Table des matières du tome CIII, fascicule 2

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Partition generation of scales

by

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Abstract. If $f, g \in {}^{\omega}\omega$ then say that $f<g$ if $(\exists m)(\forall n>m)[f(n)<g(n)]$. A scale is defined as a sequence $\langle f_\alpha |\ \alpha<\varkappa\rangle$ which is strictly increasing in this order. The partition $F\colon [\varkappa]^2\to 2$, defined for $\alpha<\beta<\varkappa$ by $F(\alpha,\beta)=\mu m(\forall n>m)[f_\alpha(n)<f_\beta(n)]$ is investigated. Conditions are found under which a scale may be derived from a partition, and under which that scale will be unbounded in ${}^{\omega}\omega$.

Introduction. Suppose that $f, g\in {}^{\omega}\omega$. We will say

1. $f\leqslant g$ b.e.p. (*by end piece*) if $(\exists n)(\forall m>n)\ [f(m)\leqslant g(m)]$. In this case we write $(f, g)=\mu n\ (\forall m>n)[f(m)\leqslant g(m)]$.

2. $f<g$ s.b.e.p. (*strictly, by end piece*) if $(\exists n)(\forall m>n)[f(m)<g(m)]$. In this case we write $(f, g)^s=\mu n(\forall m>n)[f(m)<g(m)]$.

A subset of ${}^{\omega}\omega$ which is well ordered (b.e.p.) (or an increasing enumeration of such a subset) is called a *scale*. If it is well ordered (s.b.e.p.) then it is called a *strict scale*. In this paper we will try to give some explanation of why a strict scale $\langle a_\alpha|\ \alpha<\omega_1\rangle$ might be unbounded.

It is easy to see that if $A\subseteq{}^{\omega}\omega$ is of power $\aleph_1$ then there is a strict scale $\langle a_\alpha|\ \alpha<\omega_1\rangle$ which majorizes A (s.b.e.p.) (i.e., $(\forall a\in A)(\exists\alpha<\omega_1)\ [a<a_\alpha$ s.b.e.p.]). Thus if the continuum hypothesis is assumed then there is a strict scale $\langle a_\alpha|\ \alpha<\omega_1\rangle$ which majorizes ${}^{\omega}\omega$. When the continuum hypothesis fails, this may still be the case, but it is also possible that any scale so short may be bounded.

THEOREM (Scott, Solovay [10]). *If a model of set theory satisfies* $V=L[G]$ *where* G *is a generic set of random (Solovay) reals over* L *then* ${}^{\omega}\omega\cap L$ *majorizes* ${}^{\omega}\omega$ *b.e.p.*

On the other hand, the techniques of [4] may be employed to show the following.

THEOREM. *Assume Martin's axiom (what is called A in* [5]). *Any* $B\subseteq{}^{\omega}\omega$ *of power less than* 2^{ω} *is bounded s.b.e.p.*

The existence or non-existence of unbounded scales of type ω_1 has considerable influence on the structure of sets of real numbers. Rothberger [7], [8], [9] examines these connections in some detail. Tall [12], Heckler [4], Borel [1] and Hausdorff [3] may also be of interest to the reader concerned with such questions.

Rothberger also shows [9] that the existence of an unbounded scale of type ω_1 is equivalent to the existence of an (ω_1, ω^*) gap. We say that $\langle A, B\rangle$ is a $\langle\varkappa, \lambda^*\rangle$ *gap* if $A \cup B \subseteq {}^{\omega}\omega$ and

1. $a<b$ b.e.p. whenever $a \in A$; $b \in B$,
2. A and B have order types $\varkappa$ and λ^* (= the reverse order type of λ) respectively;
3. There is no $c \in {}^{\omega}\omega$ between A and B (b.e.p.).

We observe that there is an unbounded scale of type ω_1 iff there is an unbounded strict scale of type ω_1.

Hausdorff [3] constructs an (ω_1, ω_1^*) gap by recursively choosing an incerasing (b.e.p.) sequence $\langle a_\alpha|\ \alpha<\omega_1\rangle$ and a decreasing (b.e.p) sequence $\langle b_\alpha|\ \alpha<\omega_1\rangle$ so that $a_\alpha < b_\beta$ (b.e.p.) for $\alpha, \beta<\omega_1$. Where $f(\alpha, \beta) = (a_\alpha, b_\beta)$, he chooses the sequences so that for each $\beta<\omega_1$ and $n<\omega$, $\{\alpha<\beta|\ f(\alpha, \beta)\geqslant n\}$ is finite. Now if there is a c so that $a_\alpha<c<b_\alpha$ (b.e.p.) for all α then there is an $n \in \omega$ with $\{\alpha|\ (a_\alpha, c)\leqslant n\}$ uncountable and so a $\beta<\omega_1$ with $\{\alpha<\beta|\ (a_\alpha, c)\leqslant n\}$ infinite. But then there must be an n' with $\{\alpha<\beta|\ f(\alpha, \beta)\leqslant n'\}$ infinite (which contradicts the choice of f). Thus the properties of the function f determine that $\langle\{a_\alpha|\ \alpha<\omega_1\}, \{b_\alpha|\ \alpha<\omega_1\}\rangle$ forms a gap. In §§ 2 and 3 we investigate whether anything analogous may hold for an unbounded scale.

§ 1. Unbounded and majorizing scales. Let us say that a scale $\langle g_\alpha|\ \alpha<\varkappa\rangle$ is unbounded on all infinite subsets of ω if whenever $f\colon \omega\to\omega$ is one to one then $\langle g_\alpha \circ f|\ \alpha<\varkappa\rangle$ is unbounded. If

$$(\forall h \in {}^{\omega}\omega)(\exists \alpha<\varkappa)(\forall f \in {}^{\omega}\omega)\ [f \text{ is one to one} \Rightarrow g_\alpha \circ f \nleqslant h \circ f \text{ b.e.p.}]$$

then we shall say that $\langle g_\alpha|\ \alpha<\varkappa\rangle$ is uniformly unbounded (on all infinite subsets of ω).

1.1. THEOREM. *If $\langle g_\alpha|\ \alpha<\varkappa\rangle$ is a scale then the following are equivalent.*

1. *$\langle g_\alpha|\ \alpha<\varkappa\rangle$ majorizes ${}^{\omega}\omega$.*
2. *$\langle g_\alpha|\ \alpha<\varkappa\rangle$ is uniformly unbounded.*

Proof. It is easy to see that the first statement implies the second, so suppose the second is true. If $h \in {}^{\omega}\omega$ then choose $\alpha<\varkappa$ using the uniform unboundedness condition. We claim that $h\leqslant g_\alpha$ (and so, since h is arbitrary, $\langle g_\alpha|\ \alpha<\varkappa\rangle$ majorizes ${}^{\omega}\omega$. Otherwise there must be an infinite $a\subseteq\omega$ such that $n \in a$ implies $g_\alpha(n)\leqslant h(n)$. But then if f enumerates a, $g_\alpha \circ f \leqslant h \circ f$ which contradicts the choice of α.

In the remainder of § 1 we will assume familiarity with forcing and in particular with Solovay and Tennenbaum's method of iterated forcing [11]. Specifically, we will want to iterate the forcing conditions of the following lemma.

1.2. LEMMA. *If $A\subseteq{}^{\omega}\omega$ then there is a set S_A of countable chain condition (c.c.c.) forcing conditions such that forcing with S_A introduces an f such that $f\geqslant g$ (b.e.p.) whenever $g \in A$.*

Proof. Let S_A be the set of all $\langle u, v\rangle$ where u is a function into ω from a finite subset of ω and for some finite $a\subseteq A$, $v\colon \omega\to\omega$ is defined by $v(n) = \max(\{w(n)|\ w \in a\})$. Let $\langle u, v\rangle\leqslant\langle u', v'\rangle$ mean that $u\supseteq u'$ and for $n \in \operatorname{dom}(u-u')$, $u(n)\geqslant v'(n)$.

Since $\langle u, v\rangle$ and $\langle u, v'\rangle$ are always compatible, S_A is c.c.c. It is clear that if M is a standard transitive model of ZFC, if $A \in M$ and S_A is as described and if G is S_A-generic over M then $f = \bigcup\{u|\ \langle u, v\rangle \in G\}$ has the properties claimed.

Observe that if $\varkappa$ is a regular cardinal and $\langle f_\alpha|\ \alpha<\varkappa\rangle$ is a scale which majorizes ${}^{\omega}\omega$ then any unbounded scale must have cofinality $\varkappa$. This fact will be used to prove the following.

1.3. THEOREM. *The existence of a scale (of length ω_1) which is unbounded on each subset of ω does not imply the existence of a scale (of any length) which majorizes ${}^{\omega}\omega$.*

Proof. By the preceeding remarks, it will suffice to find a model of set theory in which there are two scales of different cofinalities, each of which is unbounded on every subset of ω (where one of the scales has length ω_1).

Let M be any countable standard transitive model of set theory. Let $P, Q \in M$ be sets of forcing conditions obtained by iterating the forcing conditions of the lemma. Thus P is obtained by iterating ω_1^M times the forcing conditions S_{A_α}, where for $\alpha<\omega_1$ A_α is the set of functions added by using the conditions S_{A_β} for $\beta<\alpha$. Q is obtained in a similar manner, only the iterations are carried out ω_2^M stages.

Solovay and Tennenbaum [11] show that c.c.c. is preserved in the iteration process. It follows that if G is P-generic over M then Q is c.c.c. in $M[G]$ and hence that $P\times Q$ is c.c.c. in M. (Alternatively, $P\times Q$ may be viewed as having been constructed by an $(\omega_1+\omega_2)^M$ stage iteration, so that c.c.c. is seen to follow directly from the Solovay–Tennenbaum lemma.)

Let $G\times H$ be $P\times Q$-generic over M. Since $P\times Q$ is c.c.c, cardinals are the same in M as in $M[G\times H]$ so we will write e.g. ω_1 for $\omega_1^M = \omega_1^{M[G\times H]}$. Through only a slight abuse of language we have then that $G = \langle g_\alpha|\ \alpha<\omega_1\rangle$ and $H = \langle h_\alpha|\ \alpha<\omega_2\rangle$ are scales.

To complete the proof, we need only that G (and H) is unbounded on every subset of ω. But otherwise there are $f, h \in {}^{\omega}\omega \cap M[G\times H]$ such that $\langle g_\alpha \circ f\ |\ \alpha<\varkappa\rangle$ is bounded by $h\circ f$ (b.e.p.). But then for some $\lambda<\omega_1$, $h\circ f \in M[\langle g_\alpha|\ \alpha<\lambda\rangle]$ and $g_\lambda < h\circ f$ which contradicts the fact that g_λ is generic over $M[H][\langle g_\alpha|\ \alpha<\lambda\rangle]$.

Remark. If $\varkappa>\omega$ is a regular cardinal then there is a natural generalization of the notion of a scale to what we will call a $\varkappa$-*scale*. If M is a model of set theory, G is generic over M with respect to c.c.c. conditions and $g \in {}^{\varkappa}\varkappa$ is in $M[G]$ where $\mathrm{cf}(\varkappa)>\omega$ in M then there is an $f \in M$ such that $(\exists\alpha<\varkappa)(\forall\beta>\alpha)\, f(\beta)>g(\beta)$. Thus majorizing (unbounded) $\varkappa$-scales in M remain majorizing (unbounded) in $M[G]$. If M is taken to be a model of V = L, in the preceeding proof, then it is seen that Theorem 1.3 is true even when there are majorizing $\varkappa$-scales of length $\varkappa^+$ for all regular $\varkappa$. Professor Gödel suggested this added requirement to the author.

§ 2. The governing of strict scales. If X is a well ordered set and δ is an ordinal then $[X]^\delta$ is defined to be the set of all subsets of X with order type δ. A function

$f\colon [\gamma]^2 \to \omega$ is said to *govern a strict scale* $\langle a_\alpha |\ \alpha<\gamma\rangle$ provided that when $\alpha<\beta<\gamma$ then $f(\{\alpha,\beta\}) \geqslant (a_\alpha, a_\beta)^s$. $f\colon [\gamma]^2\to\omega$ is said to be *transitive* if

$$f(\{\alpha,\delta\}) \leqslant \max(f(\{\alpha,\beta\}), f(\{\alpha,\gamma\}))$$

whenever $\alpha<\beta<\delta<\gamma$.

The dual of a strict scale $\langle a_\alpha|\ \alpha<\gamma\rangle$ is defined as the transitive function f where

$$f(\{\alpha,\beta\}) = (a_\alpha, a_\beta)^s \quad \text{for} \quad \alpha<\beta<\gamma .$$

We note that a strict scale is governed by its dual.

$f\colon [\gamma]^2\to\omega$ is said to have the *scale property* if

$$(\forall\alpha<\gamma)(\forall n<\omega)(\exists m<\omega)(\forall A\subseteq\alpha)[f''[A\cup\{\alpha\}]^2\subseteq n \Rightarrow \overline{A}\leqslant m] .$$

In this case $\langle a_\alpha|\ \alpha<\gamma\rangle$ is called the dual of f provided that for $\alpha<\gamma$ and $n<\omega$,

$$a_\alpha(n) = \mu m[(\forall A\subseteq\alpha)[f''[A\cup\{\alpha\}]^2\subseteq n\Rightarrow \overline{A}\leqslant m]] .$$

2.1. LEMMA. *If $f\colon [\gamma]^2\to\omega$ is transitive and has the scale property then f governs its dual* (*which must therefore be a strict scale*).

Proof. We need to show that if $\alpha<\beta<\gamma$ and $k>f(\{\alpha,\beta\})$ then $a_\alpha(k)<a_\beta(k)$. Choose $A\subseteq\alpha$ so that $\overline{A} = a_\alpha(k)$ and $f''[A\cup\{\alpha\}]^2\subseteq k$. Since f is transitive, $f''[A\cup\{\alpha,\beta\}]^2\subseteq k$ and so $a_\beta(k)\geqslant a_\alpha(k)+1$.

2.2. LEMMA. *If there is a strict scale $\langle a_\alpha|\ \alpha<\gamma\rangle$ which is governed by f, then f has the scale property.*

Proof. Otherwise

$$(\exists\alpha<\gamma)(\exists n<\omega)(\forall m<\omega)(\exists A)[A\subseteq\alpha\wedge f''[A\cup\{\alpha\}]^2\subseteq n\wedge\overline{A}>m] .$$

But this is impossible since it follows that $(\forall m\in\omega)[a_\alpha(n)>m]$.

2.3. THEOREM. *Let $f\colon [\gamma]^2\to\omega$ be transitive. There is a scale which is governed by f iff f has the scale property.*

Proof. This follows from Lemmas 2.1 and 2.2.

Let $\langle a_\alpha|\ \alpha<\gamma\rangle$ be a strict scale and let f be its dual. We will say that $\langle a_\alpha|\ \alpha<\gamma\rangle$ is conservative if for $\alpha<\gamma$ and $n<\omega$,

$$a_\alpha(n) = \mu m(\forall\beta<\alpha)[f(\{\alpha,\beta\})<n\Rightarrow a_\beta(n)<m] .$$

2.4. THEOREM. *If $f\colon [\gamma]^2\to\omega$ is transitive and has the scale property then the dual of f is conservative.*

Proof: Let $\langle a_\alpha|\ \alpha<\gamma\rangle$ and $\bar f$ be the first and second duals of f and suppose α is smallest so that there is an n for which

$$a_\alpha(n)>k = \mu m(\forall\beta<\alpha)[\bar f(\{\alpha,\beta\})<n\Rightarrow a_\beta(n)<m] .$$

By the definition of a_α there is an $A\subseteq\alpha$ so that $\overline{A}>k$ and $f''[A\cup\{\alpha\}]^2\subseteq n$. By Lemma 2.1, $\bar f''[A\cup\{\alpha\}]^2\subseteq n$. But if δ is the largest member of A then $a_\delta(n)\geqslant k$ which contradicts the definition of k.

2.5. COROLLARY. *A conservative strict scale is its own second dual.*

Proof. Otherwise there must be a conservative strict scale $\langle a_\alpha|\ \alpha<\gamma\rangle$ with a second dual $\langle\bar a_\alpha|\ \alpha<\gamma\rangle$ and a smallest $\alpha<\gamma$ such that $a_\alpha\neq\bar a_\alpha$. Let f and $\bar f$ be respectively the duals of $\langle a_\alpha|\ \alpha<\gamma\rangle$ and $\langle\bar a_\alpha|\ \alpha<\gamma\rangle$. Since both scales are conservative there must be a $\beta<\alpha$ such that $f(\{\alpha,\beta\})\neq\bar f(\{\alpha,\beta\})$. Since by Lemma 2.1, f governs $\langle\bar a_\alpha|\ \alpha<\gamma\rangle$, we know that

$$(a_\alpha, a_\beta)^s = f(\{\alpha,\beta\}) > \bar f(\{\alpha,\beta\}) = (\bar a_\alpha,\bar a_\beta)^s ,$$

and consequently for some $n\in\omega$, $a_\alpha(n)<\bar a_\alpha(n)$.

But by the definition of $\bar a_\alpha$,

$$(\exists A\subseteq\alpha)[f''[A\cup\{\alpha\}]^2\subseteq n\wedge\overline{A} = \bar a_\alpha(n)] ,$$

and it follows that $a_\alpha(n)\geqslant\bar a_\alpha(n)$.

2.6. COROLLARY. *Let f be the dual of a conservative strict scale $\langle a_\alpha|\ \alpha<\gamma\rangle$. Then any scale which is governed by f* (*such as the dual of f*) *majorizes $\{a_\alpha|\ \alpha<\gamma\}$ s.b.e.p.*

Proof. Suppose $\langle b_\alpha|\ \alpha<\gamma\rangle$ is governed by f. By induction on α, if $n\in\omega$ then $a_\alpha(n)\leqslant b_\alpha(n)$.

The following two theorems suggest why the structure of f might cause all of the scales that are governed by f to be unbounded.

2.7. THEOREM. *Suppose $f\colon [\gamma]^2\to\omega$ is transitive and has the scale property. If each scale which is governed by f is unbounded then*

$$(\exists n\in\omega)(\forall m\in\omega)(\exists A\in[\gamma]^m)f''[A]^2\subseteq n .$$

Proof. Otherwise $a(n) = [\mu m(\forall A\in[\gamma]^m)[f''[A]^2\subseteq n]]+2$ defines a bound for the dual of f (s.b.e.p.).

Theorem 2.7 admits a partial converse when γ is regular.

2.8. THEOREM. *Suppose $f\colon [\omega_1]^2\to\omega$ is transitive and has the scale property. If*

$$(\forall U\in[\omega_1]^{\omega_1})(\exists n\in\omega)(\forall m\in\omega)(\exists A\in[U]^m)[f''[A]^2\subseteq n] ,$$

then every scale which is governed by f is unbounded s.b.e.p.

Proof. By Corollary 2.6, it suffices to show that the dual $\langle a_\alpha|\ \alpha<\omega_1\rangle$ of f is unbounded s.b.e.p. But suppose a is a bound for $\langle a_\alpha|\ \alpha<\omega_1\rangle$. Then there is a $U\in[\omega_1]^{\omega_1}$ and an $n\in\omega$ so that $(a_\alpha, a)^s<n$ when $\alpha\in U$. There is an $n'\geqslant n$ so that $(\forall m\in\omega)(\exists A\in[U]^m)[f''[A]^2\subseteq n']$. But this means that if $k>n'$ then

$$(\forall m\in\omega)[a(k)>m] ,$$

which is not possible.

§ 3. Partitions with large duals.

3.1. LEMMA. *If γ is countable, $\langle a_\alpha|\ \alpha<\gamma\rangle$ is a conservative strict scale and $b\in{}^\omega\omega$ then there is a conservative strict scale $\langle a_\alpha|\ \alpha<\gamma+\omega+1\rangle$ such that $a_{\gamma+\omega}\geqslant b$ s.b.e.p.*

Proof. Since γ is countable, we may extend to a conservative strict scale $\langle a_\alpha |\ \alpha<\gamma+1\rangle$. Choose $a_{\gamma+\omega}$ so that $a_{\gamma+\omega}(n)\geqslant b(n)$ for all $n\in\omega$ and so that $c(n) = a_{\gamma+\omega}(n)-a_\gamma(n)$ defines an increasing function $c\in{}^\omega\omega$. Now if we define

$$a_{\gamma+n}(k) = a_\gamma(k)+n \quad \text{for} \quad k, n\in\omega$$

then it is easily verified that $\langle a_\alpha |\ a<\gamma+\omega+1\rangle$ is a conservative strict scale.

3.2. Theorem. *If* $\{b_\alpha |\ \alpha<\omega_1\}\subseteq{}^\omega\omega$ *then there is an* $f\colon [\omega_1]^2\to\omega$ *such that*

1. *There is a scale which is governed by* f.
2. *Every scale which is governed by* f *majorizes* $\{b_\alpha |\ \alpha<\omega_1\}$.

Proof. By Corollary 2.6 it suffices to construct a conservative strict scale $\langle a_\alpha |\ \alpha<\omega_1\rangle$ which majorizes $\{b_\alpha |\ \alpha<\omega_1\}$. This is easily accomplished using Lemma 3.1 to recursively choose the sequence $\langle a_\alpha |\ \alpha<\omega_1\rangle$ so that

$$(\forall\beta<\omega_1)(\exists_\alpha<\omega_1)[b_\beta\leqslant a_\alpha] \text{ s.b.e.p.}$$

Thus if there is an unbounded (major) scale $\langle a_\alpha |\ \alpha<\omega_1\rangle$ then there is an $f\colon [\omega_1]^2\to\omega$ such that

1. There is a scale which is governed by f.
2. Every scale governed by f is unbounded (major).

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Non-finitizability of a weak second-order theory

by

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Abstract. The weak second-order theory R_2, based on the axioms for ordered fields and the continuity scheme, and Tarski's weak second-order geometry $\mathcal{E}_2'$ are shown to be not finitely axiomatizable.

Introduction. Weak second-order theories are understood here in the sense of Tarski [11] (using finite sequences). Mostowski pointed out that the weak second-order theories of familiar mathematical structures are either finitely axiomatizable or not recursively axiomatizable (with respect to the notion of weak second-order consequence), in fact, the theory of real numbers does not even have an analytic axiom system (see [6]) while the theories of natural numbers, integers, rational numbers, and complex numbers turn out to be finitely axiomatizable.

Then, Vaught proved the existence of weak second-order theories which are recursively but not finitely axiomatizable (see appendix of [8]). The axiom systems in his example, however, are just constructed to get this result by a diagonal argument, namely, they are of the form

$$\mathrm{c} = \bar{k}\to\neg\alpha_k \qquad (k\in N)$$

where c is an individual constant, $\bar{k}$ is the numeral for the number k, and α_k is the kth sentence in a recursive enumeration of all sentences or of all first-order sentences.

So, it remained an open problem to find "mathematically motivated" weak second-order theories of the same kind. Already when writing [8], the author had two candidates for such theories — which are recursively axiomatizable by definition — and he discussed them with colleagues.

The aim of this paper is to show that one of these candidates (for the other one see 7.1) is indeed not finitely axiomatizable. It is the theory R_2 based on the axioms for ordered fields and the weak second-order continuity scheme.

With this, one also gets a negative answer to the question-raised by Tarski in [12], p. 25 — if a corresponding weak second-order geometry $\mathcal{E}_2'$ is finitely axiomatizable (6.1).

The proof of our result makes use of the observation that all models of R_2 are Archimedian ordered fields. Then, a translation from R_2 into the system A_ω of