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Factorization of compact operators and finite representability of Banach spaces with applications to Schwartz spaces

by

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Abstract. Let X and Y be Banach spaces, X with a finite-dimensional decomposition. Necessary and sufficient conditions are given for every compact operator, X to X, to factor through a subspace of Y. Also derived are sufficient conditions for X uniformly finitely representable in Y to imply that every compact operator, X to X, factors through a subspace of Y. Examples of spaces X are constructed with $X \times X$ not uniformly finitely representable in X. Finally, these results are applied to Schwartz prevarieties, particularly with respect to the approximation property.

1. Introduction. Let X and Y be Banach spaces. We say that Y has the *subspace factorization property* [8] (abbreviated s.f.p) for X if each compact operator X to X factors through a subspace of Y. The s.f.p. is related to the *approximation property* (abbreviated a.p.) of Grothendieck (see [8]). In Theorem 2.3, we obtain conditions equivalent to Y having the s.f.p. for X when X has a finite-dimensional decomposition. Thess equivalent conditions are the existence of certain kinds of fragmentatione (see Section 2 for definition) in Y.

Consider the statements (i) Y has s.f.p. for X and (ii) X is uniformly finitely representable in Y. Are (i) and (ii) equivalent? When X has a finite-dimensional decomposition, (i) implies (ii) (Proposition 2.2) and conversely with certain restrictions on X or Y (Theorem 3.1). In particular, if $Y \times Y$ is isomorphic to a subspace of Y and X has a finite-dimensional decomposition, (i) and (ii) are equivalent. Figiel [8] has shown that (i) and (ii) are equivalent if $X = l_p$. (See also remarks at the end of Section 3.)

Another collection of spaces Y for which (ii) implies (i) are the galactic spaces. A Banach space Y is *galactic* if for separable spaces X, X uniformly finitely representable in Y implies that X is isomorphic to a subspace of Y. Galactic spaces are considered in Section 4.

Figiel has shown in [7] that there are reflexive Banach spaces Y with $Y \times Y$ not isomorphic to a subspace Y. In Section 4, we show that some of these examples are not even locally square, that is, $Y \times Y$ is not uni-

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formly finitely representable in Y. This is done by showing that Y is a galactic space (Theorem 4.1). The proof of Theorem 4.1 uses non-standard analysis.

The remainder of this paper applies these results to Schwartz pre-varieties of the form $\mathscr{S}_X$. For a Banach space X, $\mathscr{S}_X$ is the collection of Schwartz spaces isomorphic to a subspace of X^I, some power of X. In particular, we consider when $\mathscr{S}_X \subset \mathscr{S}_Y$ (Theorem 5.4) and we construct universal generators for some $\mathscr{S}_X$ (Corollary 5.6). Lastly, we show that for a Banach space X, X' has a.p. if the Schwartz space (X, ξ_S) has a.p. (Proposition 5.7).

An operator is a continuous linear function. A compact operator between Banach spaces maps bounded sets into relatively compact sets. To say that the operator $R: X \to Y$ factors through Z, means there are operators $U: X \to Z$ and $V: Z \to Y$ with $R = VU$. The dual of X is denoted by X' and the transpose of the operator T is written T'. We abbreviate *linear span* by lin span and *closed linear span* by cl lin span.

A sequence of finite-dimensional subspaces $\{E_n\}$ is a finite-dimensional decomposition (abbreviated f.d.d.) for X if for each $x \in X$, there is a unique sequence $\{x_n\}$ with $x_n \in E_n$ and $x = \sum_n x_n$. We reserve the letter X for a Banach space with at least a f.d.d. $\{E_n\}$. A f.d.d. is monotone if the projections: $\sum_n x_n \to \sum_1^N x_n$ all have norm one. We reserve $\lambda = (\lambda_n)$ for null sequences of reals with $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$. Also reserved is the letter T, which is always a diagonal operator on a space with a f.d.d. For instance, $T_\lambda: X \to X$ is the operator which sends $x = \sum_n x_n$ onto $T_\lambda x = \sum_n \lambda_n x_n$. The following fact will be needed, a proof is essentially given in [22], p. 40.

Fact 1.1. If the Banach space X has a monotone f.d.d. $\{E_n\}$ and λ and T_λ are as above, then T_λ is a compact operator. Furthermore, if $m < n$ are integers, letting $X_{mn} = \operatorname{lin span}\{E_{m+1}, \ldots, E_n\}$ and S_{mn} the restriction of T_λ to X_{mn}, then S_{mn} is an isomorphism with

$$\|S_{mn}\| \leqslant \lambda_{m+1} + \lambda_n \leqslant 2\lambda_{m+1}, \tag{1}$$

and

$$\|S_{mn}^{-1}\| \leqslant 2\lambda_n^{-1} - \lambda_{m+1}^{-1} \leqslant 2\lambda_n^{-1}. \tag{2}$$

If Y and Z are isomorphic Banach spaces, then the Banach–Mazur distance between Y and Z is $d(Y, Z) = \inf\{\|S\|\,\|S^{-1}\|: S: Y \to Z \text{ an isomorphism}\}$. The Banach space Y is said to be *uniformly finitely representable* (abbreviated u.f.r.) in the Banach space Z if there is a constant K such that, for each finite-dimensional subspace Y_0 of Y, there is a subspace $Z_0 \subset Z$ of the same dimension and $d(Y_0, Z_0) \leqslant K$.

The proof of Theorem 4.1 uses non-standard analysis, in particular, it uses facts about the non-standard hulls of a Banach space found in [2], [10] and [11]. For basic results about non-standard analysis see [17], [16], and [9], the last for non-standard topological vector spaces.

A *locally convex space* (abbreviated LCS) (E, ξ) is a vector space E with a Hausdorff locally convex topology ξ. Let $\mathscr{U}$ be a ξ-neighborhood basis of the origin; we will always assume that each $U \in \mathscr{U}$ is weakly closed and absolutely convex. For $U \in \mathscr{U}$, the Minkowski functional of U is ϱ_U where

$$\varrho_U(x) = \inf\{\lambda > 0: x \in \lambda U\}.$$

If $U \in \mathscr{U}$, E_U will denote the Banach space formed by taking the completion of the vector space $E/\ker \varrho_U$ with the quotient norm obtained from ϱ_U. If $V \subset U$ are elements of $\mathscr{U}$, there is a natural operator $E_V \to E_U$ induced by the identity on E. A LCS is a Schwartz space if, for each $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$, with $V \subset U$ and the natural operator $E_V \to E_U$ compact.

2. Fragmentations and factoring compact operators. For this and the next section, let X be a Banach space with a finite-dimensional decomposition $\{E_n\}$. All our results are independent of the choice of the norm on X, so we may and do assume that $\{E_n\}$ is monotone.

This section is devoted to proving Theorem 2.3, which gives conditions on Y equivalent to s.f.p. for X. First we give two preliminary results. Lemma 2.1 allows us to restrict attention to diagonal operators $T_\lambda: X \to X$, where $\lambda = (\lambda_n)$ is a null sequence with $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$. That is, every compact operator X to X factors through a subspace of Y if and only if each $T_\lambda: X \to X$ factors through a subspace of Y. (Lemma 2.1 is similar to Proposition 3.2 of [3], we give a proof of the Lemma for completeness.)

Lemma 2.1. *If $U: Z \to X$ is a compact operator between Banach spaces, then there is a null sequence $\lambda = (\lambda_n)$ with $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$ and a compact operator $V: Z \to X$ such that $U = T_\lambda V$.*

Proof. Since $\{E_n\}$ is a monotone f.d.d. for X, Q_n, the natural projection onto the lin span $\{E_1, \ldots, E_n\}$, has norm one. Let K be the closure in X of the U-image of the unit ball of Z. Since K is compact, for each $\varepsilon > 0$, there is an m so that $n \geqslant m$ and $x \in K$ imply $\|Q_n x - x\| < \varepsilon$ ([14], p.12). Let $n(0) = 0$ and inductively choose $n(k+1) > n(k)$ so that $j \geqslant n(k+1)$ and $x \in K$ imply $\|Q_j x - x\| < 2^{-k-1}$. Define $\lambda_n = k^{-1}$ for $n(k-1) < n \leqslant n(k)$. Define $V: Z \to X$ by $Vz = \sum_n \lambda_n^{-1}(Q_n - Q_{n-1})Uz = \sum_k k(Q_{n(k)} - Q_{n(k-1)})Uz$.

Suppose $\|z\| \leqslant 1$; then $\|(Q_{n(1)} - Q_{n(0)})Uz\| \leqslant \|U\|$ and, for $k > 1$, $\|(Q_{n(k)} - Q_{n(k-1)})Uz\| < 2^{-k+2}$. Hence $\|V\| < \|U\| + \sum_k k2^{-k+2} < \infty$, and $T_\lambda V = T_\lambda\big(\sum \lambda_n^{-1}(Q_n - Q_{n-1})U\big) = \sum(Q_n - Q_{n-1})U = U$. If V is not already compact, let $\mu = (\mu_n)$ be the sequence of positive reals with $\mu_n^2 = \lambda_n$. Now $T_\lambda V = T_\mu(T_\mu V)$ and $T_\mu V$ is compact. ■

The following proposition is a preliminary version of Theorem 2.3, but it is easier to give direct proof than to deduce it from the theorem. Some partial converses of Proposition 2.2 are the subject of Section 3.

PROPOSITION 2.2. *If Y has s.f.p. for X, then X is u.f.r. in Y.*

Proof. Let X_n be the linear span of the first n elements of the f.d.d. $\{E_n\}$. It suffices to show that there are subspaces $Y_n \subset Y$ with $d(X_n, Y_n)$ uniformly bounded. We complete the proof by showing that we can make $d(X_n, Y_n)$ tend to infinity as slow as we like.

Let $\lambda = (\lambda_n)$ be any sequence with $\lambda_1 = 1$ and λ_n^{-1} monotonically increasing to ∞. Since $T_\lambda\colon X \to X$ is compact, the hypothesis implies that there is a subspace W of Y and operators $U\colon X \to W$ and $V\colon W \to X$ with $T_\lambda = VU$. Let $Y_n = U(X_n) = V^{-1}(X_n)$. Thus restricting to X_n or Y_n, we have $U^{-1} = T_\lambda^{-1} V$. Thus $d(X_n, Y_n) \leqslant \|U\|\|V\|T_\lambda^{-1}$ restricted to $X_n \leqslant 2\|U\|\|V\|\lambda_n^{-1}$ by Fact 1.1. ■

DEFINITION OF FRAGMENTATION. Suppose that $\{Y_k\}$ is a sequence of finite-dimensional subspaces of Y with $Y_j \cap Y_k = \{0\}$, if $j \neq k$. Define P_j to be the projection from $\operatorname{lin\,span}\{Y_k\}_1^\infty$ onto Y_j given by $P_j(\sum_k x_k) = x_j$ if $x_k \in Y_k$. Suppose further that, for each j, $\|P_j\| < \infty$. Then each P_j has a norm perserving extension to $W = \operatorname{cl\,lin\,span}\{Y_k\}_1^\infty$, which we will also call P_j. A triple $(\{Y_k\}, \{P_k\}, W)$ satisfying all the above conditions is a *fragmentation* of Y.

Let us make the following notations. Let A be the set of all sequences $(n(k))$ of i ntegerswith $0 = n(0) < n(1) < \ldots$ Let B be the set of all sequences of positive reals (s_k) increasing to infinity with $s_1 = 1$. Since X has a monotone f.d.d. $\{E_n\}$, for each $(n(k)) \in A$ we define $Z_k = \operatorname{lin\,span}\{E_{n(k-1)+1}, \ldots, E_{n(k)}\}$. And, finally, for $(s_k) \in B$, we represent the following statement by:

(*) *For each $(n(k)) \in A$, there is a fragmentation $(\{Y_k\}, \{P_k\}, W)$ of Y with* $\max\{d(Z_k, Y_k), \|P_k\|\} = O(s_k)$.

THEOREM 2.3. *For X and Y Banach spaces, X with a f.d.d., the following are equivalent:*

(i) *Y has s.f.p. for X.*

(ii) *For each $(s_k) \in B$, (*) is true.*

(iii) *For some $(s_k) \in B$, (*) is true.*

Proof. Suppose first that (i) is true; then for each $(s_k) \in B$ and $(n(k)) \in A$, define $\lambda = (\lambda_n)$ by $\lambda_n = s_k^{-1}$ for $n(k-1) < n \leqslant n(k)$. By the hypothesis, the compact operator $T_\lambda\colon X \to X$ factors through W, a subspace of Y, by operators $U\colon X \to W$ and $V\colon W \to X$. We may and do assume that $W = \operatorname{cl} U(X)$. Now let $Y_k = U(Z_k) = V^{-1}(Z_k)$. Thus, restricting to Z_k or Y_k, we have $U^{-1} = T_\lambda^{-1} V$. Hence

$$d(Z_k, Y_k) \leqslant 2\|U\|\|V\|\lambda_{n(k)}^{-1} = 2\|U\|\|V\|s_k,$$

by Fact 1.1.

Again, let Q_n be the natural projection onto $\operatorname{lin\,span}\{E_1, \ldots, E_n\}$ in X. Since $P_k = UT_\lambda^{-1}(Q_{n(k)} - Q_{n(k-1)})V$ are the projections needed to make $(\{Y_k\}, \{P_k\}, W)$ a fragmentation and

$$\|P_k\| \leqslant 4\|U\|\|V\|\lambda_{n(k)}^{-1} = 4\|U\|\|V\|s_k,$$

by Fact 1.1. We have shown (ii).

Now suppose that (iii) is true for the sequence $(s_k) \in B$. Let (t_k) be a null sequence with $1 = t_1 \geqslant t_2 \geqslant \ldots > 0$ and $\sum_k t_k s_{k+1} < \infty$. To show (i), by Lemma 2.1, it suffices to show that each diagonal compact operator $T_\lambda\colon X \to X$ factors through a subspace of Y. Let $\lambda = (\lambda_n)$ be a null sequence with $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$ and let $\mu = (\mu_n)$ be the sequence such that $\mu_n > 0$ and $\mu_n^2 = \lambda_n$. Let $n(0) = 0$ and inductively choose $n(k+1) \geqslant n(k)$ so that $j \geqslant n(k+1)$ implies $\mu_j \leqslant t_{k+1}$.

By hypothesis, since $(n(k)) \in A$, there is a constant M and a fragmentation $(\{Y_k\}, \{P_k\}, W)$ with $d(Z_k, Y_k) \leqslant Ms_k$ and $\|P_k\| \leqslant Ms_k$. Let $J_k\colon Z_k \to Y_k$ be an isomorphism with $\|J_k\| \leqslant Ms_k$ and $\|J_k^{-1}\| \leqslant 1$. Let $T_k\colon Z_k \to Z_k$ be the restriction of T_μ to Z_k.

Define $U\colon X \to W$ and $V\colon W \to X$ by $U = \sum_k J_k T_k (Q_{n(k)} - Q_{n(k-1)})$ and $V = \sum_k T_k J_k^{-1} P_k$. Now

$$\|U\| \leqslant \sum_k \|J_k\|\|T_k\|\|Q_{n(k)} - Q_{n(k-1)}\|$$
$$\leqslant \sum_k (Ms_k)(2\mu_{n(k-1)})(2) \leqslant 4M \sum_k s_k t_{k-1} < \infty,$$
$$\|V\| \leqslant \sum_k \|T_k\|\|J_k^{-1}\|\|P_k\|$$
$$\leqslant \sum_k (2\mu_{n(k-1)})(1)(Ms_k) \leqslant 2M \sum_k t_{k-1} s_k < \infty.$$

And finally

$$VU = \Big(\sum_k T_k J_k^{-1} P_k\Big)\Big(\sum_j J_j T_j (Q_{n(j)} - Q_{n(j-1)})\Big)$$
$$= \sum_k T_k J_k^{-1} J_k T_k (Q_{n(k)} - Q_{n(k-1)}) = \sum T_k^2 (Q_{n(k)} - Q_{n(k-1)}) = T_\lambda. \ ■$$

3. Uniform finite representability and factoring compact operators.

Theorem 3.1 is a partial converse to Proposition 2.2. First we need some definitions we remind the reader that X has a monotone f.d.d. $\{E_n\}$. Let Y and Z be Banach spaces.

We say Y is *finitely stable* if there is a constant K with Y K-u.f.r. in each subspace $Y_0 \subset Y$ with $\dim(Y/Y_0) < \infty$. We say Z is *stably finitely representable* in Y if there is a constant K with Z K-u.f.r. in each subspace $Y_0 \subset Y$ with $\dim(Y/Y_0) < \infty$. We say Y is *locally square* if $Y \times Y$ is u.f.r. in Y.

THEOREM 3.1. *If X and Y are Banach spaces, X with a f.d.d. and X is u.f.r. in Y, then each of the following is sufficient to imply Y has s.f.p. for X.*

(i) *$(X \times X \times \ldots)p$ is isomorphic to a subspace of X, for some p, $1 \leqslant p \leqslant \infty$.* (By $p = \infty$ we mean the c_0-sum.)

(ii) *X is stably finitely representable in Y.*

(iii) *$(Y \times Y \times \ldots)p$ is isomorphic to a subspace of Y, for some p, $1 \leqslant p \leqslant \infty$.*

(iv) *Y is finitely stable.*

(v) *$Y \times Y$ is isomorphic to a subspace of Y.*

Proof. We will show (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (v), (iv) $\Rightarrow$ (ii) and complete the proof by showing that each of the conditions (ii) and (v) imply condition (iii) of Theorem 2.3.

(iv) $\Rightarrow$ (ii): If X is K_1-u.f.r. in Y and Y is K_2-u.f.r. in Y_0, then X is K_1K_2-u.f.r. in Y_0.

(iii) $\Rightarrow$ (v): Is obvious.

(i) $\Rightarrow$ (ii): Let Y_0 be a finite-codimensional subspace of Y. By hypothesis, we may assume $(X \times X \times \ldots)p$ is K-u.f.r. in Y. Let S be a finite rank projection with $Y_0 = \ker S$. Let E be an r-dimension subspace of X, let $m = \dim(Y/Y_0)$ and let F be the l_p-sum of $(mr+1)^2$ copies of E. Since $F \subset (X \times X \times \ldots)p$, there is a subspace $Z \subset Y$ with $d(Z, F) \leqslant K$. Let P be any projection from Y onto Z. By Lemma 4.1 of [8], there is a projection Q in Y with $QS = SQ = 0$ and $d(Q(Y), E) \leqslant K$. Now $SQ = 0$ implies $Q(y) \subset \ker S = Y_0$; thus X is K-u.f.r. in Y_0. Therefore, (ii) is true. We observe that this proof also shows (iii) $\Rightarrow$ (iv), and that (ii) is implied by the following weaker condition:

(i′) *The sequence $\{X, (X \times X)p, \ldots, (X \times \ldots \times X)p \ldots\}$ of Banach spaces are each K-u.f.r. in Y, for some K and some p, $1 \leqslant p \leqslant \infty$.*

(ii) $\Rightarrow$ Theorem 2.3, condition (ii): We actually show more, namely that (s_k) can be a bounded sequence. Let $(n(k)) \in A$ and $\{Z_k\}$ as in Theorem 2.3, and suppose X is K-u.f.r. in Y. Inductively choose $\{Y_k\}$ and $\{W_k\}$ so that

(1) $d(Y_k, Z_k) \leqslant K$,

(2) $Y_{k+1} \subset W_k$,

(3) $Y \supset W_1 \supset W_2 \supset \ldots$,

(4) $\dim(Y \setminus W_k) < \infty$,

(5) The projection from $Y_1 + \ldots + Y_k + W_k$ onto $Y_1 + \ldots + Y_k$ with kernel W_k has norm $\leqslant 1 + \delta_k$ $\left(\prod (1+\delta_k) < \varepsilon\right)$.

We can choose Y_{k+1} satisfying (1) and (2) by hypothesis, where as choosing W_{k+1} can be modelled on the standard construction of a basic sequence in any Banach space as in [14] (see p. 10).

(v) $\Rightarrow$ Theorem 2.3, condition (iii): The idea of this proof is fairly easy. We construct a sequence $\{W_k\}$ of subspaces of Y, with each W_k isomorphic to Y. By picking $Y_k \subset W_k$ we will obtain a fragmentation of Y with the desired properties. We obtain the $\{W_k\}$ by infinitely often injecting $Y \times Y \subset Y$ into the second factor of $Y \times Y \subset Y$. We pass to the details.

Let $J\colon Y \times Y \to Y$ be an isomorphism of $Y \times Y$ onto a subspace of Y. Let $Q_i\colon Y \times Y \to Y \times Y$ be the projection onto the ith factor, $i = 1, 2$. We may assume $\|Q_i\| = 1$. Let $I_i\colon Y \to Y \times Y$ be the injection of Y onto the ith factor, $i = 1, 2$. Let $M = \|J\|\,\|J^{-1}\|$.

We inductively define:

(1) $J_k^i\colon Y \to Y$ an isomorphism into a subspace of Y, for $i = 1, 2$ and $k = 1, 2, \ldots$

(2) $W_k^i = J_k^i(Y)$, for $i = 1, 2$ and $k = 1, 2, \ldots$

(3) P_k^i a projection from $W_k^1 \oplus W_k^2$ onto W_k^i, for $i = 1, 2$.

First let $J_1^i = JI_i$ and $P_1^i = JQ_iJ^{-1}$; we note that $d(W_1^i, Y) \leqslant M$ and $\|P_1^i\| \leqslant M$, for $i = 1, 2$. Define $J_{k+1}^i = J_k^2 J I_i$ and $P_{k+1}^i = J_k^2 J Q_i (J)^{-1} (J_k^2)^{-1}$ for $i = 1, 2$. Now $d(W_{k+1}^i, Y) \leqslant M d(W_k^2, Y) \leqslant M^{k+1}$ and $\|P_{k+1}^i\| \leqslant M^{k+1}$, for $i = 1, 2$.

Let $W_0 = \operatorname{lin\,span}\{W_k^1\}_{k=1}^{\infty}$ and let R_k be the natural projection from W_0 onto W_k^1. Now $R_k = P_k^1 P_{k-1}^2 \ldots P_2^2 P_1^2$ on W_0 and so $\|R_k\| \leqslant M^k \cdot M^{k-1} \cdot \ldots \cdot M \leqslant M^{k(k+1)}$.

Now let $(n(k))$ be a sequence in A, and Z_k as in Theorem 2.3. Since X is K-u.f.r. in Y, there is a subspace Y_k in W_k^1 with $d(Z_k, Y_k) \leqslant KM^k$. Furthermore, the projections $P_j\colon \operatorname{lin\,span}\{Y_k\}$ onto Y_j are just the restriction of R_j to $\operatorname{lin\,span}\{Y_k\}$; hence $\|P_k\| \leqslant M^{k(k+1)}$. Letting $W = \operatorname{cl\,lin\,span}\{Y_k\}$, the fragmentation $(\{Y_k\}, \{P_k\}, W)$ satisfies $(*)$ for $(s_k) \in B$, where $s_1 = 1$ and $s_k = M^{k(k+1)}$ for $k > 1$. ■

Remarks. 1. Since l_p satisfies condition (i), we obtain as a corollary Figiel's Theorem 7.7 [8] that Y has the s.f.p. for l_p if and only if l_p is u.f.r in Y.

2. We also obtain Figiel's result [8], Theorem 6.1, that every compact operator factors through a subspace of Y is and only if c_0 is u.f.r. in Y (i.e. Y is universal for finite-dimensional spaces). Since such an Y satisfies condition (iv).

3. It seems reasonable that Theorem 3.1 could be extended to cases where

(vi) *X is finitely stable and*
(vii) *Y is locally square.*

I do not know if this can be done, but methods used above, in particular Lemma 4.1 of [8], require too much dependence on the dimensions of Z_k.

4. It is easy to see that each of conditions (i) through (vii) imply that $X\times X$ is u.f.r. in Y. Figiel in [7], constructs examples of Banach spaces X, for which $X\times X$ is not isomorphic to a subspace of X. Actually, he shows that X is not locally square nor finitely stable. We give another proof in the next section.

5. A similar development shows that Theorem 3.1 remains true if we drop the condition that X has an f.d.d. and weaken the conclusion to every compact operator $R\colon X\to X$, which is the uniform limit of finite rank operators, factors through a subspace of Y. Hence Theorem 3.1 is true if X has the a.p.

6. Condition (ii) implies that there is a subspace Y_0 of Y such that each compact operator $T\colon X\to X$ factors through Y_0. This subspace can be constructed in a fashion similar to the spaces C_p (see [8]). The sequence $\{G_i\}_{i=1}^{\infty}$ ([8], p. 194) are chosen to be dense in the finite-dimensional subspaces of X, rather than in all Banach spaces as in [8]. The construction proceeds as in the proof of part (ii) of the theorem.

7. If $Z\oplus Z$ is isomorphic to a complemented subspace of Z, and Y has the s.f.p. for Z, then there is a constant K such that for each compact $T\colon Z\to Z$, the factorization operators U and V can be chosen to satisfy $\|U\|\|V\|\leqslant K\|T\|$. The proof combines the techniques of Theorem 3.1 (v) and (i) $\Rightarrow$ (ii) of Theorem 5.2 of [8]. Hence if such a Z has the b.a.p. ([14], p. 12), then Z is u.f.r. in each Y having the s.f.p. for Z.

4. Galactic Banach spaces. A Banach space Z will be called *galactic* if for each separable Banach space Y, then Y u.f.r. in Z implies Y is isomorphic to a subspace of Z. Roughly speaking, a galactic space Z is subspace universal for separable Banach spaces made out of Z's universe of finite-dimensional subspaces.

An obvious example of a galactic space is Hilbert space and every galactic space must contain a subspace isomorphic to l_2. Other examples include l_∞ and $C[0,1]$ as well as $L_p[0,1]$ (see remark after Lemma 4.2 below). In this section, we will construct other examples of galactic spaces which have unconditional basis and some of which are neither finitely stable nor locally square. For Y a galactic space and X with a f.d.d., clearly Y has the s.f.p. if and only if X is u.f.r. in Y.

Let $\{E_n\}$ be a sequence of finite-dimensional Banach spaces with the property, for each integer m and $\varepsilon>0$, there is N so that for each m-dimensional subspace Y of some E_n with $n\geqslant N$, we have $d(l_2^m, Y)<1+\varepsilon$. Examples of such sequences $\{E_n\}$, include $\{l_{p(n)}^{d(n)}\}$ for any sequence of integers $(d(n))$ and any sequence of reals $(p(n))$ with limit two (Corollary 2.2 of [15]).

Let $X_k=(E_k\times E_{k+1}\times\ldots)_2$ and let $X=X_1$. X satisfies:

(**) *for each integer m and $\varepsilon>0$, there is an N, so that for each m-dimensional subspace Y of X_N, we have $d(Y,l_2^m)<1+\varepsilon$.*

We observe that X has a complemented subspace isomorphic to l_2, hence by Pełczyński's decomposition method (see [14], p. 30) X is isomorphic to $X\oplus l_2$.

THEOREM 4.1. *X is a galactic space.*

Remark. Figiel has shown in [7] that if $E_n=l_{p(n)}^{d(n)}$, with $p(n)$ strictly decresing with limit two, then for some choices of $d(n)$, $X\times X$ is not isomorphic to a subspace of X. By the theorem (or by inspecting Figiel's proof [7]), X is not locally square nor finitely stable. For other results about Figiel's space see [4].

We need some simple facts about non-standard norm spaces. First so we may quote results, particularly of [10] and [11], we make the technical assumption that our nonstandard model is an enlargement which is at least $\aleph_1$-saturated. If Y is a Banach space, we will write ${}^*Y$ for the non-standard Banach space. We note that Y is a subset of ${}^*Y$ and $\|\cdot\|$ is an extension to ${}^*Y$ of the norm on Y.

Define fin $=\{x\in{}^*Y\colon \|x\|$ is finite (i.e. $\|x\|$ is infinitesimally close to a standard real number, $|||x|||)\}$ and let $\mu=\{x\in{}^*Y\colon \|x\|$ is infinitesimal or equivalently $|||x|||=0\}$. On the quotient vector space fin$/\mu=\hat{Y}$ $|||\cdot|||$ is a norm. In fact, $(\hat{Y},|||\cdot|||)$ is a standard Banach space called the *non-standard hull* of $(Y,\|\cdot\|)$. We now restate Theorem 2.3 of [10] as Lemma 4.2.

LEMMA 4.2. *If $(Y,\|\cdot\|)$ is a Banach space, then its non-standard hull $(\hat{Y},|||\cdot|||)$ is galactic.*

Remark. We now can give an easy proof that $L_p[0,1]$ is galactic. Let $Y=L_p$; then $\hat{Y}$ is isomorphic to an $L_p(\mu)$-space by Corollary 2.5 of [11]. Furthermore any separable subspace of $L_p(\mu)$ is isomorphic to a subspace of $L_p[0,1]$ ([14], p. 124). Of course, this result is well known, for example, see [14], p. 122, where a "similar" proof is given.

Proof of Theorem 4.1. Let X be as given. Since X is reflexive, by [1], Theorem 3.1 and [2], Theorem 4.1 the non-standard hull $\hat{X}$ is isomorphic to $X\oplus H$, where H is defined below. Let φ be the quotient operator: fin $\to\hat{X}$, and let $P=\{x\in\text{fin}\colon |f(x)|$ is infinitesimal for each standard f in $X'\}$. Then $H=\varphi(P)$. By the remark before Theorem 4.1, it suffices to show that H is isomorphic to Hilbert space. This will be done by showing each finite-dimensional subspace of H is close to Hilbert space of the same dimension.

Let Y be a finite-dimensional subspace of H. Let Y_1 be a subspace of $P \subset {}^*X$ of the same dimension as Y and with $\varphi(Y_1) = Y$. Let $m = \dim Y = \dim Y_1$ and let $\varepsilon > 0$; by (**), there is an N so that each m-dimensional subspace of X_N is $1+\varepsilon$ close to l_2^m. The same statement is true for m-dimensional subspaces of ${}^*X_N \subset {}^*X$. Let $P_N\colon X \to X$ be natural projection onto the lin span $\{E_1, \ldots, E_N\}$, we can write $P_N = \sum_1^M f_i \otimes x_i$, and $X_N = \ker(I - P_N)$. Now ${}^*P_N = \sum_1^M f_i \otimes x_i = P_N$ and so ${}^*X_N = \ker({}^*I - P_N)$. Letting $Y_2 = (I-P_N)(Y_1)$, and since $Y_1 \subset P$, we have $d(Y_1, Y_2)$ is infinitesimally close to one. Now $d(Y, l_2^m) \leqslant d(Y_1, Y_2) d(Y_2, l_2^m) \leqslant 1+\varepsilon$, since $Y_2 \subset {}^*X_N$. ■

Remark. A modification of the proof of Theorem 4.1 yields that the space $Y \oplus L_q$ is galactic, when $Y = (E_1 \times E_2 \times \ldots)_q$, $E_n = l_{p(n)}^{d(n)}$, limit $p(n) = q$ and $1 < q < \infty$. (Use Theorem A of [15].)

5. Schwartz spaces. In this section we explore interconnections between Banach spaces and Schwartz spaces, in particular, with respect to the approximation property. We find it convenient to use the notions of prevarieties. A collection of LCS's is a *prevariety* [1] if it is closed with respect to subspaces, products and isomorphic images. The collection $\mathscr{S}$ of all Schwartz spaces is a prevariety (see [12], p. 275).

For a Banach space Y, let $\mathscr{S}_Y$ be the set of Schwartz spaces isomorphic to a subspace of a power of Y (i.e. Y^I for some index set I). It is easy to see that $\mathscr{S}_Y$ is a prevariety; in fact, it is the intersection of $\mathscr{S}$ with $\varrho\nu(Y)$, the smallest prevariety containing Y. If $Y = l_p$, we will write $\mathscr{S}_Y = \mathscr{S}_p$.

Our first order of business is to construct some examples of spaces in $\mathscr{S}_X$, when X has a finite-dimensional decomposition $\{E_n\}$.

Example 5.1. Let $\lambda = (\lambda_n)$ be a null sequence with $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \ldots > 0$. Let T_k be the diagonal operator from X to X, corresponding to $\mu = (\mu_n)$, where $\mu_n > 0$ and $\mu_n^{2^k} = \lambda_n$ for $k = 1, 2, \ldots$ Let X_λ be the projective limit of the sequence:

$$\ldots \xrightarrow{T_3} X_2 \xrightarrow{T_2} X_1 \xrightarrow{T_1} X_0,$$

where each $X_k \equiv X$. If we let $S_k\colon X \to X_k$ be T_k, then $T_\lambda\colon X \to X_0$ is the composition $T_1 T_2 \ldots T_k S_k$. Hence there is a natural continuous operator $T_\infty\colon X \to X_\lambda$, whose image in X_0 is the same as $T_\lambda\colon X \to X_0$. It is easy to see that X_λ is a Schwartz–Fréchet space ([12], Proposition 9, p. 282) in $\mathscr{S}_X$.

Before the next example, some definitions are in order. Let X have a f.d.d. $\{E_n\}$ and let P_n be the projection $\sum_k x_k \to x_n$. If $E_n' = P_n'(X')$, then $\{E_n\}$ is said to be *shrinking* if $\{E_n'\}$ is a f.d.d. for X'. Let Y be a Banach space with norm topology ξ. Then ξ_S, the topology of uniform convergence on Y'-norm null sequences, is the strongest Schwartz topology on Y weaker than ξ [1]. If (Z, η) is a norm subspace of (Y, ξ), then η_S is the restriction of ξ_S to Z. If $A \subset Y'$, then

$$A^\circ \text{ (the absolute polar) } = \{y \in Y\colon |\langle y, a\rangle| \leqslant 1 \text{ for } a \in A\}.$$

Example 5.2. Suppose that X has a shrinking f.d.d. $\{E_n\}$; then $(X, \xi_S) \in \mathscr{S}_X$. To see this, let $K = \{x_n'\}$ be a norm null sequence in X'. Since $\{E_n\}$ is shrinking, Lemma 2.1 applies to X'. Thus there is a null sequence $\lambda = (\lambda_n)$ with $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$, and if U' is the unit ball of X', then $T_\lambda'\colon X' \to X'$ maps some multiple of U' onto a superset of K. We may assume $T_\lambda'(U') \supset K$. Since T_λ' is the transpose of $T_\lambda\colon X \to X$, and by Lemma 6, p. 39 of [18], $\big(T_\lambda'(U')\big)^\circ = T_\lambda^{-1}\big((U')^\circ\big) = T_\lambda^{-1}(U)$, where U is the unit ball of X. But $K \subset T_\lambda'(U')$, so that $K^\circ \supset T_\lambda^{-1}(U)$. We have shown that for each ξ_S-neighborhood $V = K^\circ$, of the origin, there is a ξ_S-neighborhood $W = T_\lambda^{-1}(U)$, with $W \subset V$ and X_W isomorphic to (X, ξ). (By Fact 5.3 below.) Hence $(X, \xi_S) \in \mathscr{S}_X$.

Applying our earlier results to the prevarieties of the form $\mathscr{S}_X$, we obtain Theorem 5.4. We need the following fact whose proof is easy and will not be given.

Fact 5.3. *Let (E, ξ) be a LCS, Y a Banach space with unit ball U, and $S\colon E \to Y$ be an operator. Then $W = S^{-1}(U)$ is a ξ-neighborhood with E_W isomorphic to a subspace of Y.*

Theorem 5.4. *If X has f.d.d. and if $Y \times Y$ is isomorphic to a subspace of Y. Then $\mathscr{S}_X \subset \mathscr{S}_Y$ if and only if X is u.f.r. in Y.*

Proof. Suppose $\mathscr{S}_X \subset \mathscr{S}_Y$; by the proof of Proposition 2.2, X will be u.f.r. in Y if each $T_\lambda\colon X \to X$ factors through a subspace of Y. By Example 5.1, $X_\lambda \in \mathscr{S}_X \subset \mathscr{S}_Y$, hence X_λ is isomorphic to a subspace of a power of Y. In the notation of Example 5.1, let U_k be the unit ball of X_k. Then there is a X_λ-neighborhood V and $k \geqslant 1$, with $U_k \subset V \subset U_0$ and $(X_\lambda)_V$ isomorphic to a subspace of Y^n, a product of a finite number of copies of Y. Thus, by Example 5.1 and by hypothesis, $T_\lambda\colon X \to X$ factors through a subspace of Y. Therefore, Proposition 2.2 implies X is u.f.r. in Y.

Conversely, suppose that X is u.f.r. in Y. Since $Y \times Y$ is isomorphic to a subspace of Y, X^n is u.f.r. in Y. Now X^n has a f.d.d., so Theorem 3.1 implies that Y has the s.f.p. for X^n.

Now let E be a LCS in $\mathscr{S}_X$. Thus for every neighborhood U of the origin in E, there are neighborhoods V and W with $V \subset W \subset U$, $E_V \to E_W$

compact and E_W isomorphic to a subspace of X^n. By considering $E_V \to E_W$ as a compact operator $E_V \to X^n$ and applying Lemma 2.1, we have a factorization $E_V \to X^n \to X^n$. Now the latter operator factors through a subspace of Y. Fact 5.3 yields a neighborhood G with $V \subset G \subset W$ and E_G isomorphic to a subspace of Y. Hence $E \in \mathscr{S}_Y$. ■

Remark. We can now give a complete description of the inclusion relations among the prevarieties $\mathscr{S}_p$, for $1 \leqslant p \leqslant \infty$:

(i) $\mathscr{S}_2 \subset \mathscr{S}_p$, for $1 \leqslant p \leqslant \infty$.

(ii) $\mathscr{S}_p \subset \mathscr{S}_\infty = \mathscr{S}$, for $1 \leqslant p \leqslant \infty$.

(iii) $\mathscr{S}_p \subset \mathscr{S}_q$, for $1 \leqslant q \leqslant p \leqslant 2$.

(iv) $\mathscr{S}_p \not\subset \mathscr{S}_q$, otherwise.

These inclusions are familiar, since $\mathscr{S}_p \subset \mathscr{S}_q$ if and only if l_p is isomorphic to a subspace of L_q. This follows since L_q and l_q generate the same Schwartz prevariety $\mathscr{S}_X$, L_q is a galactic space, and by Theorem 5.4. (See [14] for which l_p are subspaces of L_q.)

A *universal generator* for the prevariety $\mathscr{S}_X$, is a LCS $E \in \mathscr{S}_X$, such that each $F \in \mathscr{S}_X$ is isomorphic to a subspace of power of E. The following proposition has been proved for some special cases [13], [16], [17], and [3].

PROPOSITION 5.5. *If Y is a Banach space with $Y \times Y$ isomorphic to a subspace Y, then each $E \in \mathscr{S}_Y$ is isomorphic to a subspace of a power of (Y, ξ_S). Thus if $(Y, \xi_S) \in \mathscr{S}_Y$, then (Y, ξ_S) is a universal generator for $\mathscr{S}_Y$.*

Proof. By hypothesis, each $E \in \mathscr{S}_Y$ has a neighborhood basis $\mathscr{U}$ such that $U \in \mathscr{U}$ implies E_U is isomorphic to a subspace of Y. Thus there is a natural isomorphism E into $Y^{\mathscr{U}}$:

$$E \to \prod_{U \in \mathscr{U}} E_U \to Y^{\mathscr{U}} \to (Y, \xi_S)^{\mathscr{U}}.$$

Since the latter operator is continuous, the proof will be complete if the composition $E \to (Y, \xi_S)^{\mathscr{U}}$ is open.

Let $U \in \mathscr{U}$ and let $V \in \mathscr{U}$ so that $V \subset U$ and $E_V \to E_U$ is compact. The transpose $(E_U)' \to (E_V)'$ is also compact. Let $K \subset (E_V)'$ be the image of U° by the transpose. There is a norm null sequence $\{x_n'\}$ in $(E_V)'$ with K contained in the closed absolutely convex hull of $\{x_n'\}$ ([21], p. 111). Considering E_V as a subspace of Y, use the Hahn–Banach Theorem to extend $\{x_n'\}$ to $\{\tilde{x}_n'\}$ a null sequence in Y'. Now $W = \{\tilde{x}_n'\}^\circ$ is a ξ_S-neighborhood of Y. $W \cap E_V \subset \{x_n'\}^\circ \subset K^\circ = S^{-1}(U)$, where S: $E_V \to E_U$ is the natural operator (the last equality is Lemma 6, p. 39 [18]). Thus $E \to (Y, \xi_S)^{\mathscr{U}}$ is an isomorphism. ■

COROLLARY 5.6. *(X, ξ_S) is a universal generator for $\mathscr{S}_X$ if X has a shrinking f.d.d. and $X \times X$ is isomorphic to a subspace of X.*

Remark. Since l_p, $1 < p < \infty$, and c_0 satisfy these conditions, (l_p, ξ_S) is a universal generator for $\mathscr{S}_p$ and (c_0, ξ_S) is a universal generator for $\mathscr{S}$.

The LCS E has the *approximation property* if the identity operator $E \to E$ is in the closure of the set of finite rank operators when the space of operators: $E \to E$ is given the precompact-open topology [21]. It is known that if there is an $E \in \mathscr{S}_Y$ without the a.p., then there is a subspace of Y^n without the a.p. ([21, p. 109). Proposition 5.7 provides a converse to this result if Y has a shrinking f.d.d. (apply Example 5.2 and [21], p. 113). As a corollary, each of $\mathscr{S}_p$, $2 < p \leqslant \infty$, has a LCS without the a.p. in light of [6] and [5].*

PROPOSITION 5.7. *Let Y be a Banach space with dual Y'. Then (Y, ξ_S) has the a.p. implies that Y' has the a.p., and conversely if Y is reflexive.*

Proof. Now (Y, ξ_S) has the a.p. if and only if, for each precompact set K and open set U, there is a finite rank operator F with $(I-F)(K) \subset U$. Since ξ_S is a Schwartz topology of the dual pair (Y, Y') and bounded sets are precompact in a Schwartz space, we may assume that K runs over the scalar multiples of the unit ball of Y. By the definition of ξ_S, U may be assumed to be the polar of a norm null sequence of Y'. By Lemma 6, p. 39 of [18],

$$\big((I-F)(K)\big)^\circ = \big((I-F)'\big)^{-1}(K^\circ).$$

Thus $(I-F)(K) \subset U$ if and only if $(I-F)'(U^\circ) \subset K^\circ$. Now as U runs over the open sets of ξ_S, U° is running over the compact sets of Y', and as K runs over the precompact sets of ξ_S, K° is running over the open sets of Y'. Thus (Y, ξ_S) has the a.p. implies Y' has the a.p.

For the converse implication, we observe that the above proof is reversible if in Y' we can choose the finite rank operator to be the transpose of a finite rank operator on Y. That this is the case is an easy consequence of reflexivity. ■

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* *Added in proof*: It is possible to modify ([21], p. 109) so as to deduce the existence of a Fréchet space in $\mathscr{S}_p$, $2 < p \leqslant \infty$, without the a.p.

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Boundary limits of Green's potentials along curves II
Lipschitz domains

by

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Abstract. On a Lipschitz domain D in space, let μ be a mass distribution and u the Green potential of μ. Conditions on μ are given so that $u \not\equiv +\infty$; under the same condition we show that the boundary limits of u along curves with certain differentiability properties are zero almost everywhere.

Green's potential occurs in the study of subharmonic and superharmonic functions via Riesz decomposition theorem ([5], p. 116). Let D be an open subset of $\boldsymbol{R}^n$ having a Green's function G; Green's potential u given by a mass distribution μ is defined to be

$$u(x) = \int_D G(x, y)\, d\mu(y) \tag{0.1}$$

for every $x \in D$. When D is the unit disk in the plane, the necessary and sufficient condition for $u \not\equiv +\infty$ is

$$\int_D (1-|y|)\, d\mu(y) < +\infty;$$

under this condition u has radial limit zero at almost every point on the unit circle, see Littlewood [6]. Later in 1938, Privalov [7] proved the similar result for Green's potentials on the unit ball in $\boldsymbol{R}^n$. The nontangential limit of Green's potential need not exist at any point on the boundary, as pointed out by Zygmund, [9], pp. 644–645.

The purpose of this paper is to study the boundary limits of Green's potentials in a Lipschitz domain D in $\boldsymbol{R}^n$, $n \geqslant 3$ along curves with certain differentiability properties. The problem for $n = 2$ was studied in [11], where, with the aid of conformal mapping, we need only to study the limit of Green's potentials on $|z| < 1$ along curves with the same differentiability properties. When $n \geqslant 3$ the conformal mapping technique does not apply and it is not even obvious for which μ the Green's potential of μ is not identically $+\infty$. Our main tool is an estimate on a certain harmonic function in a cone derived from a series representation of that

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