STUDIA MATHEMATICA, T. LXIII (1978)

# Interpolating bases for spaces of differentiable functions

by

JERZY RYLL (Warszawa)

**Abstract.** The paper contains: (a) A construction of an extension operator into the space $C^p(I^d)$; (b) A construction of an interpolating basis for the space $C^p(I^d)$; (c) A characterization of Hölder continuous functions in terms of their coefficients in the decomposition with respect to the basis.

**0. Introduction.** The purpose of this paper is to give a new construction of a Schauder basis in the space $C^p(I^d)$ of $p$-times continuously differentiable functions on the $d$-dimensional cube $I^d$.

The problem of the existence of a basis in the space $C^1(I^d)$ goes back to Banach [1]. It was solved by Z. Ciesielski [2] and S. Schonefeld [10] independently. Z. Ciesielski and J. Domsta [4] constructed a basis in the space $C^p(I^d)$ for an arbitrary $p$. S. Schonefeld [11] constructed another basis in $C^p(I^d)$ (for $p = 0, \ldots, 4$ only) and in $C^p(T^d)$ (where $T$ is a one-dimensional torus). The relation between the Schonefeld bases and the Ciesielski–Domsta bases is akin to the relation between the Schauder basis and the Franklin basis in $C(I)$: the Schonefeld basis is interpolating while the Ciesielski–Domsta basis is an orthogonal system.

The basis $(\varphi_k)_{k=1}^{\infty}$ constructed in this paper (Theorem 3.2.1) has the following properties:

(i) It is an interpolating basis in $C(I^d)$.

(ii) It is a basis in each space $C^q(I^d)$ for $q = 0, \ldots, p$.

(iii) $\operatorname{diam}(\operatorname{supp}\varphi_k) \to 0$ as $k \to \infty$.

The third property is a new feature; the previously known bases do not satisfy (iii). The construction of the basis $(\varphi_k)_{k=1}^{\infty}$ leans heavily on a method of Filippov and Riabienkiĭ [5], pp. 158–165. The basic lemma (Lemma 2.2.1 below) concerns the interpolating by spline functions.

In the case $p = 0$ the construction of the basis $(\varphi_k)_{k=1}^{\infty}$ was described by the author in [9] (under the name "cube basis").

In Section 4 it is given an answer to the problem of Z. Ciesielski [3]. It is proved that derivatives of order $p$ of $f \in C^p(I^d)$ satisfy the Hölder condition with an exponent $s$ $(0 < s < 1)$ iff the sequence of coefficients of $f$ is bounded. We obtain as a corollary the result of J. Frampton and A. J. Tromba [6] that the spaces $H_{p+s}(I^d)$ and $l_\infty$ are isomorphic (for the definitions, see Preliminaries).

The author is very grateful to Z. Semadeni, Z. Ciesielski, and J. Domsta for helpful discussions.

## 1. Preliminaries

**1.1. On bases and spaces of differentiable functions.** Throughout this paper $\boldsymbol{R}$ denotes the set of reals, $\boldsymbol{N}$ the set of nonnegative integers, and $I$ the unit interval $[0,1]$; moreover, $d$ is a fixed positive integer (the dimension of the cube),

$$a = (a_1, \ldots, a_d) \in \boldsymbol{N}^d$$

is a multiindex, $|a| = \sum_{i=1}^{d} a_i$, $e_m = (\delta_{im})_{i=1}^d$ is the $m$th vector of the canonical basis of $\boldsymbol{R}^d$; $\theta = (0, \ldots, 0) \in \boldsymbol{N}^d$, $|x-y|$ denotes the Euclidean distance in $\boldsymbol{R}^d$. If $f$ is a function on $X$ and $Y$ is contained in $X$, then $f_{|Y}$ denotes the restriction of the function $f$ to $Y$. $C(X)$ denotes the space of continuous functions on the compact set $X$. $C^p(I^d)$ denotes the space of $p$-times continuously differentiable functions on the $d$-dimensional cube $I^d$ provided with the norm

$$\|f\|^{(p)} = \sup\{\|D^a f\|_\infty : |a| \leqslant p\},$$

where $D^a$ is the differential operator

$$D^a f = \frac{\partial^{|a|} f}{\partial x_1^{a_1} \ldots \partial x_d^{a_d}}$$

and $\|g\|_\infty = \sup\{|g(x)| : x \in I^d\}$.

For an $f$ in $C(I^d)$, let $\omega_f$ denote the modulus of continuity of $f$, i.e.

$$\omega_f(\delta) = \sup\{|f(x)-f(y)| : |x-y| \leqslant \delta;\ x, y \in I^d\} \quad \text{for} \quad \delta > 0.$$

If $f$ is in $C^p(I^d)$, then we define for $k \leqslant p$

$$\omega_{D^k f}(\delta) = \sup\{\omega_{D^a f}(\delta) : |a| = k\}.$$

The continuity of $f$ means that $\omega_f(\delta) \to 0$ as $\delta \to 0$; since $\omega_f$ is subadditive,

$$\omega_f(n\delta) \leqslant n\omega_f(\delta) \quad \text{for} \quad n \in \boldsymbol{N}.$$

For $0 < s < 1$, let

$$H_{p+s}(I^d) = \{f \in C^p(I^d) : \exists c > 0,\ \omega_{D^p f}(\delta) \leqslant c \cdot \delta^s\}.$$

Thus, $H_{p+s}(I^d)$ is the space of functions whose derivatives of order $p$ satisfy the Hölder condition with an exponent $s$. The norm in $H_{p+s}(I^d)$ is

$$\|f\|^{(p+s)} = \max\left(\sup\left\{\frac{|D^a f(x) - D^a f(y)|}{|x-y|^s} : |a| = p;\ x, y \in I^d\right\},\ \|f\|^{(p)}\right).$$



A sequence $(X_n)_{n=1}^\infty$ of finite-dimensional subspaces of a Banach space $X$ is called a *basis of finite-dimensional subspaces* iff each $f$ in $X$ can be written uniquely as

$$f = \sum_{n=1}^{\infty} f_n,$$

where $f_n \in X_n$ and the series converges in $X$.

A sequence $(\varphi_n)_{n=1}^\infty$ of elements of a Banach space $X$ is called a *Schauder basis* iff each $f$ in $X$ has a unique decomposition

$$f = \sum_{n=1}^{\infty} a_n(f)\varphi_n,$$

where $a_n(f)$ are scalars and the series converges in $X$ ($(a_n)_{n=1}^\infty$ is the associated sequence of coefficient functionals). If $(\varphi_n)_{n=1}^\infty$ is a Schauder basis for $X$, then $S_n$ are operators of partial sums, i.e.

$$S_n f = \sum_{i=1}^{n} a_i(f)\varphi_i,$$

and the number

$$\sup\{\|S_n f\| : \|f\| \leqslant 1,\ n \in \boldsymbol{N}\}$$

is called the *norm* of the basis $(\varphi_n)_{n=1}^\infty$. In [7] the following lemma is proved.

LEMMA 1.1.1. *If $(X_n)_{n=1}^\infty$ is a basis of finite-dimensional subspaces for $X$ and $(\varphi_i)_{i=N_n+1}^{N_{n+1}}$ are bases for $X_n$ with uniformly bounded norms, then $(\varphi_i)_{i=1}^\infty$ is a basis for $X$.*

A basis $(\varphi_n)_{n=1}^\infty$ for $C(I^d)$ is called an *interpolating basis with nodes* $(x_n)_{n=1}^\infty$ iff for each $f$ in $C(I^d)$ and each $n$ in $\boldsymbol{N}$

$$f(x_m) = S_n f(x_m) \quad \text{for} \quad m = 1, \ldots, n.$$

A basis for $C^p(I^d)$ is called *simultaneous* iff it is a basis in each space $C^q(I^d)$ for $q = 0, \ldots, p$.

**1.2. Some estimations for divided differences.** Let $T_i = [t_0^i, \ldots, t_{n_i}^i]$ be partitions of the unit interval $I$ and $0 = t_0^i < \ldots < t_{n_i}^i = 1$. Then $T_\theta = \prod_{i=1}^{d} T_i$ determines a partition of the cube $I^d$ into $n_1 \cdot \ldots \cdot n_d$ cuboids (i.e. rectangular parallelepipeds). Let $\psi\colon T_\theta \to \boldsymbol{R}$ be a function defined on the set $T_\theta$ of vertices of the cuboids. The divided differences of $\psi$ are defined by induction:

$$\Delta_{T_\theta}^{\theta}\psi = \psi,$$

$$(1.2.1) \quad \Delta_{T_\theta}^{a+e_m}\psi(t_{k_1}^1, \ldots, t_{k_d}^d) = \frac{\Delta_{T_\theta}^{a}\psi(t_{k_1}^1, \ldots, t_{k_m+1}^m, \ldots, t_{k_d}^d) - \Delta_{T_\theta}^{a}\psi(t_{k_1}^1, \ldots, t_{k_m}^m, \ldots, t_{k_d}^d)}{t_{k_m+a_m+1}^m - t_{k_m}^m},$$

where $k_m = 0, \ldots, n_m - \alpha_m - 1$; $k_i = 0, \ldots, m_i - \alpha_i$ for $i \neq m$. Note that $\Delta^{\alpha}_{T_\theta}\psi$ is defined only at the points which are not too close to the faces $\{x = (x_i)_{i=1}^d \in I^d\colon x_k = 1\}$; $\Delta^{\alpha}_{T_\theta}$ is a linear operator from $C(T_\theta)$ to $C\big(\prod_{i=1}^d \{t_0^i, \ldots, t^i_{n_i-\alpha_i}\}\big)$. We write

$$\|\Delta^{\alpha}_{T_\theta}\psi\|_\infty = \sup\Big\{|\Delta_{T_\theta}\psi(x)|\colon x \in \prod_{i=1}^d \{t_0^i, \ldots, t^i_{n_i-\alpha_i}\}\Big\}.$$

Let $W_h = \{0, \ldots, nh\}^d$, where $h = 1/n$. We write

$$(1.2.2)\qquad \Delta_h^{e_m}\psi(x) = \frac{\psi(x+e_m h) - \psi(x)}{h}, \qquad \Delta_h^{\alpha+e_m}\psi = \Delta_h^{e_m}(\Delta_h^{\alpha}\psi).$$

Thus, $\Delta_h^\alpha \psi = \alpha_1! \cdot \ldots \cdot \alpha_d! \,\Delta^\alpha_{W_h}\psi$.

LEMMA 1.2.1. *Suppose that there is a number $D$ such that*

$$(1.2.3)\qquad 1/D \leqslant \frac{t^i_{k_i+1} - t^i_{k_i}}{t^i_{k_i} - t^i_{k_i-1}} \leqslant D \quad \textit{for} \quad k_i = 1, \ldots, n_i - 1,\ i = 2, \ldots, d.$$

*Suppose that $\alpha$, $\beta$ are multiindices satisfying $\alpha_i \leqslant \beta_i$ for $i = 1, \ldots, d$. Then there exists a number $c_{\alpha\beta}(D)$ independent of $T_\theta$ and $\psi$ such that*

$$|\Delta_\theta^\beta \psi(t^1_{k_1}, \ldots, t^d_{k_d})| \leqslant c_{\alpha\beta}(D) \cdot \prod_{i=1}^d (t^i_{k_i+1} - t^i_{k_i})^{\alpha_i - \beta_i} \|\Delta^\alpha_{T_\theta}\psi\|_\infty$$

*for $\psi \in C(T_\theta)$.*

Proof. We prove the existence of $c_{\alpha\beta}(D)$ by induction on $|\beta|$, $\alpha$ being fixed. If $|\alpha| = |\beta|$, then obviously $c_{\alpha\alpha}(D) = 1$. Passing from $|\beta|$ to $|\beta|+1$, we have

$$\begin{aligned}
&|\Delta^{\beta+e_m}_{T_\theta}\psi(t^1_{k_1}, \ldots, t^d_{k_d})| \\
&\quad\leqslant \frac{|\Delta^\beta_{T_\theta}\psi(t^1_{k_1}, \ldots, t^m_{k_m+1}, \ldots, t^d_{k_d}) - \Delta^\beta_{T_\theta}\psi(t^1_{k_1}, \ldots, t^m_{k_m}, \ldots, t^d_{k_d})|}{t^m_{k_m+\beta_m+1} - t^m_{k_m}} \\
&\quad\leqslant c_{\alpha\beta}(D)\left[\left(\frac{t^m_{k_m+1} - t^m_{k_m}}{t^m_{k_m+2} - t^m_{k_m+1}}\right)^{\beta_m - \alpha_m} + 1\right](t^m_{k_m+1} - t^m_{k_m}) \times \\
&\qquad\qquad \times \prod_{i=1}^d (t^i_{k_i+1} - t^i_{k_i})^{\alpha_m - \beta_m}\|\Delta^\alpha_{T_\theta}\psi\|_\infty.
\end{aligned}$$

Hence according to (1.2.3) it suffices to take

$$c_{\alpha,\beta+e_m}(D) = c_{\alpha\beta}(D)\cdot(D^{\beta_m - \alpha_m} + 1). \ ■$$

The same formula (1.2.1) determines a linear operator $\Delta^\alpha_{T_\theta}$ from $C(I^d)$ into $C\big(\prod_{i=1}^d [0, t^i_{n_i-\alpha_i}]\big)$. The double meaning of $\Delta^\alpha_{T_\theta}$ should not cause any confusion. Let us note that

$$D^\beta \Delta^\alpha_{T_\theta} f = \Delta^\alpha_{T_\theta} D^\beta f \quad \text{for} \quad f \in C^p(I^d) \text{ and } |\beta| \leqslant p.$$

LEMMA 1.2.2. *Let $f \in C^p(I^d)$, $1 \leqslant |\alpha| \leqslant p+1$. Then*

$$(1.2.4)\qquad \|\Delta^\alpha_h f\|_\infty \leqslant h^{-1}\omega_{D^{|\alpha|-1}f}(h).$$

Proof. We proceed by induction on $|\alpha|$. If $|\alpha| = 1$, then $\alpha = e_m$ and

$$|\Delta^\alpha_h f(x)| = |\Delta^{e_m}_h f(x)| = h^{-1}|f(x+e_m h) - f(x)| \leqslant h^{-1}\omega_f(h).$$

Let $|\alpha| > 1$ and $\alpha_m > 0$. Then $\alpha = \beta + e_m$, $|\beta| = |\alpha| - 1$ and

$$\begin{aligned}
|\Delta^\alpha_h f(x)| &= |\Delta^{e_m}_h(\Delta^\beta_h f)(x)| = h^{-1}|\Delta^\beta_h f(x+e_m h) - \Delta^\beta_h f(x)| \\
&= h^{-1}\Big|\int_0^h D^{e_m}\Delta^\beta_h f(x+te_m)\,dt\Big| = h^{-1}\Big|\int_0^h \Delta^\beta_h D^{e_m} f(x+te_m)\,dt\Big| \leqslant \|\Delta^\beta_h D^{e_m} f\|_\infty.
\end{aligned}$$

But

$$\|\Delta^\beta_h D^{e_m} f\|_\infty \leqslant h^{-1}\sup\{\omega_{D^{\gamma+e_m}f}(h)\colon |\gamma| = |\beta| - 1\} \leqslant h^{-1}\omega_{D^{|\alpha|-1}f}(h).$$

Combining the above inequalities, we get (1.2.4). ■

LEMMA 1.2.3. *Let $f \in C^p(I^d)$, $0 \leqslant |\alpha| \leqslant p$; then*

$$\|\Delta^\alpha_h f\|_\infty \leqslant \sup\{\|D^\beta f\|_\infty\colon |\beta| = |\alpha|\}.$$

The proof is analogous to that in Lemma 1.2.2.

**1.3. A generalization of Rolle's theorem.** The following lemma will be needed in Section 3.

LEMMA 1.3.1. *If $|\alpha| \leqslant q$, $a^i_{k_i} \in \mathbf{R}$, $k_i = 0, \ldots, \alpha_i$, $i = 1, \ldots, d$, and sequences $(a^i_{k_i})_{k_i=0}^{\alpha_i}$ are strictly increasing, then for each $f$ in $C^q\big(\prod_{i=1}^d [a^i_0, a^i_{\alpha_i}]\big)$ satisfying $f(a^1_{k_1}, \ldots, a^d_{k_d}) = 0$ for $k_i = 0, \ldots, \alpha_i$, $i = 1, \ldots, d$, there exists a point $x^0 \in \prod_{i=1}^d [a^i_0, a^i_{\alpha_i}]$ such that $D^\alpha f(x^0) = 0$.*

Proof. If $d = 1$, then the statement of the lemma is a known generalization of Rolle's theorem. Let us assume that the lemma is true for each cube of dimension less than $d$ and let $f$ satisfy the assumption of the lemma.

(i) If $\alpha_1 = 0$, then we can consider the cube $\{a^1_0\} \times \prod_{i=2}^d [a^i_0, a^i_{\alpha_i}]$ as a cube of dimension less than $d$. We get

$$D^\alpha f = D^{(\alpha_2, \ldots, \alpha_d)}\tilde{f},$$

where

$$\tilde{f}(x_2, \ldots, x_d) = f(a^1_0, x_2, \ldots, x_d).$$

By the above assumption, there exists a point $\tilde{x}^0 = (x_2^0, \ldots, x_d^0)$ such that

$$D^{(a_2,\ldots,a_d)}\tilde{f}(\tilde{x}^0) = 0.$$

Then $x^0 = (a_1^0, x_2^0, \ldots, x_d^0)$ is the desired point.

(ii) If $a_1 > 0$, then we define a function $\bar{f}$ in $C^q\big(\prod_{i=1}^{d}[a_0^i, a_{a_i}^i]\big)$ as

$$\bar{f}(x_1, \ldots, x_d) = f(x_1, \ldots, x_d) - \sum_{k=0}^{a_1-1} W_k(x_1)\cdot f(a_k^1, x_2, \ldots, x_d),$$

where $W_k$ is the Lagrange interpolation polynomial of degree $a_1-1$ such that

(1.3.1) $$W_k(a_j^1) = \delta_{kj} \quad \text{for} \quad k, j = 0, \ldots, a_1-1.$$

By the assumption about the function $f$, we have

$$\bar{f}(a_{a_1}^1, a_{k_2}^2, \ldots, a_{k_d}^d) = 0, \quad k_i = 0, \ldots, a_i, \; i = 2, \ldots, d.$$

From the above equalities and from (i) it follows that there exists a point $\bar{x}^0 = (a_{a_1}^1, x_2^0, \ldots, x_d^0)$, which belongs to $\{a_{a_1}^1\} \times \prod_{i=2}^{d}[a_0^i, a_{a_i}^i]$, such that

(1.3.2) $$D^{a-a_1e_1}\bar{f}(\bar{x}^0) = 0.$$

Let

$$\hat{f}(x) = D^{a-a_1e_1}\bar{f}(x, x_2^0, \ldots, x_d^0).$$

It follows from (1.3.1) that

(1.3.3) $$\hat{f}(a_j^1) = D^{a-a_1e_1}f(a_j^1, x_2^0, \ldots, x_d^0) - \sum_{k=0}^{a_1-1} W_k(a_j^1)\cdot D^{a-a_1e_1}f(a_k^1, x_2^0, \ldots, x_d^0), \quad j = 0, \ldots, a_1.$$

Then, by (1.3.2), (1.3.3), $\hat{f}(a_j^1) = 0$ for $j = 0, \ldots, a_1$. Hence there exists a point $x_1^0$ in $[a_0^1, a_{a_1}^1]$ such that

$$0 = D^{a_1}\hat{f}(x_1^0) = D^{a_1e_1}(D^{a-a_1e_1}\bar{f})\,(x_1^0, \ldots, x_d^0) = D^a\bar{f}(x_1^0, \ldots, x_d^0).$$

Since $\bar{f}-f$ is a polynomial of degree $a_1-1$ with respect to the variable $x_1$, we infer that $D^{a_1e_1}f = D^{a_1e_1}\bar{f}$ and hence $D^af = D^a\bar{f}$. ■

COROLLARY 1.3.2. *If $f \in C^q(I^d)$, $f_{|W_h} = 0$, $x \in I^d$, then for each $a$ which satisfies $|a| \leqslant q$ there exists a point $x^a$ such that*

$$|x^a - x| \leqslant \sqrt{d}qh, \quad D^af(x^a) = 0.$$



## 2. Extension operators into $C^p(I^d)$

**2.1. Extension operators in the one-dimensional case.** Let $T = \{t_0, t_1, \ldots, t_n\}$, where $n \geqslant p$, $0 = t_0 < \ldots < t_n = 1$, and $\psi \in C(T)$. We define polynomials $P_i\psi$ for $i = 0, \ldots, n-p$ of degree not greater than $p$ such that

(2.1.1) $$P_i\psi(t_k) = \psi(t_k), \quad k = i, \ldots, i+p.$$

We define polynomials $Q_i\psi$ for $i = 1, \ldots, n-p$ of degree not greater than $2p+1$ such that

(2.1.2) $$\begin{aligned} D^jQ_i\psi(t_{i-1}) &= D^jP_{i-1}\psi(t_{i-1}), \\ D^jQ_i\psi(t_i) &= D^jP_i\psi(t_i). \end{aligned} \quad j = 0, \ldots, p,$$

The polynomials $P_i\psi$ and $Q_i\psi$ exist and are unique. We define the function $L_T\psi$ as

(2.1.3) $$L_T\psi(t) = \begin{cases} Q_i\psi(t) & \text{for} \quad t \in [t_{i-1}, t_i),\; i = 1, \ldots, n-p, \\ P_{n-p}\psi(t) & \text{for} \quad t \in [t_{n-p}, t_n]. \end{cases}$$

LEMMA 2.1.1. *Let $T$ satisfy* (1.2.3). *Then $L_T$ is an operator from $C(T)$ into $C^p(I)$ and*

(i) *$L_T\psi_{|T} = \psi$,*

(ii) *$D^{p+1}L_T\psi(t)$ exists for $t \in I\setminus T$,*

(iii) *there exists a number $c(p, D)$ independent of $T$ and $\psi$ such that*

(2.1.4) $$\begin{aligned} \|D^jL_T\psi\|_\infty &\leqslant c(p, D)\,\|\Delta_T^j\psi\|_\infty, \quad j = 0, \ldots, p, \\ \sup\{|D^{p+1}L_T\psi(t)|:\; t \in I\setminus T\} &\leqslant c(p, D)\,\|\Delta_T^{p+1}\psi\|_\infty. \end{aligned}$$

Proof. According to (2.1.2) the function $L_T\psi$ is in $C^p(I)$ and satisfies (i) and (ii). We are going to show that $Q_i\psi$ is of the form (2.1.9). Let $P_if = P_i(f_{|T})$; $Q_if = Q_i(f_{|T})$ for $f$ in $C(I)$. If $t$ is fixed and $\psi$ is a variable, then $Q_i\psi(t)$ becomes a linear functional on the $(p+2)$-dimensional space of all functions on the set $\{t_{i-1}, \ldots, t_{i+p}\}$. Consequently,

$$Q_i\psi(t) = \sum_{i=0}^{p+1} f_{ij}(t)\cdot\psi(t_{i+j-1}),$$

where $f_{ij}(t)$ do not depend on $\psi$. Moreover, the numbers $\psi(t_{i+j-1})$, $j = 0, \ldots, p+1$, can be expressed as linear combinations of $\Delta_T^j\psi(t_{i-1})$, $j = 0, \ldots, p+1$, with coefficients $r_{ij}(t)$ independent of $\psi$. Thus

(2.1.5) $$Q_i\psi(t) = \sum_{i=0}^{p+1} r_{ij}(t)\,\Delta_T^j\psi(t_{i-1}).$$

We define polynomials $w_{ik}$ by

$$
(2.1.6)\qquad
\begin{aligned}
w_{i0}(t) &= 1, \quad i = 1, \ldots, n-p,\\
w_{ik}(t) &= \prod_{j=0}^{k-1} (t-t_{i+j-1}), \quad k = 1, \ldots, p+1; \quad i = 1, \ldots, n-p.
\end{aligned}
$$

Then

$$(2.1.7)\qquad \Delta_T^j w_{ik}(t_{i-1}) = \delta_{jk}, \quad j, k = 0, \ldots, p+1; \ i = 1, \ldots, n-p.$$

The degree of $w_{ik}$ is equal to $k$. Hence for $k = 0, \ldots, p$

$$(2.1.8)\qquad P_{i-1}w_{ik} = w_{ik}, \quad P_i w_{ik} = w_{ik}, \quad Q_i w_{ik} = w_{ik}.$$

This means that $P_i$ are projections from $C(I)$ onto the subspace of polynomials of degree not greater than $p$.

From (2.1.5) and (2.1.7) it follows that

$$Q_i w_{ik} = \sum_{j=1}^{p+1} r_{ij} \Delta_T^j w_{ik}(t_{i-1}) = r_{ik}$$

for $k = 0, \ldots, p$; $i = 1, \ldots, n-p$. Combining this with (2.1.5) we obtain

(2.1.9)

$$Q_i\varphi = \sum_{j=0}^{p+1} v_{ij} \Delta_T^j \varphi(t_{i-1}), \quad \text{where } v_{ij} = \begin{cases} w_{ij} & \text{for } j = 0, \ldots, p,\\ Q_i w_{i,p+1} & \text{for } j = p+1.\end{cases}$$

Analogously, one can show that

$$(2.1.10)\qquad P_{n-p}\varphi = \sum_{j=0}^{p} v_{n-p,j} \Delta_T^j \varphi(t_{n-p}).$$

We are going to estimate $D^j v_{ik}(t)$ for $j = 0, \ldots, p+1$. Since $w_{ik}$ is a polynomial (2.1.6),

$$D^j w_{ik}(t) = j! \sum_{\{m_1,\ldots,m_j\}\in\{0,\ldots,k-1\}} \Big( \prod_{r\in\{0,\ldots,k-1\}\setminus\{m_1,\ldots,m_j\}} (t-t_{i+r-1})\Big).$$

By (1.2.3) we get

$$\sup\Big\{\Big|\prod_{r\in\{0,\ldots,k-1\}\setminus\{m_1,\ldots,m_j\}} (t-t_{i+r-1})\Big| : t \in [t_{i-1}, t_{i+p}]\Big\}$$
$$\leqslant |t_{i+p}-t_{i-1}|^{k-j} \leqslant c_1(p, D)\,|t_i-t_{i-1}|^{k-j}$$

and

$$(2.1.11)\qquad \sup\{|D^j w_{ik}(t)| : t \in [t_{i-1}, t_{i+p}]\} \leqslant c_2(p, D)\cdot|t_i-t_{i-1}|^{k-j}.$$



Let us note that $P_i w_{i,p+1} = (t_{i+p}-t_{i-p})\cdot w_{i+1,p}$, so

$$(2.1.12)\qquad |D^i P_i w_{i,p+1}(t_i)| \leqslant c_3(p, D)\,|t_i-t_{i-1}|^{p+1-j}.$$

Since the polynomial $Q_i w_{i,p+1}$ satisfies (2.1.2) with $\varphi = w_{i,p+1|T}$, from Hermite interpolation formula ([8], p. 98) it is of the form

$$Q_i w_{i,p+1}(t) = \sum_{k=0}^{p}\sum_{r=0}^{k} (-1)^{k-r} \frac{D^{p-k}P_i w_{i,p+1}(t_i)\,(p+k-r)!\,(t-t_{i-1})^{p+1}\cdot(t-t_i)^{p-r}}{(p-k)!\,(k-r)!\,p!\,(t_i-t_{i-1})^{p+1+k-r}}.$$

If we write $w(t) = (t-t_{i-1})^{p+1}\cdot(t-t_i)^{p-r}$, then

$$\sup\{|D^j w(t)| : t \in [t_{i-1}, t_i]\} \leqslant c_4(p)\,|t_i-t_{i-1}|^{2p+1-j-r}.$$

So, combining the above inequalities and (2.1.12), we have an estimation

$$
(2.1.13)\qquad
\begin{aligned}
&\sup\{|D^j Q_i w_{i,p+1}(t)| : t \in [t_{i-1}, t_i]\}\\
&\leqslant c_5(p, D) \sum_{k=0}^{p}\sum_{r=0}^{k} |t_i-t_{i-1}|^{k+1}\,|t_i-t_{i-1}|^{-p-1-k+r}\,|t_i-t_{i-1}|^{2p+1-r-j}\\
&\leqslant c_6(p, D)\cdot|t_i-t_{i-1}|^{p+1-j}.
\end{aligned}
$$

From (2.1.9), (2.1.10), (2.1.11), and (2.1.13) we obtain

$$\sup\{|D^j Q_i\varphi(t)| : t \in [t_{i-1}, t_i]\} \leqslant c_7(p, D) \sum_{k=j}^{p+1} |t_i-t_{i-1}|^{k-j}\,|\Delta_T^k \varphi(t_{i-1})|,$$

$$\sup\{|D^j P_{n-p}(t)| : t \in [t_{n-p}, t_n]\} \leqslant c_7(p, D) \sum_{k=j}^{p} |t_{n-p}-t_{n-p-1}|^{k-j}\,|\Delta_T^k \varphi(t_{n-p})|.$$

Combining the above inequalities and Lemma 1.2.1, we obtain (2.1.4). ∎

**2.2. Extension operators in the multi-dimensional case.** Let $T_i = \{t_0^i, \ldots, t_{n_i}^i\}$ for $i = 1, \ldots, d$ be partitions of $I$ $(0 = t_0^i < \ldots < t_{n_i}^i = 1)$. We write

$$T_\varepsilon = \prod_{i=1}^{d} T_{i,\varepsilon_i} \quad \text{for} \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0, 1\}^d,$$

where $T_{i,0} = T_i$; $T_{i,1} = I$. If $L\colon C(T_j)\to C(I)$ is any linear operator and $\varepsilon$ is such that $\varepsilon_j = 0$, then we can define an operator $L_j^\varepsilon\colon C(T_\varepsilon)\to C(T_{\varepsilon+e_j})$ by the formula

$$(2.2.1)\qquad (L_j^\varepsilon\varphi)(x_1, \ldots, x_d) = \big(L(\varphi(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d))\big)(x_j),$$

where $\varphi \in C(T_\varepsilon)$ and $(x_1, \ldots, x_d) \in T_{\varepsilon+e_j}$, i.e. $L_j^\varepsilon$ is the operator $L$ applied to the function $\varphi$ regarded as a function of the $j$th variable only. Obviously, if

$$\|L\varphi\|_\infty \leqslant \|\Delta_{T_j}^{\alpha_j}\varphi\|_\infty \quad \text{for} \quad \varphi \in C(T_j),$$

then

$$\|L_j^{\varepsilon}\psi\|_\infty \leqslant \|\Delta_{T_\theta}^{a_j e_j}\psi\|_\infty \quad \text{for} \quad \psi \in C(T_\varepsilon).$$

Formula (2.2.1) will be applied to the operators $L_{T_i}\colon C(T_i)\to C(I)$, $i = 1, \ldots, d$ (2.1.9).

We define an operator $L = L_{T_\theta}\colon C(T_\theta)\to C(I^d)$ by the formula

$$L = L_{T_1,1}^{\varepsilon^1}\circ \ldots \circ L_{T_i,i}^{\varepsilon^i}\circ \ldots \circ L_{T_d,d}^{\varepsilon^d}, \tag{2.2.2}$$

where $\varepsilon^i = \sum_{j=i+1}^{d} e_j$. The operator $L_{T_i,i}^{\varepsilon}$ is of the form

$$L_{T_i,i}^{\varepsilon}\psi(x_1, \ldots, x_d) = \begin{cases} \sum\limits_{n=0}^{p+1} v_{km}(x_i)\,\Delta_{T_i}^{ke_m}\psi(x_1, \ldots, x_{i-1}, t_k^i, \ldots, x_d), & t_k^i \leqslant x_i < t_{k+1}^i, \\ & k = 0, \ldots, n_i-p-1, \\ \sum\limits_{m=0}^{p} v_{km}(x_i)\,\Delta_{T_i}^{ke_m}\psi(x_1, \ldots, x_{i-1}, t_{n_i-p}^i, \ldots, x_d), & t_{n_i-p} \leqslant x_i \leqslant 1, \end{cases}$$

where $\psi \in C(T_\varepsilon)$, $(x_1, \ldots, x_d) \in T_{\varepsilon+e_i}$, and $\varepsilon_i = 0$. The operators $L_{T_i,i}^{\varepsilon}$ commute in the sense:

if $i \neq j$, $\varepsilon$ is such that $\varepsilon_i = \varepsilon_j = 0$, then

$$L_{T_i,i}^{\varepsilon+e_j}\circ L_{T_j,i}^{\varepsilon}\psi(x_1, \ldots, x_d) = L_{T_j,j}^{\varepsilon+e_i}\circ L_{T_i,i}^{\varepsilon}\psi(x_1, \ldots, x_d)$$

for $\psi \in C(T_\varepsilon)$, $(x_1, \ldots, x_d) \in T_{\varepsilon+e_j+e_i}$.

Let $\varepsilon$ be such that $\varepsilon_{i_0} = \varepsilon_{i_1} = 0$ for a pair of indices $i_0 < i_1 \leqslant d$. Let $f_j$ be a function on $I$ for $j = 0, 1$. The operator $M_i^\eta$ is defined by the formula

$$M_j^\eta\psi(x_1, \ldots, x_d) = f_j(x_{i_j})\,\Delta_T^{r_j e_{i_j}}\psi(x_1, \ldots, x_{i_j-1}, t_k^{i_j}, x_{i_j+1}, \ldots, x_d) \tag{2.2.3}$$

for $x_{i_j} \in [t_k^{i_j}, t_{k+1}^{i_j})$; $(x_1, \ldots, x_d) \in T_{\eta+e_{i_j}}$, where $\eta = \varepsilon$ or $\eta = \varepsilon + e_{i_{1-j}}$ and $j = 0, 1$. It is clear that

$$\begin{aligned}(M_0^{\varepsilon+e_{i_1}}M_1^{\varepsilon}\psi)(x_1, \ldots, x_d) &= (M_1^{\varepsilon+e_{i_0}}M_0^{\varepsilon}\psi)(x_1, \ldots, x_d) \\ &= f_0(x_{i_0})\cdot f_1(x_{i_1})(\Delta_{T_0}^{r_0e_{i_0}+r_1e_{i_1}}\psi)(x_1, \ldots, x_{i_0-1}, t_{k_0}^{i_0}, x_{i_0+1}, \ldots, x_{i_1-1}, t_{k_1}^{i_1}, \\ &\qquad x_{i_1+1}, \ldots, x_d)\end{aligned}$$

for $x_{i_j} \in [t_{k_j}^{i_j}, t_{k_j+1}^{i_j}]$, $j = 0, 1$; $(x_1, \ldots, x_d) \in T_{\varepsilon+e_{i_0}+e_{i_1}}$. Since $L_{T_i,i}^{\varepsilon}$ are sums of operators of the form (2.2.3), they commute too. Let us note that if $i \neq j$, then the operator $\Delta_{T_\theta}^{e_j}$ commutes with the operator $L_{T_i,i}^{\varepsilon}$.

LEMMA 2.2.1. *Let $T_i$ satisfy (2.1.1) for $i = 1, \ldots, d$. Then $L$ is an operator from $C(T_\theta)$ into $C^p(I^d)$ and*



(i) $L\psi_{|T_\theta} = \psi$;

(ii) *the derivative $D^{(p+1)e_i}L\psi(x)$ exists for any $x$ in $I^d$ such that $x_i \in I\setminus T_i$ for $i = 1, \ldots, d$;*

(iii) *the derivative $D^aL\psi(x)$ exists for any $x$ in $I^d$ and $a$ such that* $\max a_i \leqslant p$;

(iv) *there exists a number $c(p, d, D)$ not depending on $T_\theta$ and $\psi$ such that*

$$\|D^aL\psi\|_\infty \leqslant c(p, d, D)\|\Delta_{T_\theta}^{a}\psi\|_\infty \quad \textit{for} \quad \max a_i \leqslant p,$$

$$\sup\{|D^{(p+1)e_i}L\psi(x_1, \ldots, x_d)|\colon x_i \in I\setminus T_i\} \leqslant c(p, d, D)\|\Delta_{T_\theta}^{(p+1)e_i}\psi\|_\infty.$$

Proof. For $\varphi \in C(T_i)$ the functions $L_{T_i}\varphi$ and their derivatives are of the form

$$D^{a_i}L_{T_i}\varphi(t) = \sum_{j=a_i}^{p+1} D^{a_i}v_{kj}\,\Delta_{T_i}^{j}\varphi(t_k^i) \tag{2.2.4}$$

for $t$ in the closure of $J_{k,i}$, $k = 0, \ldots, n_i-p$,

$$D^{p+1}L_{T_i}\varphi(t) = D^{p+1}v_{k,p+1}(t)\,\Delta_{T_i}^{p+1}\varphi(t_k^i) \tag{2.2.5}$$

for $t$ in $J_{k,j}$, $k = 0, \ldots, n_i-p$, where

$$J_{k,i} = \begin{cases}(t_k^i, t_{k+1}^i), & k = 0, \ldots, n_i-p-1, \\ (t_{n_i-p}^i, t_{n_i}^i), & k = n_i-p.\end{cases}$$

Let us note that

$$D^{a_ie_i}\circ(L_{T_i,i}^{\varepsilon}) = (D^{a_i}\circ L_{T_i})_i^{\varepsilon}.$$

Let $\psi \in C(T_\theta)$, let $a$ be such that $\max a_i \leqslant p$. According to (2.2.2), we have

$$\begin{aligned}D^{a_1e_1}L &= D^{a_1e_1}L_{T_1,1}^{\varepsilon^1}\circ L_{T_2,2}^{\varepsilon^2}\circ \ldots \circ L_{T_d,d}^{\varepsilon^d} = (D^{a_1}L_{T_1})_1^{\varepsilon^1}\circ L_{T_2,2}^{\varepsilon^2}\circ \ldots \circ L_{T_d,d}^{\varepsilon^d} \\ &= L_{T_2,2}^{\varepsilon^1}\circ \ldots \circ L_{T_d,d}^{d-1}\circ(D^{a_1}L_{T_1})_1^{\varepsilon^d}.\end{aligned}$$

Thus $D^{a_1e_1}L$ exists and is $p$-times differentiable with respect to the variables $x_2, \ldots, x_d$. If we apply the above procedure to all variables $x_1, \ldots, x_d$, then we obtain

$$D^aL\psi = D^{a_1e_1}\circ \ldots \circ D^{a_de_d}\circ L\psi = (D^{a_1}L_{T_1})_1^{\varepsilon^1}\circ \ldots \circ(D^{a_d}L_{T_d})_d^{\varepsilon^d}.$$

Hence the function $L\psi$ is in $C^p(I^d)$. From the above and (2.2.4) we obtain

$$D^aL\psi(x_1, \ldots, x_d) = \sum_{\beta_1=a_1}^{p+1} \ldots \sum_{\beta_d=a_d}^{p+1} D^{a_1}v_{k_1\beta_1}(x_1)\cdot \ldots \cdot D^{a_d}v_{k_d\beta_d}(x_a)\,\Delta_{T_\theta}^{\beta}\psi(t_{k_1}^1, \ldots, t_{k_d}^d)$$

for $x_i$ the closure of $J_{k,i}$, $i = 1, \ldots, d$. Therefore from (2.1.9) and (2.1.11) we infer

$$\|D^\alpha L\varphi\|_\infty \leqslant \sup\Big\{\sum_{\beta_1=\alpha_1}^{p+1}\dots\sum_{\beta_d=\alpha_d}^{p+1} c_1(p,d,D)\cdot\prod_{i=1}^{d}|t^i_{k_i+1}-t^i_{k_i}|^{\beta_i-\alpha_i}\times$$
$$\times|\Delta^\beta_{T_0}\varphi(t^1_{k_1},\dots,t^d_{k_d})|:\ k_i=0,\dots,n_i-p;\ i=1,\dots,d\Big\}.$$

Now we apply Lemma 1.2.1 and obtain (iv) in the case where $\max\alpha_i\leqslant p$.

In an analogous way one can show, using (2.2.5), the estimation (iv) for $\alpha=(p+1)e_i$.

## 3. Constructions of bases

**3.1. Projections in the space $C^q(I^d)$.** In this section $p$ and $d$ are fixed integers ($p\geqslant 0$, $d\geqslant 1$). The number $c(p,d,1)$ will shortly be denoted by $c$. We recall that $W_h=\{0,h,\dots,nh\}^d$, where $1/h=n\geqslant p$. Obviously, $\{0,h,\dots,nh\}$ satisfies (1.2.3) with $D=1$. We define an operator $G_h$ from $C(I^d)$ to $C^p(I^d)$ as an extension of the function $g$ restricted to $W_h$:

$$G_h g = L_{W_h}(g|_{W_h})\quad\text{for}\quad g\in C(I^d). \tag{3.1.1}$$

By Lemmas 1.2.3 and 2.2.1 we have for $g\in C^q(I^d)$ and $q=0,\dots,p$

$$\|G_h g\|^{(q)}\leqslant c\cdot\|g\|^q\quad\text{and}\quad G_h g|_{W_h}=g|_{W_h}.$$

This means that the operator $G_h$ is a continuous projection on the space $C^q(I^d)$ for $q=0,\dots,p$. Let us take a multiindex $\alpha$ satisfying $|\alpha|\leqslant q$ and a function $g$ in the space $C^q(I^d)$. If $x,y\in(I\setminus W_h)^d$, then

$$\begin{aligned}&|D^\alpha G_h g(x)-D^\alpha G_h g(y)|\\ &\leqslant\sum_{i=1}^{d}|D^\alpha G_h g(x_1,\dots,x_i,y_{i+1},\dots,y_d)-D^\alpha G_h g(x_1,\dots,x_{i-1},y_i,\dots,y_d)|\\ &\leqslant\sum_{i=1}^{d}\Big|\int_{x_i}^{y_i}D^{\alpha+e_i}G_h g(x_1,\dots,x_{i-1},t,y_{i+1},\dots,y_d)\,dt\Big|\\ &\leqslant d^{1/2}\cdot|x-y|\cdot\sup\{\|D^\beta G_h g\|_\infty:\ \beta=|\alpha|+1\}.\end{aligned}$$

Let $\alpha$, $g$ be as before and $x,y\in I^d$. We choose sequences $(x^k)$, $(y^k)$ such that $\lim x^k=x$, $\lim y^k=y$ and $x^k,y^k\in(I\setminus W_h)^d$. Obviously, we have

$$\begin{aligned}&|D^\alpha G_h g(x)-D^\alpha G_h g(y)|\\ &=\lim|D^\alpha G_h g(x^k)-D^\alpha G_h g(y^k)|\leqslant d^{1/2}|x-y|\cdot\sup\{\|D^\beta G_h g\|_\infty:\ |\beta|=|\alpha|+1\}.\end{aligned}\tag{3.1.2}$$

From this inequality and Lemmas 1.2.2 and 2.2.1 it follows that

$$\omega_{D^{|\alpha|}G_h g}(\delta)\leqslant d^{1/2}\cdot c\cdot\delta\cdot\omega_{D^{|\alpha|}g}(h)\cdot h^{-1}. \tag{3.1.3}$$

According to Corollary 1.3.2, for each $\alpha$ with $|\alpha|\leqslant q$ and each $x\in I^d$ there exists a point $x^\alpha$ in $I^d$ such that

$$|x-x^\alpha|\leqslant qh\sqrt{d}\quad\text{and}\quad D^\alpha G_h g(x^\alpha)=D^\alpha g(x^\alpha).$$



Combining (3.1.2) and Lemmas 1.2.2 and 2.2.1, we get

$$\begin{aligned}&|D^\alpha G_h g(x)-D^\alpha g(x)|\leqslant|D^\alpha G_h g(x)-D^\alpha G_h g(x^\alpha)|+|D^\alpha g(x^\alpha)-D^\alpha g(x)|\\ &\leqslant\sqrt{d}\cdot c\cdot qh\cdot\omega_{D^{|\alpha|}g}(h)\cdot h^{-1}+\omega_{D^{|\alpha|}g}(qh\sqrt{d})\leqslant\sqrt{d}(qc+q)\,\omega_{D^{|\alpha|}g}(h).\end{aligned}$$

So

$$\|D^\alpha G_h g-D^\alpha g\|_\infty\leqslant\sqrt{d}(qc+q)\,\omega_{D^{|\alpha|}g}(h). \tag{3.1.4}$$

Now let $(h_n)_{n=1}^\infty$ be a sequence convergent to 0 and such that

$$h_n\cdot h_{n+1}^{-1}\in N\quad(\text{i.e. } W_{h_{n+1}}\subset W_{h_n}),\ n=1,2,\dots$$

We write

$$W_n=W_{h_n};\quad V_n=W_n\setminus W_{n-1};\quad V=\bigcup_{n=1}^\infty V_n\quad(W_0=\emptyset).$$

We arrange the elements of $V$ into a sequence $(v_k)_{k=1}^\infty$ so that $v_k\in V_n$ for $(h_{n-1}^{-1}+1)^d<k\leqslant(h_n^{-1}+1)^d$ ($h_0=-1$). We write

$$N_n=\{k\in N:\ v_k\in V_n\}=\{k\in N:\ (h_{n-1}^{-1}+1)^d<k\leqslant(h_n^{-1}+1)^d\},\quad n=1,\dots$$

The operators $B_n$ and $R_n$ from the space $C^q(I^d)$ to itself are defined by induction:

$$\begin{aligned}&R_0=\mathrm{id},\quad B_n=G_{h_n}\circ R_{n-1}\quad(n=1,\dots),\\ &R_n=R_{n-1}-B_n=\mathrm{id}-\sum_{k=1}^{n}B_k\quad(n=1,\dots).\end{aligned}\tag{3.1.5}$$

LEMMA 3.1.1. *The operators $B_n$ are orthogonal projections, i.e.*

$$B_nB_m=\begin{cases}0, & m\neq n,\\ B_n, & m=n.\end{cases}$$

Proof. In virtue of (3.1.5) and Lemma 2.2.1 we have for $f$ in $C^q(I^d)$

$$R_nf|_{W_n}=R_{n-1}f|_{W_n}-G_{h_n}R_{n-1}f|_{W_n}=0.$$

Hence

$$B_{n+1}f|_{W_n}=G_{h_{n+1}}R_nf|_{W_n}=0$$

and

$$G_{h_m}B_{n+1}f=0\quad\text{for}\quad m=1,\dots,n.$$

For a fixed $n$, let

$$m(n)=\inf\{m\in N:\ B_mB_n\neq 0\ \text{and}\ m\neq n\}.$$

Then $m(n) \leqslant \infty$ and

$$B_{m(n)}B_n = G_{h_{m(n)}}\Big(B_n - \sum_{k=1}^{m(n)-1} B_k B_n\Big) = \begin{cases} 0 & \text{if} \quad m(n) < n, \\ G_{h_{n(m)}}(B_n - B_n B_n) & \text{if} \quad m(n) > n. \end{cases}$$

Hence $m(n) > n$ and

$$B_n B_n = G_{h_n}\Big(B_n - \sum_{k=1}^{n-1} B_k B_n\Big) = G_{h_n} B_n = G_{h_n} G_{h_n} R_{n-1} = G_{n_n} R_{n-1} = B_n.$$

Thus $m(n) = +\infty$, i.e. $B_n B_m = 0$ for $n \neq m$. ■

LEMMA 3.1.2. *Let*

$$\omega_{D^q G_{h_n} g}(\delta) \leqslant \delta \cdot b \cdot \omega_{D^q g}(h_n) \cdot h_n^{-1}$$

*for $g$ in $C^q(I^d)$ and $n = 1, \ldots$, $b$ being a positive constant. Then*

$$\omega_{D^q R_n f}(\delta) \leqslant \omega_{D^q f}(\delta) + \delta \cdot b \cdot \sum_{k=1}^{n} (b+1)^{n-k} \omega_{D^q f}(h_k) h_k^{-1}. \tag{3.1.6}$$

Let $\omega_n = \omega_{D^q R_n f}$. We are going to prove (3.1.6) by induction on $n$. For $n = 1$, inequality (3.1.6) is just the same as (3.1.7). Let us assume that (3.1.6) is true for some $n$. Then

$$\omega_{n+1}(\delta) \leqslant \omega_n(\delta) + \delta \cdot b \cdot \omega_n(h_{n+1}) h_{n+1}^{-1} \leqslant \omega_0(\delta) + \delta \cdot b \sum_{k=1}^{n} (b+1)^{n-k} \omega_0(h_k) h_k^{-1} +$$

$$+ \delta \cdot b \Big(\omega_0(h_{n+1}) + h_{n+1} b \sum_{k=1}^{n} (b+1)^{n-k} \omega_0(h_k) h_k^{-1}\Big) h_{n+1}^{-1}$$

$$= \omega_0(\delta) + \delta \cdot b \Big((b+1) \sum_{k=1}^{n} (b+1)^{n-k} \omega_0(h_k) h_k^{-1} + \omega_0(h_{n+1}) h_{n+1}^{-1}\Big)$$

$$= \omega_0(\delta) + \delta \cdot b \sum_{k=1}^{n+1} (b+1)^{n+1-k} \omega_0(h_k) h_k^{-1}. \ ■$$

**3.2. Bases in $C^q(I^d)$.** We recall that $c = c(p, d, 1)$. Let $A$ and $M$ be fixed integers such that

$$A \geqslant c d^{1/2} + 1, \qquad M \geqslant 2. \tag{3.2.1}$$

We define a sequence $(h_n)_{n=1}^{\infty}$ by

$$h_n = A^{-n} M^{-n}. \tag{3.2.2}$$

Let us note that if $t_n \to 0$ as $n \to \infty$, then

$$M^{-n} \sum_{k=1}^{n} t_k M^k \to 0 \quad \text{as} \quad n \to \infty. \tag{3.2.3}$$



We define functions $\varphi_k^{(M)}$ in $C^p(I^d)$ extending canonically the functions $\tilde{\varphi}_{v_k}^{(M)}$, where

$$\tilde{\varphi}_{v_k}^{(M)}(w) = \begin{cases} 0, & w \in V_n \setminus \{v_k\}, \\ 1, & w = v_k, \end{cases} \qquad \text{for } k \in N_n.$$

Technically,

$$\varphi_k = \varphi_k^{(M)} = L_{W_n}(\tilde{\varphi}_{v_k}^{(M)}) \quad \text{for} \quad k \in N_n. \tag{3.2.4}$$

THEOREM 3.2.1. *The sequence $(\varphi_k)_{k=1}^{\infty}$ is a simultaneous interpolating basis for $C^p(I^d)$ with nodes $(v_k)_{k=1}^{\infty}$.*

Proof. Let $q$ be fixed, $0 \leqslant q \leqslant p$. We write

$$E_n = B_n(C^q(I^d)).$$

We prove the theorem in two steps. First we show that $(E_n)_{n=1}^{\infty}$ is a basis of finite-dimensional subspaces in $C^q(I^d)$. Then we prove that for each $n$ the sequence $(\varphi_k)_{k \in N_n}$ is a basis in $E_n$ and the norms are uniformly bounded with respect to $n$. Hence, by Lemma 1.1.1, $(\varphi_k)_{k=1}^{\infty}$ is a basis in $C^q(I^d)$.

Let $f \in C^q(I^d)$. By (3.1.5) we have

$$f = \sum_{m=1}^{n} B_m f + R_n f. \tag{3.2.5}$$

We are going to prove that

$$\|R_n f\|^{(q)} \to 0 \quad \text{as} \quad n \to \infty.$$

According to (3.1.4) and (3.1.5) we have for $|\alpha| \leqslant q$

$$\|D^\alpha R_n f\|_\infty = \|D^\alpha R_{n-1} - f\, D G_{h_n} R_{n-1} f\|_\infty \leqslant q A \omega_{D^{|\alpha|} R_{n-1}}(h_n). \tag{3.2.6}$$

Since for $k \leqslant q-1$

$$\omega_{D^k R_{n-1} f}(h_n) \leqslant h_n \sup\{\|D^\beta R_{n-1} f\|_\infty \colon |\beta| = k+1\},$$

it is enough to prove that

$$\omega_{D^q R_n f}(h_{n+1}) \to 0 \quad \text{as} \quad n \to 0. \tag{3.2.7}$$

Using (3.1.3), (3.2.1), (3.2.2), and Lemma (3.1.2), we infer that

$$\omega_{D^q R_n f}(h_{n+1}) \leqslant \omega_{D^q f}(h_{n+1}) + h_{n+1}(A-1) \sum_{k=1}^{n} A^{n-k} \omega_{D^q f}(h_k) h_k^{-1} \tag{3.2.8}$$

$$\leqslant \omega_{D^q f}(h_{n+1}) + M^{-n-1} A^{-n} \sum_{k=1}^{n} A^{n-k} A^k M^k \omega_{D^q f}(h_k)$$

$$\leqslant M^{-n-1} \sum_{k=1}^{n+1} \omega_{D^q f}(h_k)\, M^k.$$

Since $\omega_{D^q f}(h_k)\to 0$ as $k\to 0$ and (3.2.3), we get (3.2.7). Hence $f = \sum_{n=1}^{\infty} B_n f$. The projections $B_n$ are orthogonal so the decomposition is unique.

Obviously, $(\varphi_k)_{k\in N_n}$ is a basis in $E_n$. We are going to estimate the norm of this basis. Let

$$v_k = \big(v_k(1), \ldots, v_k(d)\big) \in V_n.$$

Then the support of $\varphi_k$ is contained in

$$\prod_{i=1}^{d} [v_k(i)-p-1,\ v_k(i)+p+1].$$

Consequently, for $x$ in $I^d$ the cardinality of the set

$$N_n(x) = \{k \in N_n:\ \varphi_k(x) \neq 0\}$$

is not greater than $(2p+2)^d$. Let $g \in E_n$ and $U \subset N_n$. Then $g = \sum_{k\in N_n} a_k\varphi_k$, where $a_k = g(v_k)$. We have to estimate the norm of $S_U g = \sum_{k\in U} a_k\varphi_k$:

$$\Big|D^\alpha\Big(\sum_{k\in U} a_k\varphi_k\Big)(x)\Big| = \Big|\sum_{k\in U\cap N_n(x)} a_k D^\alpha\varphi_k(x)\Big|$$
$$\leqslant (2p+2)^d\cdot\sup\{|a_k|:\ k\in N_n\}\cdot\sup\{\|D^\alpha\varphi_k\|_\infty:\ k\in N_n\},$$

for $|\alpha|\leqslant q$ and therefore

(3.2.9) $$\|S_U g\|^{(q)} \leqslant (2p+2)^d \sup\{|a_k|:\ k\in N_n\}\sup\{\|\varphi_k\|^{(q)}:\ k\in N_n\}.$$

We are going to estimate these upper bounds. We fix $k$ in $N_n$. Lemmas 2.2.1 and 1.2.1 imply that

$$\|D^\alpha\varphi_k\|_\infty \leqslant c\,\|\Delta^\alpha_{h_n}\tilde{\varphi}_{v_k}\|_\infty \leqslant c\cdot c_{\theta\alpha}(1)\,\|\Delta^\theta_{h_n}\tilde{\varphi}_{v_k}\|_\infty h_n^{-|\alpha|}.$$

Hence

(3.2.10) $$\|\varphi_k\|^{(q)} \leqslant c(q)\cdot h_n^{-q}.$$

Since $g\in E$, we have $g|_{W_{n-1}} = 0$. Let $x\in I^d$, $|\alpha|<q$ and $x^\alpha$ be such as in Corollary 1.3.2. According to (3.1.2) we have

$$|D^\alpha g(x)| = |D^\alpha G_{h_n} g(x)| = |D^\alpha G_{h_n} g(x) - D^\alpha G_{h_n} g(x^\alpha)|$$
$$\leqslant d^{1/2} q h_{n-1}\sup\{\|D^\beta g\|_\infty:\ |\beta| = |\alpha|+1\}.$$

So

(3.2.11) $$|a_k| = |D^\theta g(v_k)| \leqslant d^{1/2} q h_{n-1}\sup\{\|D^\beta g\|_\infty:\ |\beta| = 1\}\leqslant \ldots$$
$$\ldots\leqslant d^{q/2} q^q\cdot h_{n-1}^q \sup\{\|D^\beta g\|_\infty:\ |\beta| = q\}\leqslant d^{q/2}q^q\cdot h_{n-1}^q\|g\|^{(q)}.$$

From (3.2.9), (3.2.10), and (3.2.11) it follows that

$$\|S_U g\|^{(q)} \leqslant c_1\cdot h^{-q}\cdot h_{n-1}^q\|g\|^{(q)} = c_2\|g\|^{(q)}.$$



Hence the norm of the basis $(\varphi_k)_{k\in N_n}$ is not greater than $c_2$. ■

**4. An isomorphism of the spaces $H_{p+s}(I^d)$ and $l_\infty$.** Troughout this section $s$ is a fixed real number such that $s\in(0,1)$, while $A$ and $M$ are integers satisfying (3.2.1) and $M > A^{s/(1-s)}$. Let $(\varphi_k)_{k=1}^\infty$ be a basis for $C^p(I^d)$ constructed in Section 3.2 for the given $M$, let $(a_k)_{k=1}^\infty$ be the associated sequence of coefficient functionals.

LEMMA 4.1.1. *There exists a number $c_3$ such that the conditions $f\in C^p(I^d)$ and $\omega_{D^p f}(\delta)\leqslant \delta^s$ imply*

(4.1.1) $$|a_k(f)|\leqslant c_3 h_n^{p+s}\quad \textit{for}\quad k\in N_n.$$

Proof. From (3.2.6) and (3.2.8) it follows that

(4.1.2) $$\sup\{\|D^\alpha R_n f\|_\infty:\ |\alpha| = p\}\leqslant pA\omega_{D^p R_{n-1} f}(h_n)$$
$$\leqslant M^{-n}\sum_{k=1}^{n}\omega_{D^p f}(h_k)\,M^k \leqslant M^{-n}\sum_{k=1}^{n} h_k^s M^k = M^{-n}\sum_{k=1}^{n} M^{-ks}A^{-ks}M^k$$
$$= M^{-n}M^{1-s}A^{-s}\frac{(M^{n(1-s)}A^{-ns}-1)}{(M^{1-s}A^{-s}-1)}\leqslant \frac{M^{1-s}A^{-s}}{M^{1-s}A^{-s}-1}(M^{-n}A^{-n})^s = c_1 h_n^s.$$

Since $R_n f$ vanishes on $W_n$, by Corollary 1.3.2 for each $x$ in $I^d$ and $\alpha$ satisfying $|\alpha|\leqslant p$ there exists a point $x^\alpha$ in $I^d$ such that $|x^\alpha - x|\leqslant ph_n$ and $D^\alpha R_n(x^\alpha) = 0$. Hence

$$|D^\alpha R_n f(x)| = |D^\alpha R_n f(x) - D^\alpha R_n f(x^\alpha)|\leqslant ph_n\cdot\sup\{\|D^\beta R_n f\|_\infty:\ |\beta| = |\alpha|+1\}.$$

Combining this with (4.1.2), we get for each $k$ in $N_{n+1}$

$$|R_n f(v_k)|\leqslant p^p h_n^p\cdot\sup\{\|D^\beta R_n f\|_\infty:\ |\beta| = p\}\leqslant c_2 h_n^{p+s}$$

but if $i\in N\setminus N_{n+1}$, then $\varphi_i|_{W_{n+1}} = 0$ and hence

$$R_n f(v_k) = B_n f(v_k) + \sum_{i\in N\setminus N_{n+1}} a_i(f)\varphi_i(v_k) = B_n f(v_k) = a_k(f).$$

Since $h_n = h_{n+1}\in AM$, we obtain the desired estimation. ■

LEMMA 4.1.2. *There exists a number $c_{10}$ such that if $f\in C^p(I^d)$ and $|a_k(f)|\leqslant h_n^{p+s}$ for $k\in N_n$; $n = 1, \ldots,$ then*

(4.1.3) $$\omega_{D^p f}(\delta)\leqslant c_{10}\cdot\delta^s\quad \textit{for}\quad \delta>0.$$

Proof. Let $|\alpha|\leqslant p$, $x\in I^d$. Then

$$|D^\alpha R_n f(x)|\leqslant \sum_{m=n+1}^{\infty}\sum_{k\in N_m}|a_k(f)|\,|D^\alpha\varphi_k(x)|$$
$$\leqslant \sum_{m=n+1}^{\infty}(2p+2)^d\sup\{|a_k(f)|:\ k\in N_m\}\cdot\sup\{\|\varphi_k\|^{(p)}:\ k\in N_m\}.$$

Hence by (3.2.10)

$$\|R_n f\|^{(p)} \leqslant c_4 \sum_{m=n+1}^{\infty} h_m^s = c_4 h_n^s \sum_{m=1}^{\infty} (A^{-s}M^{-s})^m = c_5 \cdot h_n^s. \tag{4.1.4}$$

By (3.1.2) and Lemmas 2.2.1 and 1.2.1 we have

$$\begin{aligned}\omega_{D^p\varphi_k}(\delta) &= \omega_{D^p G_{h_m}\varphi_k}(\delta) \leqslant \delta d^{1/2} \sup\{\|D^\beta \varphi_k\|_\infty \colon |\beta| = p+1\} \\ &\leqslant \delta d^{1/2} c \cdot \sup\{\|\Delta_{h_m}^\beta \tilde{\varphi}_{v_k}\|_\infty \colon |\beta| = p+1\} \leqslant \delta c_6 h_m^{-p-1} \quad \text{for} \quad k \in N_m.\end{aligned} \tag{4.1.5}$$

Let $x, y \in I^d$. Then there exists an $n$ such that

$$h_{n+1} < |x-y| \leqslant h_n.$$

For $a$ with $|a| = p$ we obtain

$$\begin{aligned}&|D^a f(x) - D^a f(y)| \\ &\leqslant \sum_{m=1}^{n} \sum_{k \in N_n} |a_k(f)|\, |D^a \varphi_k(x) - D^a \varphi_k(y)| + 2\|D^a R_n f\|_\infty \\ &\leqslant \sum_{m=1}^{n} 2(2p+2)^d \cdot \sup\{|a_k(f)| \colon k \in N_m\} \cdot \sup\{\omega_{D^p\varphi_k}(|x-y|) \colon k \in N_m\} + \\ &\qquad + 2\|D^a R_b f\|^{(p)} \\ &\leqslant c_7 \sum_{m=1}^{n} h_m^{p+s} h_n h_m^{-p-1} + 2c_5 h_n^s \leqslant c_8 M^{-n} A^{-n} \sum_{m=1}^{n} (M^{1-s} A^{1-s})^m \\ &\leqslant c_8 M^{-n} A^{-n} M^{1-s} A^{1-s} \frac{(M^{1-s}A^{1-s})^n}{M^{1-s}A^{1-s} - 1} \leqslant c_9 h_n^s = c_{10} h_{n+1}^s < c_{10} |x-y|^s,\end{aligned}$$

and

$$\omega_{D^p f}(\delta) \leqslant c_{10} \cdot \delta^s. \ \blacksquare$$

Obviously for $\delta \geqslant \sqrt{d} \cdot p h_m$ and $k \in N_m$ $\omega_{D^p\varphi_k}(\delta) = \omega_{D^p\varphi_k}(\sqrt{d} \cdot p h_m)$. From above and (4.1.5) it follows that for $k \in N_n$ we have

$$\|\varphi_k\|^{(p+s)} \leqslant c_{11} \cdot h_n^{-p-s}.$$

But

$$\|\varphi_k\|^{(p+s)} \geqslant h_n^{-p-s}.$$

Let $\psi_k = \varphi_k (\|\varphi_k\|^{(p+s)})^{-1}$ (for $k = 1, \ldots$). Lemmas 4.1.1 and 4.1.2 imply

THEOREM 4.1.3. *Let $f \in C^p(I^d)$, $f = \sum_{k=1}^{\infty} a_k \psi_k$. The following conditions are equivalent:*

(i) $\omega_{D^p f}(\delta) = O(\delta^s)$ *as* $\delta \to 0$,

(ii) $|a_k| = O(1)$ *as* $k \to \infty$.



THEOREM 4.1.4. *The spaces $H_{p+s}(I^d)$ and $l_\infty$ are isomorphic as linear topological spaces.*

Proof. Let $f \in H_{p+s}(I^d)$. Then

$$f = \sum_{k=1}^{\infty} a_k(f) \varphi_k \quad (\text{in } C^p(I^d)).$$

We define $\xi_k = a_k(f) \cdot h_n^{-p-s}$ for $k \in N_n$, $n = 1, 2, \ldots$, and

$$Tf = (\xi_k)_{k=1}^{\infty}.$$

Since $f \in H_{p+s}(I^d)$, we have $\omega_{D^p f}(\delta) \leqslant b \cdot \delta^s$ and, by Lemma 4.1.1, $(\xi_k)_{k=1}^\infty \in l_\infty$. If $\|f\|^{(p+s)} \leqslant 1$ $(b \leqslant 1)$, then $|a_k(f)| \leqslant c_2 \cdot h_n^{p+s}$ for $k \in N_n$, i.e. $\|Tf\| \leqslant c_2$. Obviously, $T$ is a one-to-one operator. We shall show that $T$ maps $H_{p+s}(I^d)$ onto $l_\infty$. Let $(\xi_k)_{k=1}^\infty \in l_\infty$ and $\|(\xi_k)_{k=1}^\infty\| \leqslant 1$. We are looking for a function $f$ such that $Tf = (\xi_k)_{k=1}^\infty$. Let

$$a_k = \xi_k \cdot h_n^{p+s} \quad \text{for} \quad k \in N_n,$$

$$f_n = \sum_{i=1}^{n} \sum_{k \in N_i} a_k \varphi_k \quad \text{for} \quad n = 1, \ldots$$

By (4.1.4)

$$\|f_n - f_m\|^{(p)} = \|R_m f_n\|^{(p)} \leqslant c_5 \cdot h_m^s.$$

So $(f_n)_{n=1}$ is a Cauchy sequence and

$$f = \lim_{n \to 0} f_n = \sum_{n=1}^{\infty} \sum_{k \in N_n} a_k \varphi_k \in C^p(I^d).$$

By Lemma 4.1.2, $\omega_{D^p f}(\delta) \leqslant c_{10} \cdot \delta^s$ and so $f \in H_{p+s}(I^d)$. ■

## References


[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] Z. Ciesielski, *A construction of a basis in $C^1(I^2)$*, Studia Math. 33 (1969), pp. 243–247.

[3] — *Problem* 1, Proceedings of the Conference on Approximation Theory (Poznań 1972), Warszawa 1975.

[4] Z. Ciesielski and J. Domsta, *Construction of orthonormal bases in $C^m(I^d)$ and $W_p^m(I^d)$*, Studia Math. 41 (1972), pp. 211–224.

[5] A. F. Filippov and W. S. Rjabenkiĭ, *On stability of difference equations*, Moskva 1956 (Russian).

[6] J. Frampton and A. J. Tromba, *On the classification of spaces of Hölder continuous functions*, J. Functional Analysis 10 (1972), pp. 336–345.

[7] W. B. Johnson, H. Rosenthal, M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. 9 (1971), pp. 488–506.

[8] W. N. Malozemow, *Simultaneous approximation of functions and their derivatives*, Leningrad 1973 (Russian).
[9] J. Ryll, *Schauder bases for the space of continuous functions on n-dimensional cube*, Comm. Math. 17 (1973), pp. 201–213.
[10] S. Schonefeld, *Schauder bases in spaces of differentiable functions*, Bull. Amer. Math. Soc. 75 (1969), pp. 586–590.
[11] — *A study of products and sums of Schauder bases in Banach spaces*, Disertation, Purdue University, 1969.

*Received December 16, 1975* (1105)

STUDIA MATHEMATICA, T. LXIII. (1978)

# The chain rule for differentiable measures*

by

HUI-HSIUNG KUO (Amherst, N. Y.)

**Abstract.** The chain rule for differentiable measures is proved. It states that if $\nu$ is an $H$-differentiable measure on a Banach space $B$ and $\theta$ is a suitable transformation, then the composition $\mu = \nu\circ\theta$ is also $H$-differentiable and the derivative is given by $D\mu(dx) = \theta'(x)^* D\nu\circ\theta(dx) + \sum_n \langle\theta''(x)(\theta'(x)^{-1}e_n, \cdot), e_n\rangle\mu(dx)$, where $\{e_n;\ n = 1, 2, \ldots\}$ is an orthonormal basis of $H$.

**1. Introduction.** The notion of differentiable measure has been introduced in [5], [6], [8]. It plays an important role in Schwartz' distribution theory on Banach spaces. See, for instance, papers [1], [3], [10]. In particular, it has been shown in [10], Theorem 8, that a harmonic distribution can be represented by a smooth measure. However, in infinite dimensional spaces, there is no canonical way to represent a smooth measure by a smooth function.

In order to study distribution theory on infinite dimensional manifolds, one has to define differentiability for measures on manifolds. This obviously requires a fundamental theorem for differentiable measures, namely, the chain rule. Unlike the chain rule for differentiable functions, that for differentiable measures takes a non-trivial form and has some rather unexpected applications. For example, one can consider a Dirichlet form associated with a Borel measure on a Riemann–Wiener manifold. In case the measures is differentiable and has logarithmic derivative ([13], p. 121), we can use the chain rule to produce a self-adjoint operator associated with this Dirichlet form. This will be done in [12] and the subsequent papers. We remark that the number operator on a Riemann–Wiener manifold can be constructed in this way [11].

We would like to thank the referee for pointing out several ambiguous statements and arguments in the original version of this paper and for making some suggestions to generalize the original results.

*Research supported by NSF Grant MPS 73-08624 A02.