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Some results in the metric theory of tensor products

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Abstract. Using estimates, based on Khintchine's inequality, for multilinear forms we obtain similar estimates for tensor norms of trigonometric polynomials. From these estimates we obtain new proofs of Varopoulos results in the descriptive theory of tensor algebras. We also present an operator from the disc algebra $A(D)$ into l^1 which has no isometric extension as an operator from $C(T)$ into l^1.

0. Introduction. In this paper we shall study some problems in the metric theory of tensor products. Our main result is an estimate for tensor norms of trigonometric polynomials. Using this estimate, we can then give new proofs of the results of Varopoulos in the descriptive theory of the tensor algebra $V(T^2) = C(T)\hat{\otimes} C(T)$. We can also prove various generalizations of these results. We obtain our estimates for tensor norms of trigonometric polynomials by passing over from norm estimates on tensor products over finite spaces, and we start the paper by proving in Section 1 the necessary results for tensor products over discrete spaces. The starting point for our estimates is the following theorem of Littlewood (by probabilists called *Khintchine's inequality*): Let I be the unit interval, and let $\{r_k(x)\}_{k=1}^{\infty}$ be the Rademacher functions. Let further $\{a_k\}_{k=1}^{n}$ be complex numbers. Let

$$f(x) = \sum a_k r_k(x), \quad 1 \leqslant k \leqslant n,\ x \in I;$$

then $f \in L^1(I)$, $f \in L^2(I)$, and

$$\|f\|_{L^1} \geqslant (2^{1/2})^{-1} \cdot \|f\|_{L^2} \quad ([10]). \tag{0.1.1}$$

Our estimates in the metric theory of tensor products are contained in the first three sections of the paper, while in the last section we study the algebra $V^{++}(T^2) = A(D)\hat{\otimes} A(D)$. Our main result in this section is that the imbedding of V^{++} into V is not isometric, though we are not able to decide whether V^{++} is a closed subspace of V.

1. Tensor products over discrete spaces. 1. We shall assume the reader to be acquainted with the theory of tensor products of Banach spaces as presented e.g. in Grothendieck [2] or Varopoulos [11]. We

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shall here follow the notations of Grothendieck [2], so that $A\hat{\otimes}B$ and $A\check{\otimes}B$ will denote respectively the projective and the injective tensor products of the Banach spaces A and B.

Let further (X, dx) be a measure space with a positive measure dx and let A be a Banach space. We shall denote by $\mathscr{L}^p(X, A)$ the set of all measurable A-valued functions $a(x)$ on X, such that

$$\Big(\int_X \|a(x)\|_A^p\,dx\Big)^{1/p} = \|a(x)\|_{\mathscr{L}^p(X,A)} < \infty. \tag{1.1.1}$$

If X is a discrete space and dx is counting measure, we shall write $l^p(X, A)$.

In this section we shall study spaces of the type

$$\mathscr{L}^p(X, \mu)\check{\otimes} l^1(D_1)\check{\otimes} l^1(D_2)\check{\otimes}\ldots\check{\otimes} l^1(D_s),$$

where the D_k are discrete spaces.

The particular cases $l^1(Z)\check{\otimes} l^1(Z)$, resp. $\mathscr{L}^p(X, dx)\check{\otimes} l^1(Z)$ were studied by resp. Littlewood and Orlicz. They proved essentially that

$$\mathscr{L}^p(X, dx)\check{\otimes} l^1(D) \subset \mathscr{L}^p\big(X, l^2(D)\big), \tag{1.1.2}$$

and that the inclusion has norm $\leqslant \sqrt{3}$ ([6], [9]).

Now the theorems of Littlewood and Orlicz follow directly from inequality (0.1.1) of the introduction, and using a straightforward induction argument it is easy to see that this inequality may be generalized as follows:

Lemma (1.1) (Davie [1]). *Let $I^s = I_1\times I_2\times\ldots\times I_s$ be the unit cube in $\mathbf{R}^n$, and let $r_k(x_j)$ be the k-th Rademacher function on I_j. Let further $A\sim\{a_{k_1,k_2,\ldots,k_s}\}$, $1\leqslant k_i\leqslant K_i$, be a finite array of (real or complex) numbers. Defining*

$$A(x) = A(x_1, x_2, \ldots, x_s) = \sum_{1\leqslant k_i\leqslant K_i} a_{(k_1,k_2,\ldots,k_s)} r_{k_1}(x_1) r_{k_2}(x_2)\ldots r_{k_s}(x_s), \tag{1.1.3}$$

we have

$$A(x) \in \mathscr{L}^2(I^s),$$

and

$$\|A(x)\|_{\mathscr{L}^1(I_s)} \geqslant 2^{-s/2}\|A(x)\|_{\mathscr{L}^2(I^s)} = 2^{-s/2}\Big(\sum |a_k|^2\Big)^{1/2}. \tag{1.1.4}$$

Replacing now inequality (0.1.1) by (1.1.4) in the arguments of Littlewood and Orlicz, one has the following

Theorem (1.2). *Let (X, μ) be a measure space, let $\{D_i\}_{i=1}^s$ be discrete spaces, and let $D = D_1\times D_2\times\ldots\times D_s$; then*

$$\mathscr{L}^p(X, \mu)\check{\otimes} l^1(D_1)\check{\otimes}\ldots\check{\otimes} l^1(D_s) \subset \mathscr{L}^p\big(X, l^2(D)\big), \quad 1\leqslant p\leqslant\infty, \tag{1.1.5}$$

and the inclusion has norm $< 2^{s/2}$.

2. In a later paper [3], Hardy and Littlewood, starting from (1.1.2) studied the spaces $l^p\check{\otimes} l^q$, $1\leqslant p\leqslant 2$, $1\leqslant q\leqslant 2$, and proved in particular that

$$l^p(D_1)\check{\otimes} l^1(D_2) \subset l^p\big(D_2, l^2(D_1)\big) \tag{1.2.1}$$

and that the inclusion has norm $\leqslant\sqrt{3}$.

Using Minkowski's inequality one has

$$l^p\big(D_1, l^2(D_2)\big) \subset l^2\big(D_2, l^p(D_1)\big) \tag{1.2.2}$$

and by Hölder's inequality one has also

$$l^2\big(D_2, l^p(D_1)\big)\cap l^p\big(D_2, l^2(D_1)\big) \subset l^r(D_1\times D_2) \tag{1.2.3}$$

where $r = 4p/(2+p)$, and hence (Hardy and Littlewood)

$$l^p(D_1)\check{\otimes} l^1(D_2) \subset l^r(D_1\times D_2), \quad r \text{ as above.} \tag{1.2.4}$$

Combining the above results, one has by induction the following corollary.

Corollary (1.3). *Let $\{D_i\}_{i=1}^s$ be discrete spaces, let $D = D_1\times D_2\times\ldots\times D_s$, and let $r = 2s/(s+1)$; then*

$$l^1(D_1)\check{\otimes} l^1(D_2)\check{\otimes}\ldots\check{\otimes} l^1(D_s) \subset l^r(D). \tag{1.2.5}$$

If the spaces D_i are not only discrete, but also finite, then $l^2(D)$ and $l^1(D)$ are isomorphic as linear spaces, and the inclusion of $l^2(D)$ into $l^1(D)$ has norm $\sqrt{|D|}$, ($|D| = \mathrm{card}(D)$). One has therefore also the following corollary of (1.1.5).

Theorem (1.4) (Varopoulos). *Let (X, μ) be a measure space, and let $\{D_i\}_{i=1}^s$ be finite spaces. Denoting the cardinality of $D = D_1\times D_2\times\ldots\times D_s$ by $|D|$, we have*

$$\begin{aligned}\mathscr{L}^1(X, \mu)\check{\otimes} l^1(D_1)\check{\otimes}\ldots\check{\otimes} l^1(D_s) &\subset \mathscr{L}^1\big(X, l^1(D)\big)\\ &= \mathscr{L}^1(X)\hat{\otimes} l^1(D) = \mathscr{L}^1(X\times D),\end{aligned} \tag{1.2.6}$$

where the inclusion has norm $\leqslant 2^{s/2}\cdot\sqrt{|D|}$.

Using the duality of the spaces c_0 and l^1 and of the projective and injective tensor norms, one has the following corollaries of the above results.

Corollary (1.5). *Let (X, μ) be a measure space, let $\{D_i\}_{i=1}^s$ be discrete spaces, and let $1\leqslant q\leqslant\infty$; then*

$$\mathscr{L}^q\big(X, l^2(D)\big) \subset \mathscr{L}^q(X)\hat{\otimes} C_0(D_1)\hat{\otimes}\ldots\hat{\otimes} C_0(D_s) \tag{1.2.7}$$

and also

$$C\big(X, l^2(D)\big) \subset C(X)\hat{\otimes} C_0(D_1)\hat{\otimes}\ldots\hat{\otimes} C_0(D_s). \tag{1.2.8}$$

Corollary (1.6). *Let $\{D_i\}$ be discrete spaces, and let $r' = 2s/(s-1)$; then*

(1.2.9) $$l^{r'}(D) \subset C_0(D_1)\otimes C_0(D_2)\otimes \dots \otimes C_0(D_s).$$

Corollary (1.7). *Let D_i be finite spaces; then*

(1.2.10) $$C(X\times D) \subset C(X)\otimes l^\infty(D_1)\otimes l^\infty(D_2) \dots \otimes l^\infty(D_s)$$

and the norm of the inclusion is $\leqslant 2^{s/2}\cdot\sqrt{|D|}$.

2. Tensor norms of trigonometric polynomials. 1. In this section we shall find estimates for the tensor norm of a trigonometric polynomial

(2.1.1) $$p(s,t) = \sum_{m=-M}^{M}\sum_{n=-N}^{N} a_{mn}e^{i(ms+nt)},$$

and of higher dimensional analogues. It will turn out that these estimates depend only on $\min(N, M)$, say $\min(N, M) = N$, and that we may just as well consider the function

(2.1.2) $$f(s,t) = \sum_{n=-N}^{N} a_n(s)e^{int}, \quad \|a_n\|_\infty \leqslant 1,$$

and obtain the same estimates.

Let now $n\in Z^s$, $n = (n_1, n_2, \dots, n_s)$, and let $t\in T^s$, $t = (t_1, t_2, \dots, t_s)$. We shall write

(2.1.3) $$nt = n_1t_1+n_2t_2+\dots+n_st_s.$$

We further define $|n|\in(Z^+)^s$, by

(2.1.4) $$|n| = (|n_1|, |n_2|, \dots, |n_s|),$$

and for $n\in Z^s$, $N\in(Z^+)^s$, we shall say that $|n|\leqslant N$ if $(N-|n|)\in(Z^+)^s$.

We can now make the following definition.

Definition (2.1). Let X be compact space, and let $N\in(Z^+)^s$. We shall say that the function $f(x,t)$, $x\in X$, $t\in T^s$, is a *trigonometric polynomia of degree $\leqslant N$ on $X\times T^s$* (or shorter $f\in P_N(X\times T^s)$) if

(2.1.5) $$f(x,t) = \sum_{|n|\leqslant N} a_n(x)e^{int}, \quad a_n(x)\in C(X),\ n\in Z^s,\ t\in T^s.$$

For $N\in(Z^+)^s$ we shall further write $\operatorname{prod}(N) = N_1\cdot N_2\cdot\ldots\cdot N_s$. We can now state our main theorem for trigonometric polynomials.

Theorem (2.2). *Let X be a compact space, let $N\in(Z^+)^s$, and let $f\in P_N(X\times T^s)$; then*

$$f\in V_s(X\times T^s) = C(X)\otimes C(T_1)\otimes C(T_2)\otimes\dots\otimes C(T_s) \quad (\textit{trivially}),$$

and

(2.1.6) $$\|f\|_V \leqslant 4^s\cdot(\operatorname{prod}(N))^{1/2}\cdot\|f\|_\infty.$$

To prove Theorem (2.2), we shall use a lemma which reduces the given problem to a problem for finite spaces. We shall first introduce some notations. We shall denote by P_N the space of ordinary (one-variable, periodic) trigonometric polynomials of degree $\leqslant N$, and we shall denote by E_M the set of Mth roots of unity, considered as a subspace of the unit circle T. We shall further denote by $R_M\colon C(T)\to C(E_M)$ the restriction mapping, and we shall denote by R_M^N the restriction of R_M to P_N. We shall prove the following lemma.

Lemma (2.3). *Let N and M be natural numbers such that $M > (\pi/\sqrt{2})\cdot N$, and let P_N, E_M, R_M^N be as above. Then there exists a map*

(2.1.7) $$S_M^N\colon C(E_M)\to C(T)$$

such that

$$S_M^N\circ R_M^N = \operatorname{Id}(P_N),$$

and such that

$$\|S_M^N\|\leqslant(1-N^2\pi^2/2M^2)^{-1}.$$

To prove the lemma, we shall use the following result of Grothendieck [2].

Proposition (2.4). *Let $\varepsilon>0$ be a real number, let A be a closed subspace of the Banach space B, let C be a $C(K)$-space, and let*

$$T\colon A\to C$$

be a compact operator. Then there exists

$$T'\colon B\to C$$

such that $T'|_A = T$, and such that $\|T'\| < \|T\|+\varepsilon$.

2. Proof of Lemma (2.3). Let us, in accordance with Proposition (2.4), denote the spaces $R_M^N(P_N)$, $C(E_M)$, and $C(T)$, resp. by A, B, and C. Let further

$$T\colon A\to C$$

be the interpolating polynomial, i.e. T is the inverse of R_M^N (defined on $A = \operatorname{Im}(R_M^N(P_N))$) followed by the inclusion of P_N into $C(T)$. Since A is finite-dimensional, T is of finite rank and a fortiori compact. By Proposition (2.4), T can then be extended to a mapping of B into C with norm arbitrarily close.

We shall prove that $\|T\| < k(M,N)^{-1} = (1-N^2\pi^2/2M^2)^{-1}$, and this is equivalent to the following inequality

(2.2.1) $$\max\{|p(m)|;\ m\in E_M\} > k(M,N)\cdot\max\{|p(t)|;\ t\in T\}, \quad p\in P_N.$$

Towards this, let $p\in P_N$ be such that $\|p\|_{C(T)} = |p(t_0)| = p(t_0) = 1$, and put $q = (p+\bar{p})/2$. We have then $|q| = |\mathrm{Re}\{p\}| \leqslant |p|$, and $q \in P_N$, so it suffices to prove (2.2.1) for q. We take now $m \in E_M$ with $|m-t_0| \leqslant \pi/M$, and we estimate $q(m)$. We write therefore

$$(2.2.2)\qquad q(m) = q(t_0) + \int_{t_0}^{m} q'(x)\,dx$$
$$= q(t_0) + [(x-m)q'(x)]_{t_0}^{m} - \int_{t_0}^{m} (x-m)q''(x)\,dx.$$

Now, t_0 is a max for q so the integrated term is 0. By Bernstein's inequality we have therefore

$$(2.2.3)\qquad q(m) \geqslant 1 - N^2(m-t_0)^2/2 \geqslant 1 - N^2\pi^2/2M^2.$$

We see that there are two inequalities in (2.2.3), both having cases of equality. However, the cases of inequality are different, so the combined inequality is strict. By compactness of the unit ball of P_N we have therefore $\|T\| < k(M, N)$, and this proves the lemma.

3. Proof of Theorem (2.2). To simplify notations we consider only the case $s = 1$, but the general case is proved in exactly the same way. We have therefore

$$(2.3.1)\qquad f(x,t) = \sum a_n(x)e^{int}, \qquad -N \leqslant n \leqslant N,\ a_n(x) \in C(X),\ t \in T.$$

Let $M = 5N$ and let $E_M \subset T$ be as in Lemma (2.3). Let further $R(f)$ be the restriction of f to $X \times E_M$. Since $\operatorname{card}(E_M) = M$, we have by Corollary (1.6)

$$(2.3.2)\qquad \|R(f)\|_{V(X\times E_M)} \leqslant (2M)^{1/2}\cdot\|R(f)\|_\infty.$$

We observe next that for $M = 5N$ we have $k(M,N) > (1-10/50) = 4/5$, so by Lemma (2.3) we can find $S_M^N\colon C(E_M) \to C(T)$ with $\|S_M^N\| \leqslant 5/4$.

We define now $S\colon V(X\times E_M) \to V(X\times T)$ by $S = \mathrm{Id}\big(C(X)\big)\otimes S_M^N$ and by standard properties of tensor norms we have then $\|S\| \leqslant 5/4$. On the other hand, we have also defined S such that $S\circ R$ is the identity operator for all trigonometric polynomials of degree $\leqslant N$ on $X\times T$, and we have therefore

$$(2.3.3)\qquad \|f\|_V = \|S\circ R(f)\|_V \leqslant \|S\|\cdot\|R(f)\|_V$$
$$\leqslant (5/4)\cdot(10N)^{1/2}\cdot\|R(f)\|_\infty \leqslant 4\cdot N^{1/2}\cdot\|f\|_\infty.$$

This proves the case $s = 1$. In the general case we define, for each i, $M_i = 5N_i$, E_{M_i} as above, and an S_i of norm $\leqslant 5/4$, and we obtain then the estimate (2.1.6).

4. Another proof of Theorem (2.2). In the preceding proof we chose $M = 5N$ in order to obtain the constant 4 in (2.3.3), and it is rather

easy to see that with the restriction-extension method used above, this constant cannot be very much improved. On the other hand, we have very little information about the possible representations of f. In applications it is, however, generally less important to have good constants than to have good information on the representation. We shall therefore sketch another proof of Theorem (2.2) that will give us a representation with polynomials of a suitable order.

Towards this we define a "de la Vallée Poussin kernel" $V_{k,N}(x)$ as follows

$$V_{k,N}(x) = \big(k/(k-1)\big)\mathscr{K}_{kN}(x) - \big(1/(k-1)\big)\mathscr{K}_N(x),$$

where k is an integer > 1, and $\mathscr{K}_M(x)$ is the Fejér kernel. We next choose $M = 2(k+1)N+1$, and we denote the normalized $(=1)$ Haar measure on E_M by dm. We need the following facts:

(i) For a polynomial of degree $\leqslant (k+1)N$, we have

$$\int_{E_M} p(m)\,dm = \int_T p(t)\,dt,$$

where dt is normalized $(=1)$ Haar measure on T.

(ii) For any $f \in C(E_M)$ we define $V(f) \in C(T)$ by

$$\int_{E_M} V_{k,N}(t-m)f(m)\,dm$$

and we have then $\|V(f)\|_\infty \leqslant \big((k+1)/(k-1)\big)\cdot\|f\|_\infty$, and furthermore $V(f)$ is a polynomial of degree $\leqslant kN$.

(iii) If $f \in C(E_M)$ is the restriction to E_M of the polynomial p of degree $\leqslant N$ on T, then $V_{k,N}\cdot f$ is a polynomial of degree $\leqslant (k+1)N$, and therefore $V(f) = p$.

Let now f be a trigonometric polynomial of degree $\leqslant N$ on $X\times T$, let $k > 1$ be an integer and define M as above. We restrict f to $X\times E_M$, and we choose a "good" representation

$$f(x,m) = \sum g_k(x)h_k(m)$$

of f in $V(X\times E_M)$. By the above facts we have now

$$f(x,t) = \sum g_k(x)V(h_k)(t),$$

which is then a "good" representation of f with polynomials of degree $\leqslant kN$, and we have therefore the following theorem.

THEOREM (2.5). *Let $f(x,t)$ be a polynomial of degree $\leqslant N$ on $X\times T$. Then there exists a representation*

$$f(x,t) = \sum g_k(x)h_k(t)$$

with

$$\sum \|g_k\| \cdot \|h_k\| \leqslant \big((k+1)/(k-1)\big) \cdot \|f\|_V$$

and all h_k's are polynomials of degree $\leqslant kN$.

The importance of Theorem (2.5) is that we can now obtain estimates for trigonometric polynomials also in other algebras. Choosing $k = 2$, disregarding the 1 in the definition of M, and using Bernstein's inequality, we have e.g. the following

THEOREM (2.6). *Let X be a compact space and let f be a trigonometric polynomial of degree $\leqslant N$ on $X \times T$; then*

$$f \in V_a = C(X) \otimes A_a(T),$$

and

$$\|f\|_{V_a} \leqslant 3 \cdot (6N)^{1/2+a}.$$

Similarly we have, by using well-known estimates for the Hilbert transform on polynomials, the following result.

THEOREM (2.7). *Let X be a compact space, and let f be an analytic trigonometric polynomial of degree $\leqslant N$ on $X \times T$; then*

$$f \in V_A = C(X) \otimes A(D)$$

and

$$\|f\|_{V_A} \leqslant 6 \cdot N^{1/2} \cdot \log N.$$

3. The descriptive theory of tensor algebras. 1. In this paragraph we shall combine Theorems (2.2) and (2.5) with some standard results of approximation theory to obtain some results in the descriptive theory of tensor algebras. Our main result is the following theorem.

THEOREM (3.1). *Let $f(x, t)$ be a function on $X \times T^s$ such that f qua function of t_i has p_i continuous derivatives and such that*

$$D_i^{p_i}(f) = \left(\frac{\partial}{\partial t_i}\right)^{p_i}(f)$$

qua function of t_i belongs (uniformly) to the Lipschitz' class a_i. If, moreover,

$$\sum \frac{1}{p_i + a_i} = A < 2,$$

then $f \in V_s(X \times T^s)$.

To prove the theorem we shall need the following lemma from approximation theory ([7], p. 87).

LEMMA (3.2). *Let $F(t)$ be a function on T^s such that f qua function of t_i has p_i continuous derivatives, and such that $D_i^{p_i}(f)$ qua function of t_i has modulus of continuity $\omega_i(h)$. Let further $N \in (Z^+)^s$. Then there exists a trigonometric polynomial P on T^s of degree $\leqslant N$, such that*

$$\|F - P\|_\infty \leqslant K \cdot \Big\{ \sum_i N_i^{-p_i} \cdot \omega_i(1/N_i) \Big\}, \tag{3.1.1}$$

K depending possibly on s, p_i and ω_i, but not on N.

Proof of Theorem (3.1). We shall denote constants by K_1, K_2, etc. Let $N_k = (N_{1k}, N_{2k}, \ldots, N_{sk})$ with $N_{ik} = [(2)^{k/(p_i+a_i)} + 1]$, ($[x]$ being the integral part of x) and let P_k be a trigonometric polynomial on $X \times T^s$ satisfying (3.1.1). By the choice of N_k and by the assumptions on f, we have then

$$\|f - P_k\|_\infty \leqslant K_1 \cdot 2^{-k}.$$

The polynomials P_k converge uniformly to f, and we have therefore

$$f(x, t) = P_1(x, t) + \sum \big(P_k(x, t) - P_{k-1}(x, t)\big).$$

We shall prove that under the given assumptions on P_k,

$$\sum \|P_k - P_{k-1}\|_{V_s} < \infty \tag{3.1.2}$$

and this will clearly prove the theorem. Towards this we observe that $P_k - P_{k-1}$ is a trigonometric polynomial on $X \times T^s$, of degree $\leqslant N_k$, and that

$$\|P_k - P_{k-1}\|_\infty \leqslant \|P_k - f\|_\infty + \|f - P_{k-1}\|_\infty \leqslant K_2 \cdot 2^{-k}.$$

By Theorem (2.2) we have therefore

$$\|P_k - P_{k-1}\|_{V_s} \leqslant K_3 \cdot 2^{-k} \cdot prod(N_k)^{1/2},$$

and by the choice of N_k, we have $prod(N_k) \leqslant K_4 \cdot 2^{Ak}$, and by comparison with a geometric series we see that (3.1.2) is convergent.

Remark. Scrutinizing the proof of Theorem (3.1) we see that it suffices in fact to assume that the functions $\omega_i(h)$ defined in Lemma (3.2) satisfy

$$\int_0^1 \frac{\omega_i(h)}{h^{a_i+1}}\, dh < \infty$$

for some a_i such that

$$\sum \frac{1}{p_i + a_i} = 2. \quad \text{(Cf. [12] I, p. 241.)}$$

By exactly the same means as above, but using Theorems (2.6) or (2.7) instead of Theorem (2.2) in our estimates, we have also the following theorems.

THEOREM (3.3). *Let $f(x, t)$ be a function on $X \times T^s$ such that f qua function of t_i has p_i continuous derivatives, and such that $D_i^{p_i}(f)$ qua function*

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of t_i belongs to the Lipschitz class β_i. If, moreover,

$$\sum \frac{(1/2)+a_i}{p_i+\beta_i} < 1,$$

then

$$f \in C(X) \hat{\otimes} A_{a_1}(T) \hat{\otimes} \ldots \hat{\otimes} A_{a_s}(T).$$

THEOREM (3.4). *Let $f(x, t)$ be an analytic function on $X \times T^s$ satisfying the smoothness conditions of Theorem* (3.1); *then*

$$f \in C(X) \hat{\otimes} A(D_1) \hat{\otimes} \ldots \hat{\otimes} A(D_s).$$

We conclude this section with the remark that the sharpening of Theorem (3.1) obtained in the Remark is a best possible result, and that a similar sharpening of Theorem (3.4) is of course possible. However, such a sharpening will contain a logarithm in the numerator of the integral, and we do not know if it is best possible.

4. The algebra $A(D)\hat{\otimes}A(D)$. 1. The algebra $V(T^2) = C(T)\hat{\otimes}C(T)$ has been an important tool in the study of the Wiener algebra $A(T)$. It has therefore been suggested by Varopoulos that the algebra $V^{++}(T^2) = A(D)\hat{\otimes}A(D)$ (where $A(D)$ is the disc algebra) could be used as a mean of studying the algebra $A^+(T)$ of analytic functions with absolutely convergent Taylor series.

Now it is well known that if E_i, $i = 1, 2$, are closed subspaces of F_i, $i = 1, 2$, then $E_1 \hat{\otimes} E_2$ is in general not a closed subspace of $F_1 \hat{\otimes} F_2$, and it is often a difficult matter to decide when it is. In our case we have the following; $V^{++}(T^2)$ is the algebra of all functions $F(z, w)$ in $A(D^2)$ that have a representation

$$F(z, w) = \sum f_k(z) g_k(w), \quad f_k, g_k \in A(D) \tag{4.1.1}$$

with

$$\sum \|f_k\| \cdot \|g_k\| < \infty. \tag{4.1.2}$$

The norm is as always the inf of (4.1.2) over all representations (4.1.1). Now it is easy to see that the closure of $V^{++}(T^2)$ in $V(T^2)$ is simply $V(T^2) \cap A(D^2)$, which we shall denote $V^+(T^2)$, and which is the algebra of all $F(z, w) \in A(D^2)$, which on T^2 have a representation

$$F(e^{is}, e^{it}) = \sum f_k(e^{is}) g_k(e^{it}), \quad f_k, g_k \in C(T) \tag{4.1.3}$$

with

$$\sum \|f_k\|_\infty \|g_k\|_\infty < \infty, \tag{4.1.4}$$

thus f_k and g_k are only assumed to be continuous.

In this section we shall study the relations between V^{++} and V^+, and we shall prove that the embedding is not isometric, even though we shall not be able to decide whether it is closed or not.

2. *The non-isometry of V^{++} in V.* Let c_2 be the algebra of continuous functions on two points. We shall denote the algebra $A(D)\hat{\otimes}c_2$ by A_2, and the algebra $C(T)\hat{\otimes}c_2$ by C_2. We have then the following theorem.

THEOREM (4.1). *Let $T\colon A_2 \to C_2$ be the natural embedding. Then there exists $F \in A_2$ such that*

$$\|TF\|_{C_2} \leqslant (1+\pi^{-2})^{-1/2} \cdot \|F\|_{A_2}.$$

By duality, Theorem (4.1) is equivalent to the following

THEOREM (4.1)′. *There exists $\Delta \in A_2'$ such that for every extension $\Lambda \in C_2'$ of Δ,*

$$\|\Lambda\|_{C_2'} \geqslant (1+\pi^{-2})^{1/2} \cdot \|\Delta\|_{A_2'}.$$

It is in this form that we shall first prove it. Towards this we observe that an element F of A_2 can be represented by a pair (f_1, f_2), $f_i \in A(D)$. Likewise an element Δ of A_2' can be represented as a pair

$$\Delta = (\tilde{\mu}_1, \tilde{\mu}_2), \qquad \tilde{\mu}_i \in M(T)/H_0^1,$$

and an element Λ of C_2' can be represented by a pair

$$\Lambda = (\mu_1, \mu_2), \qquad \mu_i \in M(T).$$

An extension of Δ to C_2' is therefore a choice (μ_1, μ_2) of measures such that $\mu_i|_{A(D)} = \tilde{\mu}_i$.

Furthermore, the norm of Δ in A_2' is given by

$$\begin{aligned}\|\Delta\|_{A_2'} &= \sup\Big\{\Big|\alpha \int_T f d\tilde{\mu}_1 + \beta \int_T f d\tilde{\mu}_2\Big|;\ |\alpha| \leqslant 1,\ |\beta| \leqslant 1,\ f \in A(D),\ \|f\| \leqslant 1\Big\}\\ &= \sup\Big\{\Big|\int_T f d\tilde{\mu}_1\Big| + \Big|\int_T f d\tilde{\mu}_2\Big|;\ f \in A(D),\ \|f\| \leqslant 1\Big\}\\ &= \sup\{\|\tilde{\mu}_1 + e^{i\theta}\tilde{\mu}_2\|_{M(T)/H_0^1};\ 0 \leqslant \theta \leqslant 2\pi\},\end{aligned}$$

and likewise, for any $\Lambda \in C_2'$, we have

$$\|\Lambda\|_{C_2'} = \sup\{\|\mu_1 + e^{i\theta}\mu_2\|_{M(T)};\ 0 \leqslant \theta \leqslant 2\pi\}.$$

We shall consider now the elements $F_a = (1, az)$, $0 \leqslant a \leqslant 1$, in A_2, and the elements $\Delta_b = (dt, be^{-it}dt)$, $0 \leqslant b$, dt is normalized Haar measure on T, $\Delta_b \in A_2'$. We shall prove Theorem (4.1)′ in the following more precise form.

THEOREM (4.1)″. (i) *If $0 \leqslant b \leqslant 1/2$, then $\|\Delta_b\| = 1$.*

(ii) *If $1/2 \leqslant b$, then $\|\Delta_b\| = b + (1/4b)$.*

(iii) *Every extension Λ_b of Δ_b has norm $\geqslant \big(1+(4b^2/\pi^2)\big)^{1/2}$.*

Proof of Theorem (4.1)″. We shall first prove (i) and (ii). Towards this we observe that, by the definition of norm, we have

$$\|\Lambda_b\|_{A_2'} = \sup\{|f(0)|+b|f'(0)|;\ f\in A(D),\ \|f\|_{A(D)}\leqslant 1\}.$$

Let $f\in A(D)$, $\|f\|\leqslant 1$. Putting

$$g(z) = \big(f(z)-f(0)\big)/\big(1-\overline{f(0)}f(z)\big),$$

we have by Schwartz' lemma

$$|g'(0)| = |f'(0)|/\big(1-|f'(0)|^2\big)\leqslant 1,$$

and hence $|f'(0)|\leqslant 1-|f(0)|^2$.

Therefore,

$$\|\Lambda_b\|\leqslant \sup\{x+b(1-x^2);\ 0\leqslant x\leqslant 1\} = \begin{cases}1 & \text{if } 0\leqslant b\leqslant 1/2,\\ b+(1/4b) & \text{if } b\geqslant 1/2.\end{cases}$$

On the other hand, the function $f(z)=1$ certainly satisfies $|f(0)|=1$ and this proves (i). To finish the proof of (ii) we take

$$f_b(z) = \big(z+(1/2b)\big)/\big(1+(z/2b)\big)$$

and then

$$|f_b(0)|+b|f_b'(0)| = b+(1/4b).$$

To prove (iii) we shall need the following lemma, proved in [4].

Lemma (4.2). *Let X be a locally compact space and let $\mu_1,\mu_2\in M(X)$; then*

$$\begin{aligned}\max\{\|\mu_1+e^{i\theta}\mu_2\|;\ 0\leqslant\theta\leqslant 2\} &\geqslant (2\pi)^{-1}\int_0^{2\pi}\|\mu_1+e^{i\theta}\mu_2\|\,d\theta\\ &\geqslant (2\pi)^{-1}\int_0^{2\pi}\big|\|\mu_1\|+e^{i\theta}\|\mu_2\|\big|\,d\theta\\ &\geqslant \big(\|\mu_1\|^2+(4/\pi^2)\|\mu_2\|^2\big)^{1/2}.\end{aligned}$$

Let now $\Lambda_b=(\mu_1,\mu_2)=\Big(\big(1+h_1(t)\big)dt,\ \big(be^{-it}+h_2(t)\big)dt\Big)$, $h_1,h_2\in H_0^1$, be any extension of Λ_b to C_2. Then

$$\int 1\,d\mu_1 = 1,\qquad \int e^{it}\,d\mu_2 = b,$$

so $\|\mu_1\|\geqslant 1$, $\|\mu_2\|\geqslant b$. By Lemma (4.2) we have then

$$\|\Lambda_b\|\geqslant\big(1+(4b^2/\pi^2)\big)^{1/2}.$$

This completes the proof of Theorem (4.1)″. Theorem (4.1)′ follows from the particular case $b=1/2$, and Theorem (4.1) follows then by duality. For more information about this problem we refer to [13].

3. *The descriptive theory for $A(D)\hat{\otimes}A(D)$*. In Section 2 we considered trigonometric polynomials on $X\times T^s$, and were able to obtain estimates also for analytic trigonometric polynomials on $X\times T^s$. However, in that section we did not consider the most natural class of analytic polynomials, namely the analytic polynomials on $T\times T$. The main reason for this is that in the estimates obtained there, the dependence on the x-variable was completely irrelevant. Consequently, in the descriptive theory of Section 3 no assumptions were made on the behaviour of $F(x,t)$ as a function of x. In the case of analytic polynomials on $T\times T$ we can no longer do this, and our main estimate is the following theorem.

Theorem (4.3). *Let*

$$F(s,t) = \sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}\exp\{i(ms+nt)\};$$

then

$$F(s,t)\in V^{++}$$

and

$$\|F\|_{V^{++}}\leqslant C\cdot(\log M)\,(\log N)\,\big(\min(M,N)\big)^{1/2}\cdot\|F\|_\infty,$$

where C is an absolute constant.

To prove Theorem (4.3) we start from Theorem (2.5) to get a "good" representation of F using only polynomials of degree $\leqslant 2M$ in s and $\leqslant 2N$ in t. This gives us the factor $\big(\min(M,N)\big)^{1/2}$ in the estimate. We then "cut" this representation to obtain a representation using only analytic polynomials. This gives us the factors $\log M$ and $\log N$ and this proves the theorem.

Using Theorem (4.3) we can now prove the following result in the descriptive theory of V^{++}.

Theorem (4.4). *Let $F(z,w)\in A(D^2)$ and assume that $F(e^{is},e^{it})$ as a function of s belongs to Λ_a, some $a>0$, and as a function of t belongs to Λ_β, $\beta>1/2$. Then $F(z,w)\in V^{++}$.*

Since the proof of this theorem is only a slight modification of the proof of Theorem (3.1), we shall only indicate the proof. We start as usual by approximating F successively by polynomials P_k of degree (m_k,n_k), where $m_k=(2^k)^{\beta/a}$, $n_k=2^k$, and we observe then that we have

$$\begin{aligned}\|P_k-P_{k-1}\|_{V^{++}} &\leqslant C\cdot(\beta/a)\cdot k^2\cdot 2^{k/2}\,[(2^{k\beta/a})^{-a}+2^{(-\beta)k}]\\ &= C_1\cdot k^2\cdot 2^{(1/2-\beta)k},\end{aligned}$$

and therefore the series $\sum(P_k-P_{k-1})$ is convergent in V^{++}, and this proves the theorem.

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A version of the Harris–Spitzer "random constant velocity" model for infinite systems of particles

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Abstract. In this paper a one-dimensional system of infinitely many elastic particles is considered. If the initial positions and velocities are independent random variables, then the actual motion of the 0th particle converges to the Gaussian process, which is in general non-Markovian.

0. Introduction. We shall consider a system of particles with equal masses (point masses) on the real line. This system will be one with a "random constant velocity", i.e. the position $x_k(t)$ of the kth trajectory at time $t \geqslant 0$ (if the particles can penetrate each other) is described by the formula

$$x_k(t) = x_k + v_k \cdot t \quad \text{for} \quad k = 0, \pm 1, \ldots, \ t \geqslant 0,$$

where $\{x_k - k\}_{-\infty}^{+\infty}$, $\{v_k\}_{-\infty}^{+\infty}$ are independent systems of independent random variables identically distributed in each of the systems.

We shall consider the billiard-ball case, i.e. whenever two particles meet we assume that they collide *elastically*, that is, the collision conserves the energy and momentum. This implies that they simply exchange trajectories.

If $E[x_k - k] = 0$ and $E[v_k] = 0$, we define, by the deterministic theorem of Harris [6], the actual motion of the kth elastic particle $y_k(t)$.

We restrict our attention to the trajectory $y(t) = y_0(t)$. In this model, which we call *model* D, we shall prove the convergence of the finite-dimensional distributions of the processes $Y_A(t) := y(At)/A^{1/2}$, $t \geqslant 0$, to the joint distributions of the Gaussian process $X(t)$, as $A \to \infty$, with

$$E[X(t)] = 0,$$

$$E[X(t) \cdot X(s)] = \min(t, s) E|v| - E[\min(tu^-, sv^-) + \min(tu^+, sv^+)];$$

here u and v are independent random variables, with the same distribution law as v_k's, and

$$a^- := -\min(0, a), \quad a^+ := \max(0, a).$$