We conclude

$$x_V \to \frac{a-b}{1+\mu} =: x \in X,$$

$$y_V = -\mu_V x_V \to -\mu x =: y \in X.$$

Since $\mathrm{gr}(F)$ is closed we have $x-a \in F(x)$, $y-b \in F(y)$. From (iv) we get

$$x-a = y-b \in F(x) \cap F(y),$$

and the theorem is proved.

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Uniform shape and uniform Čech homology and cohomology groups for metric spaces

by

D. Doitchinov (Sofia)

Abstract. It is proved that the uniform Čech homology and cohomology groups of a metric space, namely those defined by means of all uniform coverings of the space, are invariant with respect to uniform shape equivalence understood in the sense of the paper [3] as well as in the sense of the paper [4].

In paper [3] a concept of uniform shape for metric spaces was introduced. Another notion of uniform shape equivalence, called uniform fundamental equivalence, for complete metric spaces was defined earlier in [4]. The purpose of this paper is to establish two theorems showing that the uniform Čech homology and cohomology groups, namely those defined by means of all uniform coverings of the space, are invariant with respect to uniform shape equivalence understood as either of the notions mentioned above. As these two theorems are proved in essentially the same way, only the proof of one of them will be given in detail.

§ 1. Uniform and double-uniform shapes for metric spaces. Let us recall the definition of uniform shape for metric spaces given in [3]. Every metric space X can be considered as uniformly embedded in a complete metric space M which is an absolute uniform neighbourhood extensor for metric spaces — such a space will be called a UANE-space. The family of all open neighbourhoods of X in M will be denoted by $\underline{U}(X, M)$.

If X and Y are subsets of the UANE-spaces M and N, respectively, then a *uniform shape map*

$$f\colon \underline{U}(X, M) \to \underline{U}(Y, N)$$

is a collection of uniformly continuous maps $f\colon U \to V$, where $U \in \underline{U}(X, M)$, $V \in \underline{U}(Y, N)$, provided that the following conditions are satisfied:

a) for every $V \in \underline{U}(Y, N)$ there exist a $U \in \underline{U}(X, M)$ and a $f\colon U \to V$ with $f \in \underline{f}$;

b) if $f \in \underline{f}$, $f\colon U \to V$, $U' \subset U$, $V' \supset V$, $U' \in \underline{U}(X, M)$, $V' \in \underline{U}(Y, N)$, and if $f' = jf|_{U'}$, where $j\colon V \to V'$ is the inclusion map, then $f' \in \underline{f}$;

c) if $f_1, f_2 \in \underline{f}$, f_1, f_2: $U \to V$, then there exists a $U' \subset U$, $U' \in \underline{U}(X, M)$ such that the restriction maps $f_1|_{U'}$ and $f_2|_{U'}$ are uniformly homotopic in V.

The composition $\underline{gf}$ of two shape maps $\underline{f}$: $\underline{U}(X, M) \to \underline{U}(Y, N)$ and $\underline{g}$: $\underline{U}(Y, N) \to \underline{U}(Z, P)$ is defined by the family of all maps gf with $f \in \underline{f}$, $g \in \underline{g}$, which have a sense. The identity shape map $\underline{i}_{(X,M)}$: $\underline{U}(X, M) \to \underline{U}(X, M)$ consists of all inclusion maps in the neighbourhood system $\underline{U}(X, M)$.

Two uniform shape maps $\underline{f}, \underline{g}$: $\underline{U}(X, M) - \underline{U}(Y, N)$ are said to be *uniformly homotopic*, and this is denoted by $\underline{f} \underset{U}{\simeq} \underline{g}$, if for every two maps $f \in \underline{f}$ and $g \in \underline{g}$ with f, g: $U \to V$ there is a $U' \in \underline{U}(X, M)$, $U' \subset U$, such that the restrictions $f|_{U'}$ and $g|_{U'}$ are uniformly homotopic in V. The uniform homotopy of shape maps, as can easily be seen, is an equivalence relation which is also compositive.

The metric spaces X and Y are said to be *uniformly shape-equivalent* if there are two UANE-spaces M and N uniformly including X and Y, respectively, and two uniform shape maps $\underline{f}$: $\underline{U}(X, M) \to \underline{U}(Y, N)$, $\underline{g}$: $\underline{U}(Y, N) \to \underline{U}(X, M)$ such that $\underline{gf} \underset{U}{\simeq} \underline{i}_{(X,M)}$, $\underline{fg} \underset{U}{\simeq} \underline{i}_{(Y,N)}$.

Then the following statements can be proved:

A. The relation of uniform shape equivalence defined above does not depend on the UANE-spaces M and N, or on the respective inclusions of X and Y into M and N, either. Moreover, it is a proper equivalence relation.

B. Uniform shape equivalence is a relation which is both weaker than the usual uniform homotopy equivalence and stronger than shape equivalence for metric spaces in the sense of Fox [1].

C. Uniform shape equivalence coincides with usual uniform homotopy equivalence for the class of UANE-spaces.

The class of all metric spaces uniformly shape-equivalent to a given metric space X is called *uniform shape* of X and is denoted by $\mathrm{USh}\,X$.

If in the construction described above $\underline{U}(X, M)$ is not the system of all neighbourhoods of X in M but the family of its uniform neighbourhoods only, then one gets another equivalence relation — let us call it *double-uniform shape equivalence*. In the case of complete metric spaces it coincides with uniform fundamental equivalence in the sense of [4]. The class of all metric spaces double-uniform shape equivalent to a given metric space X will be called *double-uniform shape* of X and will be denoted by $\mathrm{UUSh}\,X$.

In order to formulate the main assertions of this paper let us denote by $H_m(X; G)_U$ (resp. $H^m(X; G)_U$) the m-dimensional Čech homology (cohomology) group over any coefficient group G, defined on the directed set of all uniform coverings of the metric space X and called uniform Čech homology (cohomology) group.

In § 3 we will prove the following theorems:

THEOREM 1. *If* $\mathrm{USh}\,X = \mathrm{USh}\,Y$, *then* $H_m(X; G)_U = H_m(Y; G)_U$ *and* $H^m(X; G)_U = H^m(Y; G)_U$ *for any abelian group G and each m.*

THEOREM 2. *If* $\mathrm{UUSh}\,X = \mathrm{UUSh}\,Y$, *then* $H_m(X; G)_U = H_m(Y; G)_U$ *and* $H^m(X; G)_U = H^m(Y; G)_U$ *for any abelian group G and each m.*

§ 2. Similar extension of uniform coverings. If $\beta = \{B_\lambda |\ \lambda \in \Lambda\}$ is a covering of a subspace A of a space X, then the family $\tilde{\beta} = \{\tilde{B}_\lambda |\ \lambda \in \Lambda\}$ of sets in X defined on the same indexing set Λ is called an *extension* of β if $B_\lambda = \tilde{B}_\lambda \cap A$ for each $\lambda \in \Lambda$. Moreover, when the nerves of β and $\tilde{\beta}$ coincide, $\tilde{\beta}$ is said to be a *similar extension* of β.

We need the following

LEMMA 2.1. *Let X be a metric space and* $A \subset X$. *If* β *is an uniform covering of the subspace A, then there exists a similar extension* $\tilde{\beta}$ *of* β *which is a uniform covering of some uniform neighbourhood V of A in X. Moreover, if* ε *is a Lebesgue number of* β, *then* $\tilde{\beta}$ *can be constructed in such a manner that* $\frac{1}{6}\varepsilon$ *is its Lebesgue number.*

Proof. If $\beta = \{B_\lambda |\ \lambda \in \Lambda\}$ and if $\tilde{B}'_\lambda = \{x \in X |\ \varrho(x, B_\lambda) < \varrho(x, A \setminus B_\lambda\}$, where ϱ is the metric in X, then, as it is known [2], $\tilde{B}'_\lambda$ is open in X and the family $\tilde{\beta}' = \{\tilde{B}'_\lambda |\ \lambda \in \Lambda\}$ is a similar extension of β.

Let ε be a Lebesgue number of the covering β. It is easy to see that the $\frac{1}{2}\varepsilon$-neighbourhood $O_{\varepsilon/2}A = \{x \in X |\ \varrho(x, A) < \frac{1}{2}\varepsilon\}$ of A in X is included in the union $\bigcup_{\lambda \in \Lambda} \tilde{B}'_\lambda$. Indeed, for every $x \in X$ with $\varrho(x, A) < \frac{1}{2}\varepsilon$ there is a $y_1 \in A$ with $\varrho(x, y_1) < \frac{1}{2}\varepsilon$. On the other hand, the ε-neighbourhood $\{y \in A |\ \varrho(y, y_1) < \varepsilon\}$ of y_1 in A is included in some $B_{\lambda_1} \in \beta$. Then it is clear that $\varrho(x, B_{\lambda_1}) < \frac{1}{2}\varepsilon$ and $\varrho(x, A \setminus B_{\lambda_1}) \geqslant \frac{1}{2}\varepsilon$. Hence $x \in \tilde{B}'_{\lambda_1}$ and thus $O_{\varepsilon/2}A \subset \bigcup_{\lambda \in \Lambda} \tilde{B}'_\lambda$.

We set $V = O_{\varepsilon/6}A = \{x \in X |\ \varrho(x, A) < \frac{1}{6}\varepsilon\}$, $\tilde{B}_\lambda = \tilde{B}'_\lambda \cap V$. It is clear that the family $\tilde{\beta} = \{\tilde{B}_\lambda |\ \lambda \in \Lambda\}$ is a similar extension of β. We will show also that it is a uniform covering of the uniform neighbourhood V of A in X. Indeed, let x_1 and x_2 be two points in V with $\varrho(x_1, x_2) < \frac{1}{6}\varepsilon$. There are two points y_1, y_2 in A with $\varrho(x_1, y_1) < \frac{1}{6}\varepsilon$, $\varrho(x_2, y_2) < \frac{1}{6}\varepsilon$. It follows that $\varrho(y_1, y_2) < \frac{1}{2}\varepsilon$. But

$$\{y \in A |\ \varrho(y, y_1) < \varepsilon\} \subset B_{\lambda_1}$$

for some $\lambda_1 \in \Lambda$; hence $y_2 \in B_{\lambda_1}$. Since $\varrho(x_2, y_2) < \frac{1}{6}\varepsilon$, we have $\varrho(x_2, B_{\lambda_1}) < \frac{1}{6}\varepsilon$. On the other hand,

$$\varrho(x_2, A \setminus B_{\lambda_1}) \geqslant \varrho(y_1, A \setminus B_{\lambda_1}) - \varrho(y_1, x_2).$$

Because of the inequalities $\varrho(y_1, A \setminus B_{\lambda_1}) \geqslant \varepsilon$ and

$$\varrho(y_1, x_2) \leqslant \varrho(y_1, y_2) + \varrho(y_2, x_2) < \tfrac{1}{2}\varepsilon + \tfrac{1}{6}\varepsilon = \tfrac{2}{3}\varepsilon,$$

one gets $\varrho(x_2, A \setminus B_{\lambda_1}) > \frac{1}{6}\varepsilon$. Then $\varrho(x_2, B_{\lambda_1}) < \varrho(x_2, A \setminus B_{\lambda_1})$; hence $x_2 \in \tilde{B}_{\lambda_1}$, and so $\{x \in V |\ \varrho(x, x_1) < \frac{1}{6}\varepsilon\} \subset \tilde{B}_{\lambda_1}$. Thus it is shown that $\tilde{\beta}$ is a uniform covering of V with the Lebesgue number $\frac{1}{6}\varepsilon$, and the lemma is proved.

When β is a uniform covering of the subspace A of a metric space X and ε is a Lebesgue number of β, then the family $\tilde{\beta}$ constructed above will be called the *standard similar* ε*-extension* of β in X.

Besides, one can easily see that if β and β' are two uniform coverings of A with the Lebesgue numbers ε and ε', respectively, such that β' is a refinement of β and $\varepsilon' \leqslant \varepsilon$, then the standard similar ε'-extension $\tilde{\beta}'$ of β' is a refinement of the standard similar ε-extension $\tilde{\beta}$ of β.

§ 3. **Proof of Theorems 1 and 2.** All the statements in this section will be formulated and proved for homology groups only, because analogous statements and proofs for cohomology groups are obtainable in an obvious way.

Further we need two lemmas — the first of them is obvious and the latter is proved in a standard manner.

LEMMA 3.1. *Let* $\{X_\alpha, \pi_\alpha^{\alpha'}, A\}$ *and* $\{Y_\beta, \pi_\beta^{\beta'}, B\}$ *be inverse systems with inverse limits* X_∞ *and* Y_∞, *respectively, and let* $\varphi_\infty, \psi_\infty\colon X_\infty \to Y_\infty$ *be two homomorphisms defined by the order-preserving functions* $\varphi, \psi\colon B \to A$ *and by the homomorphisms* $\varphi_\beta\colon X_{\varphi(\beta)} \to Y_\beta$, $\psi_\beta\colon X_{\psi(\beta)} \to Y_\beta$, *respectively. If for every* $\beta \in B$ *there is an* $\alpha \in A$, $\alpha > \varphi(\beta), \psi(\beta)$, *such that*

$$\varphi_\beta \pi_{\varphi(\beta)}^{\alpha} = \psi_\beta \pi_{\psi(\beta)}^{\alpha},$$

then $\varphi_\infty = \psi_\infty$.

LEMMA 3.2. *Let* X *and* Y *be metric spaces and let the mappings* $f, g\colon X \to Y$ *be uniformly homotopic. Then for every uniform covering* β *of* Y *there exists a uniform covering* α *of* X *which is a refinement of the two coverings* $f^{-1}(\beta)$ *and* $g^{-1}(\beta)$ *and has the following property*:

if X_α, $X_{f^{-1}(\beta)}$, $X_{g^{-1}(\beta)}$, Y_β *are the nerves of the corresponding coverings,* $f^\beta\colon X_{f^{-1}(\beta)} \to Y_\beta$, $g^\beta\colon X_{g^{-1}(\beta)} \to Y_\beta$ *are the simplicial inclusion maps, and* $p\colon X_\alpha \to X_{f^{-1}(\beta)}$, $q\colon X_\alpha \to X_{g^{-1}(\beta)}$ *are simplicial projections, then for the induced homomorphisms of the corresponding homology groups we have*

$$f_*^\beta p_* = g_*^\beta q_*.$$

Now we will obtain Theorems 1 and 2 as consequences of some propositions which follow below.

In the sequel X, Y, Z are always metric spaces, and M, N, P are complete metric UANE-spaces. Everywhere we use the definitions and notations given in § 1. As has been said, we have to do with the uniform Čech homology groups $H_m(X; G)_U$, defined, for any abelian group G and for each m, as the inverse limit of the inverse system $\{H_m(X_\alpha; G), (\pi_\alpha^{\alpha'})_*, A\}$. Here A is the directed set (ordered by refining) of all uniform coverings of the space X, X_α is the nerve of the covering α and $\pi_\alpha^{\alpha'}\colon X_{\alpha'} \to X_\alpha$ is a projection.

PROPOSITION 3.1. *Every uniform shape map* $\underline{f}\colon \underline{U}(X, M) \to \underline{U}(Y, N)$ *induces, for any abelian group* G *and each integer* m, *a homomorphism* $\underline{f}_*\colon H_m(X; G)_U \to H_m(Y; G)_U$ *which only depends on the spaces* X, Y, M, N *(and the embeddings of* X *and* Y *into* M *and* N, *respectively) and on the shape map* $\underline{f}$ *itself.*

Proof. We will prove something more, namely we will show that, if $B' = \{\beta_n \mid n = 1, 2, \ldots\}$ is any monotonous sequence which is cofinal in the directed set B of all uniform coverings of Y, then the required homomorphism $\underline{f}_*$ can be defined by means of a map

$$\{H_m(X_\alpha; G), (\pi_\alpha^{\alpha'})_*, A\} \to \{H_m(Y_{\beta_n}; G), (\pi_\beta^{\beta_{n-1}})_*, B\}$$

where A is of course the directed set of the uniform coverings of X.

And so, let the sequence $B' = \{\beta_n \mid n = 1, 2, \ldots\}$ be monotonous and cofinal in B, and, for every n, let ε_n be a Lebesgue number of the covering β_n. Further let $\tilde{\beta}_n$ be the standard similar ε_n-extension of β_n in N and let V_n be the uniform neighbourhood of Y in N which is covered by $\tilde{\beta}_n$. We can assume that $\varepsilon_{n+1} \leqslant \varepsilon_n$; hence $V_{n+1} \subset V_n$. In view of the properties of a shape map one finds, for every n, a $U_n \in \underline{U}(X, M)$ and a $f_n\colon U_n \to V_n$, $f_n \in \underline{f}$. Besides, there is such a $U_n' \in \underline{U}(X, M)$ with $U_n' \subset U_n \cap U_{n+1}$, that $f_n|_{U_n'} \underset{U}{\simeq} f_{n+1}|_{U_n'}$ in V_n. Let us denote by σ_n the uniform covering of X induced by the covering $f_n^{-1}(\tilde{\beta}_n)$ of U_n, and by γ_n that induced by $f_{n+1}^{-1}(\tilde{\beta}_n)$ on X. Then, by means of Lemma 3.2, one can find a uniform covering α_n of X which refines both σ_n and γ_n and is such that the equality

$$(f_n^{\beta_n} \pi_{\sigma_n}^{\alpha_n})_* = (f_{n+1}^{\beta_n} \pi_{\gamma_n}^{\alpha_n})_* \tag{1}$$

holds. Here $f_n^{\beta_n}\colon X_{\sigma_n} \to (V_n)_{\tilde{\beta}_n} = Y_{\beta_n}$ and $f_{n+1}^{\beta_n}\colon X_{\gamma_n} \to (V_n)_{\tilde{\beta}_n} = Y_{\beta_n}$ are the simplicial inclusion mappings of the corresponding nerves. We can assume besides that $\alpha_{n+1} > \alpha_n$. Then in the diagram

$$\begin{tikzcd}
 & X_{\sigma_n} \arrow[dr, "f_n"] & \\
X_{\alpha_n} \arrow[ur, "\pi"] \arrow[r, "\pi"] & X_{\gamma_n} \arrow[r, "f_{n+1}"] & Y_{\beta_n} \\
X_{\alpha_{n+1}} \arrow[u, "\pi"] \arrow[r, "\pi"] & X_{\sigma_{n+1}} \arrow[u, "\pi"] \arrow[r, "f_{n+1}"] & Y_{\beta_{n+1}} \arrow[u, "\pi"']
\end{tikzcd}$$

where the projection $\pi_{\gamma_n}^{\sigma_{n+1}}$ is supposed to be defined by the projection $\pi_{\beta_n}^{\beta_{n+1}}$, we have

$$f_{n+1}^{\beta_n} \pi_{\gamma_n}^{\sigma_{n+1}} = \pi_{\beta_n}^{\beta_{n+1}} f_{n+1}^{\beta_{n+1}} \quad \text{and} \quad (\pi_{\gamma_n}^{\sigma_{n+1}} \pi_{\sigma_{n+1}}^{\alpha_{n+1}})_* = (\pi_{\gamma_n}^{\alpha_n} \pi_{\alpha_n}^{\alpha_{n+1}})_*.$$

Hence, in view of the equality (1), one gets

$$(f_n^{\beta_n} \pi_{\sigma_n}^{\alpha_n} \pi_{\alpha_n}^{\alpha_{n+1}})_* = (\pi_{\beta_n}^{\beta_{n+1}} f_{n+1}^{\beta_{n+1}} \pi_{\sigma_{n+1}}^{\alpha_{n+1}})_*. \tag{2}$$

If we set $\varphi_n = (f_n^{\beta_n} \pi_{\sigma_n}^{\alpha_n})_*$, then the equality (2) is written as follows:

$$\varphi_n (\pi_{\alpha_n}^{\alpha_{n+1}})_* = (\pi_{\beta_n}^{\beta_{n+1}})_* \varphi_{n+1}.$$

This shows that the order-preserving function $\varphi\colon B \to A$ given by $\varphi(\beta_n) = \alpha_n$ and the homomorphisms $\varphi_n\colon H_m(X_{\alpha_n}; G) \to H_m(Y_{\beta_n}; G)$ define a map

$$\{H_m(X_\alpha; G), (\pi_\alpha^{\alpha'})_*, A\} \to \{H_m(Y_{\beta_n}; G), (\pi_{\beta_n}^{\beta_{n+1}})_*, B'\}.$$

If the inverse limit of the countable inverse system $\{H_m(Y_{\beta_n}; G), (\pi_{\beta_n}^{\beta_{n+1}})_*, B'\}$ is the group $H_{B'}$, then this map defines a homomorphism

$$\varphi_\infty\colon H_m(X; G)_U \to H_{B'}.$$

On the other hand, the injection

$$\{H_m(Y_\beta; G), (\pi_\beta^{\beta'}), B\} \to \{H_m(Y_{\beta_n}; G), (\pi_{\beta_n}^{\beta_{n+1}})_*, B'\}$$

induces, as is known, a group isomorphism $i_{B'}\colon H_m(Y; G) \to H_{B'}$. We set $f_* = i_{B'}^{-1}\varphi_\infty$ and thus get a homomorphism of the group $H_m(X; G)$ into the group $H_m(Y; G)$.

We will show that the homomorphism f_* does not depend on the arbitrariness in the construction described above. First of all we suppose that a monotonous sequence $B' = \{\beta_n \mid n = 1, 2, \dots\}$ cofinal in the set B is given, and we will prove that f_* is independent of the choice of ε_n, U_n, f_n and α_n. And so, let two homomorphisms $\varphi'_\infty, \varphi''_\infty\colon H_m(X; G) \to H_{B'}$ be defined in the manner described, the first one — by means of some ε'_n, U'_n, f'_n, α'_n, the second one — by means of some ε''_n, U''_n, f''_n, α''_n. If $\varepsilon_n = \max(\varepsilon'_n, \varepsilon''_n)$ and if V_n denotes the $\frac{1}{6}\varepsilon_n$-neighbourhood of Y in N, then one can assume that $f'_n\colon U'_n \to V_n$ and $f''_n\colon U''_n \to V_n$. Besides, it is clear that the standard similar ε_n-extension $\tilde{\beta}_n$ of β_n is at the same time a similar extension of the standard similar ε'_n-extensions and ε''_n-extensions of β_n. Taking $U_n \in \underline{U}(X, M)$ with $U_n \subset U'_n \cap U''_n$ in such a manner that $f'_n|_{U_n} \underset{U}{\simeq} f''_n|_{U_n}$, one finds, according to Lemma 3.2, for every n such an $\alpha_n \in A$ that the equality

$$(f_n'^{\beta_n}\pi_{\sigma'_n}^{\alpha_n})_* = (f_n''^{\beta_n}\pi_{\sigma''_n}^{\alpha_n})_*$$

holds. Here σ'_n and σ''_n are the coverings of X induced by the coverings $f_n'^{-1}(\tilde{\beta}_n)$ and $f_n''^{-1}(\tilde{\beta}_n)$, respectively, while $f_n'^{\beta_n}$ and $f_n''^{\beta_n}$ are the corresponding inclusion mappings of the nerves. One can assume besides that α_n is a refinement both of α'_n and of α''_n. Thus we get

$$(f_n'^{\beta_n}\pi_{\sigma'_n}^{\alpha'_n})_*(\pi_{\alpha'_n}^{\alpha_n})_* = (f_n''^{\beta_n}\pi_{\sigma''_n}^{\alpha''_n})_*(\pi_{\alpha''_n}^{\alpha_n})_*,$$

which shows, according to Lemma 3.1, that $\varphi'_\infty = \varphi''_\infty$.

It remains to see that the definition of the homomorphism f_* does not depend on the choice of the sequence $\{\beta_n \mid n = 1, 2, \dots\}$. Let $B^1 = \{\beta_n^1 \mid n = 1, 2, \dots\}$ and $B^2 = \{\beta_n^2 \mid n = 1, 2, \dots\}$ be two monotonous sequences both cofinal in B. For every n let us denote by β_n^3 the uniform covering of Y consisting of all the sets $V' \cap V''$, where $V' \in \beta_n^1$, $V'' \in \beta_n^2$, by ε_n a Lebesgue number of β_n^3 chosen so that $\varepsilon_{n+1} < \varepsilon_n$, by $\tilde{\beta}_n^1$, $\tilde{\beta}_n^2$ and $\tilde{\beta}_n^3$ the standard similar ε_n-extensions of β_n^1, β_n^2 and β_n^3, respectively, and by V_n the $\frac{1}{6}\varepsilon_n$-neighbourhood of Y in N. Further we find

a $U_n \in \underline{U}(X, M)$ and an $f_n\colon U_n \to V_n$, $f_n \in \underline{f}$, and denote by σ_n^1, σ_n^2 and σ_n^3 the uniform coverings of X induced by the coverings $f_n^{-1}(\tilde{\beta}_n^1)$, $f_n^{-1}(\tilde{\beta}_n^2)$ and $f_n^{-1}(\tilde{\beta}_n^3)$, respectively. Finally, we denote by B^3 the sequence $\{\beta_n^3 \mid n = 1, 2, \dots\}$ and by H_{B^i} the inverse limit of the inverse system $\{H_m(Y_{\beta_n^i}; G), (\pi_{\beta_n^i}^{\beta_{n+1}^i})_*, B^i\}$ for $i = 1, 2, 3$.

Let $\varphi_\infty^1\colon H_m(X; G)_U \to H_{B^1}$ and $\varphi_\infty^2\colon H_m(X; G)_U \to H_{B^2}$ be two homomorphisms, the first of them defined by means of the sequence B^1 and an order-preserving function $\varphi^1\colon B^1 \to A$ and the second by means of the sequence B^2 and another order-preserving function $\varphi^2\colon B^2 \to A$. As has already been proved, we can assume that $\varphi^1(\beta_n^1) = \varphi^2(\beta_n^2) = \alpha_n$ and that the homomorphisms φ_∞^1 and φ_∞^2 are induced by the homomorphisms

$$\varphi_n^1 = (f_n^{\beta_n^1}\pi_{\sigma_n^1}^{\alpha_n})_* \quad\text{and}\quad \varphi_n^2 = (f_n^{\beta_n^2}\pi_{\sigma_n^2}^{\alpha_n})_* \quad (n = 1, 2, \dots),$$

respectively. On the other hand, the function $\varphi^3\colon B^3 \to A$ given by $\varphi^3(\beta_n^3) = \alpha_n$ and the homomorphism $\varphi_n^3 = (f_n^{\beta_n^3}\pi_{\sigma_n^3}^{\alpha_n})_*$ $(n = 1, 2, \dots)$ induce another homomorphism $\varphi_\infty^3\colon H_m(X; G)_U \to H_{B^3}$. If the projection $\pi_{\sigma_n^1}^{\sigma_n^3}$ is supposed to be defined by the projection $\pi_{\beta_n^1}^{\beta_n^3}$, then we have $f_n^{\beta_n^1}\pi_{\sigma_n^1}^{\sigma_n^3} = \pi_{\beta_n^1}^{\beta_n^3} f_n^{\beta_n^3}$, which, together with the equality $(\pi_{\sigma_n^1}^{\alpha_n})_* = (\pi_{\sigma_n^1}^{\sigma_n^3})_*(\pi_{\sigma_n^3}^{\alpha_n})_*$, implies the equality

$$\varphi_n^1 = (\pi_{\beta_n^1}^{\beta_n^3})_*\varphi_n^3.$$

But we have

$$(\pi_{\beta_n^1}^{\beta_n^3})_* = p_{\beta_n^1} i_{B^1} i_{B^3}^{-1} p_{\beta_n^3}^{-1},$$

where $p_{\beta_n^1}\colon H_{B^1} \to H_m(Y_{\beta_n^1}; G)$, $p_{\beta_n^3}\colon H_{B^3} \to H_m(Y_{\beta_n^3}; G)$ are the projections and $i_{B_1}\colon H_m(Y; G)_U \to H_{B_1}$, $i_{B^3}\colon H_m(Y; G)_U \to H_{B^3}$ are the group isomorphisms induced by the corresponding injections. Hence one gets

$$\varphi_n^1 = p_{\beta_n^1} i_{B^1} i_{B^3}^{-1} p_{\beta_n^3}^{-1} \varphi_n^3,$$

which implies that $\varphi_\infty^1 = i_{B^1} i_{B^3}^{-1}\varphi_\infty^3$.

So $i_{B^1}^{-1}\varphi_\infty^1 = i_{B^3}^{-1}\varphi_\infty^3$. Analogously one sees that $i_{B^2}^{-1}\varphi_\infty^2 = i_{B^3}^{-1}\varphi_\infty^3$, and hence $i_{B^1}^{-1}\varphi_\infty^1 = i_{B^2}^{-1}\varphi_\infty^2$.

Thus, the correctness of the definition of f_* is proved.

In addition to the proved Proposition 3.1 the following remark is to be noted. Every usual uniformly continuous map $f\colon X \to Y$, as can easily be seen, induces a uniform shape map $\underline{f}\colon \underline{U}(X, M) \to \underline{U}(Y, N)$, and all the uniform shape maps induced by f are uniformly homotopic. The construction of f_* given in the proof above shows that if f is induced by f, then f_* coincides with the homomorphism f_* induced by f.

PROPOSITION 3.2. *If* $\underline{f}$: $\underline{U}(X, M)\to\underline{U}(Y, N)$ *and* $\underline{g}$: $\underline{U}(Y, N)\to\underline{U}(Z, P)$ *are uniform shape maps and if* $\underline{i}$: $\underline{U}(X, M)\to\underline{U}(X, M)$ *is the identity uniform shape map, then*

$$(\underline{gf})_* = \underline{g}_*\underline{f}_* \quad \textit{and} \quad \underline{i}_* = \textit{identity}.$$

Proof. The fact that $\underline{i}_*$ is the identity isomorphism being clear, it remains to prove the equality $(\underline{gf})_* = \underline{g}_*\underline{f}_*$.

Let A, B, Γ be directed sets of the uniform coverings of the spaces X, Y, Z, respectively, and let $\Gamma' = \{\gamma_n|\ n = 1, 2, \ldots\}$ be a monotonous sequence cofinal in Γ.

Let us suppose further that the homomorphism $\underline{g}_*$ is defined by means of the mappings $g_n \in \underline{g}$, g_n: $V_n'\to W_n$, where W_n is, for every n, a neighbourhood of Z covered by a similar extension $\tilde{\gamma}_n$ of γ_n. If $2\varepsilon_n$ is a Lebesgue number of the covering $g_n^{-1}(\tilde{\gamma}_n)$ of V_n', we set

$$V_n(y) = \{y' \in Y|\ \varrho(y', y)<\varepsilon_n\}, \qquad \beta_n = \{V_n(y)|\ y \in Y\}.$$

Then β_n is a uniform covering of Y. For the elements $\tilde{V}_n(y)$ of the standard similar ε_n-extension $\tilde{\beta}_n$ of β_n we have

$$\tilde{V}_n(y)\subset\{y' \in N|\ \varrho(y', V_n(y))<\tfrac{1}{6}\varepsilon_n\}\subset\{y' \in N|\ \varrho(y', y)<2\varepsilon_n\};$$

hence, $\tilde{\beta}_n$ is a refinement of $g_n^{-1}(\tilde{\gamma}_n)$.

We can assume that the numbers ε_n are taken in such a manner that we have $\varepsilon_n<1/n$ and $\varepsilon_{n+1}<\varepsilon_n$ for every n, and that the order-preserving function ψ: $\Gamma'\to B$ in the definition of $\underline{g}_*$ is given by $\psi(\gamma_n) = \beta_n$. If λ_n is the covering of Y induced by the covering $g_n^{-1}(\tilde{\gamma}_n)$ of V_n', and if $g_n^{\gamma_n}$: $Y_{\lambda_n}\to(W_n)_{\tilde{\gamma}_n} = Z_{\gamma_n}$ is the inclusion map of the nerve of λ_n into the nerve of γ_n, then $\underline{g}_*$ is defined by means of the homomorphisms $\psi_n = (g_n^{\gamma_n}\pi_{\lambda_n}^{\beta_n})_*$.

Now the sequence $B' = \{\beta_n|\ n = 1, 2, \ldots\}$ is monotonous and cofinal in B. We denote by V_n the neighbourhood of Y in N covered by $\tilde{\beta}_n$ and note that $V_n\subset V_n'$. One supposes that $\underline{f}_*$ is defined by an order-preserving function φ: $B'\to A$ and some mappings f_n: $U_n\to V_n$, $f_n \in \underline{f}$, inducing the homomorphisms $\varphi_n = (f_n^{\beta_n}\pi_{\sigma_n}^{\alpha_n})_*$ where $\alpha_n = \varphi(\beta_n)$, σ_n is the covering of X induced by $f_n^{-1}(\tilde{\beta}_n)$ and $f_n^{\beta_n}$: $X_{\sigma_n}\to(V_n)_{\tilde{\beta}_n} = Y_{\beta_n}$ is the inclusion map of the nerve of σ_n into the nerve of β_n.

On the other hand, we have $g_n f_n \in \underline{gf}$. Therefore, if we set $\tau(\gamma_n) = \alpha_n$, if μ_n is the covering of X, induced by $(g_n f_n)^{-1}(\tilde{\gamma}_n)$, and if $(g_n f_n)^{\gamma_n}$: $X_{\mu_n}\to(W_n)_{\tilde{\gamma}_n} = Z_{\gamma_n}$ is the corresponding simplicial inclusion map, then the order-preserving function τ: $\Gamma'\to A$ and the homomorphisms $\tau_n = ((g_n f_n)^{\gamma_n}\pi_{\mu_n}^{\alpha_n})_*$ define the homomorphism $(\underline{gf})_*$.

Consider the diagram

$$\begin{tikzcd}
X_{\alpha_n} \arrow[r, "\pi"] \arrow[drr, "\pi"'] & X_{\sigma_n} \arrow[r, "f_n"] & (V_n)_{\tilde{\beta}_n} = Y_{\beta_n} \arrow[r, "\pi"] & Y_{\lambda_n} \arrow[r, "g_n"] & (W_n)_{\tilde{g}_n} = Z_{\gamma_n} \\
 & & X_{\mu_n} \arrow[urr, "g_n f_n"'] & &
\end{tikzcd}$$

If we first define a projection of the nerve of $\tilde{\beta}_n$ into the nerve of $g_n^{-1}(\tilde{\gamma}_n)$, and by means of that projection we define the projection $\pi_{\lambda_n}^{\beta_n}$: $Y_{\beta_n}\to Y_{\lambda_n}$, then it is easy to see that the simplicial mappings $g_n^{\gamma_n}\pi_{\lambda_n}^{\beta_n}f_n^{\beta_n}\pi_{\sigma_n}^{\alpha_n}$ and $(g_n f_n)^{\gamma_n}\pi_{\mu_n}^{\alpha_n}$ are contiguous. Hence we have $\psi_n\varphi_n = \tau_n$.

Let $H_{B'}$ and $H_{\Gamma'}$ be the inverse limits of the inverse systems

$$\{H_m(Y_{\beta_n}; G), (\pi_{\beta_n}^{\beta_{n+1}})_*, B'\} \quad \text{and} \quad \{H_m(Z_{\gamma_n}; G), (\pi_{\gamma_n}^{\gamma_{n+1}})_*, \Gamma'\},$$

respectively, let φ_∞: $H_m(X; G)_U\to H_{B'}$, ψ_∞: $H_m(Y; G)_U\to H_{\Gamma'}$ and τ_∞: $H_m(X; G)_U\to H_{\Gamma'}$ be the homomorphisms defined above, and let $i_{B'}$: $H_m(Y; G)_U\to H_{B'}$ and and $i_{\Gamma'}$: $H_m(Z; G)_U\to H_{\Gamma'}$ be the isomorphisms induced by the corresponding injections. From the proved equality $\tau_n = \psi_n\varphi_n$ one gets

$$\tau_\infty = \psi_\infty i_{B'}^{-1}\varphi_\infty;$$

hence $i_{\Gamma'}^{-1}\tau_\infty = i_{\Gamma'}^{-1}\psi_\infty i_{B'}^{-1}\varphi_\infty$, or, which is the same,

$$(\underline{gf})_* = \underline{g}_*\underline{f}_*.$$

PROPOSITION 3.3. *If* $\underline{f}, \underline{g}$: $\underline{U}(X, M)\to\underline{U}(Y, N)$, *then* $\underline{f}\underset{U}{\simeq}\underline{g}$ *implies* $\underline{f}_* = \underline{g}_*$.

Proof. Let A and B be again the directed sets of the uniform coverings of the spaces X and Y, respectively. We assume that a sequence $B' = \{\beta_n|\ n = 1, 2, \ldots\}$, monotonous and cofinal in B, is given. One supposes the homomorphisms $\underline{f}_*$ and $\underline{g}_*$, the first of them defined by means of an order-preserving function φ: $B'\to A$ and some functions f_n: $U_n'\to V_n$, $f_n \in \underline{f}$, and the second by means of another order-preserving function ψ: $B'\to A$ and some functions g_n: $U_n''\to V_n$, $g_n \in \underline{g}$, where V_n is a neighbourhood of Y in N covered by a similar extension $\tilde{\beta}_n$ of β_n. Let σ_n be the covering of X induced by the covering $f_n^{-1}(\tilde{\beta}_n)$ of V_n and let λ_n be that induced by $g_n^{-1}(\tilde{\beta}_n)$. Then the homomorphisms $\varphi_n = (f_n^{\beta_n}\pi_{\sigma_n}^{\varphi(\beta_n)})_*$ define $\underline{f}_*$ and the homomorphisms $\psi_n = (g_n^{\beta_n}\pi_{\lambda_n}^{\psi(\beta_n)})_*$ define $\underline{g}_*$. Here of course $f_n^{\beta_n}$: $X_{\sigma_n}\to Y_{\beta_n}$ and $g_n^{\beta_n}$: $X_{\lambda_n}\to Y_{\beta_n}$ are the simplicial inclusion maps of the nerves.

For each n there is a $U_n \in \underline{U}(X, M)$ with $U_n\subset U_n' \cap U_n''$ such that the restriction maps $f_n|_{U_n}$ and $g_n|_{U_n}$ are uniformly homotopic. Then, according to Lemma 3.2, one finds, for each n, a uniform covering α_n of X such that the equality

$$(f_n^{\beta_n}\pi_{\sigma_n}^{\alpha_n})_* = (g_n^{\beta_n}\pi_{\lambda_n}^{\alpha_n})_*$$

holds. We can assume that α_n is a refinement both of $\varphi(\beta_n)$ and of $\psi(\beta_n)$, so that one gets

$$\varphi_n(\pi^{\alpha_n}_{\varphi(\beta_n)})_* = \psi_n(\pi^{\alpha_n}_{\psi(\beta_n)})_*.$$

Then one concludes by means of Lemma 3.1 that $f_* = g_*$.

Now Theorem 1 follows immediately from Propositions 3.1, 3.2 and 3.3. As regards Theorem 2, it follows from the same propositions, accordingly reformulated for double-uniform shape maps. Their proof remains essentially the same — it suffices to remark that standard similar ε-extension $\tilde{\beta}$, constructed in §2, of a uniform covering β of a subspace always covers uniform neighbourhood of that subspace.

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INSTITUTE OF MATHEMATICS AND MECHANICS
BULGARIAN ACADEMY OF SCIENCES
Sofia

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Paracompact box products in forcing extensions

by

Judy Roitman (Wellesley, Mass.)

Abstract. In an iterated ccc extension with length $\varkappa$, where $\varkappa$ has uncountable cofinality, $\square_{i<\omega} X_i$ is paracompact if each X_i is compact metrizable; if in addition $\varkappa$ is regular and no bigger than the cardinality of the continuum in the ground model, then $\square_{i<\omega} X_i$ is paracompact if each X_i is compact first countable.

§ 0. Introduction. The question of when box products are normal is an old one (see, e.g., [R_1]). Van Douwen and Kunen each showed that the box product of countably many spaces need not be normal if the spaces are not compact or are of large character. Known positive results are all consistency results and proceed by proving paracompactness. Thus attention has focused between the parameters of: is there an absolute proof that at least $\square^{\omega}(\omega+1)$ is paracompact? Is it consistent that some $\square_{i<\omega} X_i$ is not normal where each X_i is compact first countable?

The positive consistency results have been: that MA $\Rightarrow$ the box product of countably many compact first countable spaces is paracompact (Rudin, Kunen); that $\exists$ a λ-scale $\Rightarrow \square^{\omega}(\omega+1)$ is paracompact (Williams); that $\exists$ a λ-scale $\Rightarrow \square_{i<\omega} X_i$ is paracompact if each X_i is compact metrizable (Van Douwen. This is an improvement of Williams' result using a different technique).

Using a criterion inspired by Williams' method, we show that the converse of Van Douwen's result is false, and that, in fact, in many models of set theory both with and without λ-scales, the box product of countably many compact first countable spaces is paracompact.

More precisely, we have that if $\mathrm{cf}(\varkappa)>\omega$, then in a forcing extension by a ccc iterated algebra of length $\varkappa$ in a ground model M, the following hold:

$\square_{i<\omega} X_i$ is paracompact if each X_i is compact first countable of weight $\leqslant \mathrm{cf}(\varkappa)$; hence if each X_i is compact metrizable.

$\square_{i<\omega} X_i$ is paracompact if each X_i is compact first countable and $M \models \varkappa = \mathrm{cf}(\varkappa) \geqslant c$; in particular if $M \models (\varkappa$ regular uncountable and $c = \omega_1)$.