iii) of Lemma A.2 and let $\varphi\in\mathscr{C}_+$ be given by (A.5). Then

$$\sum x_j^0 \leqslant \sum y_i^0, \qquad \sum x_j^1 \leqslant \sum y_i^1,$$

and by an integration

$$\sum x_j^1 \int \min(x_j^0/x_j^1, t)\,dw(t) \leqslant \sum y_i^1 \int \min(y_i^0/y_i^1, t)\,dw(t).$$

Combining these three inequalities, we get (A.6).

Verifying the "only if" part of (i), suppose that $\gamma_X\leqslant\gamma_Y$ implies (A.6). Let $\bar{x}_1 = (x_1, 1)$, $\bar{x}_2 = (x_2, 1)$, $\bar{y}_1 = \bar{y}_2 = \big((x_1+x_2)/2, 1\big)$. Then $Y = \Theta X$ with $\Theta\in\mathscr{S}$. Hence, by (A.6),

$$\varphi(x_1)+\varphi(x_2) \leqslant 2\varphi\big((x_1+x_2)/2\big),$$

which proves the concavity. To see that φ is positive, let X and Y consist of the single vectors $\bar{x} = (x, 1)$ and $\bar{y} = (2x, 2)$, respectively. Then $\gamma_X\leqslant\gamma_Y$, so that $\varphi(x)\leqslant 2\varphi(x)$, i.e. $\varphi\geqslant 0$. Thus $\varphi\in\mathscr{C}_+$.

The necessity part of (ii), finally, follows immediately by taking in (A.6) the particular function $\varphi(x) = \min(x, t)$. We then obtain condition (iii) of Lemma A.2. ■

Remark A.1. These lemmata can be extended in various directions. Firstly, it is not necessary to restrict oneself to the quadrant $\boldsymbol{R}_+^2$. In fact, there exists an analogue of Lemma A.1 for arbitrary $X = (\bar{x}_j)_{j=1}^n$, $Y = (\bar{y}_i)_{i=1}^m$ in $\boldsymbol{R}^2$. In particular, such a result applies to complex vectors $(x_1, \ldots, x_n)\in\boldsymbol{C}^n$, $(y_1, \ldots, y_m)\in\boldsymbol{C}^m$.

Above we have been confined to vectors $\bar{x}_j$ $(j\in J)$ which are, or can be, arranged in non-decreasing order. However, by an argument similar to that of the proof of Lemma A.2 in the transition from the finite to the non-finite case, Lemma A.2 can be extended to X and Y being countable unions of monotonic sets.

In formulating Lemma A.1 we could also have used $\bar{\gamma}_{X,P}\geqslant\bar{\gamma}_{Y,P}$, which obviously is equivalent to $\gamma_{X,P}\leqslant\gamma_{Y,P}$ (cf. Figure A.1). In passing from Lemma A.1 to Lemma A.2 we used the translates into the second quadrant γ_X and γ_Y of $\gamma_{X,P}$ and $\gamma_{Y,P}$, having the axes as asymptotes. Here the assumption (A.1) enabled us to consider the non-finite case. Translating in the same way $\bar{\gamma}_{X,P}$ and $\bar{\gamma}_{Y,P}$, we obtain what we define as $\bar{\gamma}_X$ and $\bar{\gamma}_Y$. Assuming $\sum x_j^0$ or $\sum x_j^1$ to be finite, also in this case the sequences are allowed to be non-finite. Arguing exactly as in the proof of Lemma A.2, we find that the following conditions are equivalent:

(i) $\Theta X \geqslant Y$ with $\Theta \in \mathscr{S}$,

(ii) $\bar{\gamma}_X \geqslant \bar{\gamma}_Y$.

(iii) $\sum_{j\in J} \max(x_j^0, tx_j^1) \geqslant \sum_{i\in I} \max(y_i^0, ty_i^1)$, $t > 0$.

If $\sum \bar{x}_j = \sum \bar{y}_i$, then (ii) and (iii) are equivalent to

(i') $\Theta X = Y$ with $\Theta \in \mathscr{S}$.

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