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The K-functional for rearrangement invariant spaces

by

ALBERTO TORCHINSKY* (Urbana, Ill.)

Abstract. This paper deals with the explicit computation of the K-functional for classes of rearrangement invariant spaces. As an application we discuss an extension of the characterization of sublinear operations of weak type in terms of rearrangements due to A. P. Calderón.

Let B_0 and B_1 be real linear spaces continuously embedded in a real topological vector space V; we will call the pair (B_0, B_1) an *interpolation pair*. Suppose that in each of the spaces B_j there is a function $f \to |f|_j$, which we shall call the *quasinorm* of f in B_j, verifying the following properties

(i) $|f|_j \geqslant 0$;

(ii) $|\lambda f|_j = |\lambda| |f|_j$ for all real λ;

(iii) $|f+g|_j \leqslant c(|f|_j + |g|_j)$ where c is a constant independent of f, and g.

The space $B_0 + B_1 = \{f \in V: f = f_0 + f_1, f_i \in B_i\}$ endowed with the quasinorm $|f| = \inf(|f_0|_0 + |f_1|_1)$ is also a real linear space and its embedding in V is continuous. In $B_0 + B_1$ we define a family of equivalent quasinorms by means of the relation, also known as *Peetre's K-functional*,

$$K(t, f, B_0, B_1) = \inf(|f_0|_0 + t|f_1|_1)$$

where $t > 0$ and as above the inf is taken over $f = f_0 + f_1$ with f_j in B_j.

It is our purpose to compute by elementary methods the K-functional corresponding to interpolation pairs where the B_j are r.i. spaces $\Lambda(C, X)$ or $M(X)$ defined in § 1 below. Some particular instances of this result are known [B2], [Ho], [K], [O], [Sh1] and [Sh2]. However, the result is apparently new for arbitrary functions C which are either convex or concave. As an application we discuss an extension of the interesting results of A. P. Calderón concerning the characterization of sublinear operations of weak type in terms of rearrangements [C], Theorem 8,

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to sublinear operations of either strong type or mixed weak and strong type between L^p classes. This will be a particular case of Theorems 6.1 and 7.1. Once this is done the corresponding interpolation results follow readily, see [C], [HV], [Ho] and [T].

The reader may also be interested in applications to some important kinds of operations, such as singular integrals, with L^p replaced by appropriate classes. The first such class replacing L^∞ was introduced by Calderón in [C], Appendix, this class was later called $L^{\infty,1}$ in [B1]. Further results involving H^p and B.M.O. are given in [FRS] [CT], [H] and [R].

The notation we use will be introduced as we go along. We only call attention to the fact that the letter c will denote a constant which need not be the same in different occurrences.

1. The r.i. spaces $\Lambda(X)$, $M(X)$ and $\Lambda(\varphi_X, C)$.

1.1. In what follows the letter C is reserved for a non-decreasing function $C(t)$ defined from $[0, \infty]$ into $[0, \infty]$, $C(0) = 0$, such that

(1) C is non-trivial, i.e. $C \not\equiv 0$ or $C \not\equiv \infty$ for $t \in (0, \infty]$;

(2) $C(t)$ is left-continuous;

(3) $C(t)/t^p$ increases in the wide sense for some $p > 0$.

The inverse of C is defined on $[0, \infty]$ by

$$C^{-1}(t) = \inf\{s\colon C(s) > t\}, \qquad \inf \emptyset = \infty.$$

It is easily seen that C^{-1} is a monotone non-decreasing function from $[0, \infty]$ into $[0, \infty]$ which is right continuous and

$$C\big(C^{-1}(t)\big) \leqslant t \leqslant C^{-1}\big(C(t)\big), \qquad t \geqslant 0.$$

Let (M, μ) be a positive measure space. For a μ-measurable function f defined on M we set

$$|f|_C = \inf\Big\{\varepsilon > 0\colon \int_M C(|f|/\varepsilon)\,d\mu \leqslant C(1)\Big\}.$$

Then $B = \{f\colon |\lambda f|_C < \infty$ for some real $\lambda\}$ is a quasinormed real linear space. Indeed, (i) and (ii) above are clear. As for (iii) when $p > 1$ in (3) this is done in [Z], pp. 173–175. If not, let $|f|_C = \varepsilon$, $|g|_C = \eta$ and $\delta > 0$ be given.

Then

$$\int C(|f+g|/\varepsilon+\eta+\delta)\,d\mu$$
$$\leqslant \int C\big([(\varepsilon+\delta)|f|/(\varepsilon+\delta)(\varepsilon+\eta+\delta)]+[(\eta+\delta)|g|/(\eta+\delta)(\varepsilon+\eta+\delta)]\big)\,d\mu$$
$$\leqslant \int C(|f|/\varepsilon+\delta) \vee C(|g|/\eta+\delta)\,d\mu \leqslant 2C(1).$$

But since $C(at) \leqslant a^p C(t)$, $0 < a \leqslant 1$, we have that

$$C(t/2^{1/p}) \leqslant C(t)/2$$

and consequently

$$\int C(|f+g|/2^{1/p}(\varepsilon+\eta+\delta))\,d\mu \leqslant C(1).$$

Whence

$$|f+g|_C \leqslant 2^{1/p}(|f|_C+|g|_C)$$

since $\delta > 0$ is arbitrary.

LEMMA 1.1. *If f and g are disjointly supported functions, then*

$$|f|_C+|g|_C \leqslant 2|f+g|_C.$$

Proof. Since on account of our assumption $|f+g| = |f|+|g|$, it follows that $C(|f|/\varepsilon) \leqslant C(|f+g|/\varepsilon)$ for any $\varepsilon > 0$ and consequently

$$\int C(|f|/\varepsilon)\,d\mu \leqslant \int C(|f+g|/\varepsilon)\,d\mu.$$

Similarly for g. Thus $|f|_C \leqslant |f+g|_C$, $|g|_C < |f+g|_C$ and our conclusion follows.

1.2. In the sequel we shall only consider totally σ-finite measure spaces (M, μ) and the spaces V of equivalence classes of real valued measurable functions on M, the equivalence being that of coincedence μ-almost everywhere. To define the r.i. spaces that interest us we recall the notation $m(f, \lambda)$ for the *distribution function* of a measurable function f on (M, μ), to wit

$$m(f, \lambda) = \mu(\{x \in M\colon |f(x)| > \lambda\}), \qquad \lambda > 0.$$

If $m(f, \lambda)$ is finite for λ sufficiently large we denote by $f^*(t)$ the unique non-negative non-increasing left-continuous function on $[0, \infty]$ such that the sets $\{|f| > \lambda\}$ and $\{f^* > \lambda\}$ are equimeasurable. f^* is called the *non-increasing rearrangement* of the function f. The following properties of f^* will be used latter on

(i) If $|f| \leqslant |g|$, then $f^* \leqslant g^*$;

(ii) If $t = t_1+t_2$, then $(f+g)^*_{\frac{1}{2}}(t) \leqslant f^*(t_1)+g^*_{\frac{1}{2}}(t_2)$.

These and further properties are discussed in [C].

1.3. A Banach space X of real valued, measurable functions on a possibly infinite interval $I = [0, l)$ is said to be a *function space* if the following conditions are satisfied,

(iii) $|f| \leqslant |g|$ a.e. and $g \in X$, then $f \in X$ and $\|f\|_X \leqslant \|g\|_X$;

(iv) If $\{f_m\}_{m=1}^{\infty} \subset X$, $\|f_m\|_X \leqslant c$ and $0 < f_m \nearrow f$, then $f \in X$ and $\|f\| \leqslant c$.

A function space X is said to be a *rearrangement invariant space* (r.i. space) if whenever $f \in X$ and f' is any function on I equiveasurable with f, then $f' \in X$ and $\|f\|_X = \|f'\|_X$. Examples of r.i. spaces include the classes B discussed above and the Lorentz spaces Λ, M and $L(p, q)$. Also if X, Y are r.i. spaces, so is $X+Y$.

The fundamental function $\varphi_X(t) = \varphi(t)$ of a r.i. space is defined as $\varphi(t) = \|\chi_{[0,t]}\|_X$ where $\chi_{[0,t]}$ is characteristic function of the interval $[0, t]$.

The following properties will be used in what follows.

(v) $\varphi(t)$ is a continuous increasing function, which is absolutely continuous away from the origin and $\frac{d\varphi}{dt}(t) \leqslant \varphi(t)/t$ a.e.;

(vi) X has an equivalent r.i. norm such that the fundamental function $\varphi_{X_0}(t) = \varphi_0(t)$ is concave, and, moreover,

$$\varphi(t) \leqslant \varphi_0(t) \leqslant \varphi(2t), \qquad 0 < t < \infty.$$

Let $\Lambda(x) = \left\{f\colon f^* \text{ exists and } \|f\|_{\Lambda(x)} = \int\limits_0^\infty f^*(t)\varphi(t)\,\frac{dt}{t} < \infty\right\}$;

$$M(X) = \left\{f\colon \|f\|_{M(X)} = \sup_{t>0}\big(f^*(t)\varphi(t)\big) < \infty\right\};$$

and

$$\Lambda(\varphi, C) = \{f\colon |f|_{\Lambda(\varphi,C)} = |f^*\varphi|_C < \infty\}$$

where the norm is taken over $([0, \infty), dt/t)$.

These properties are discussed in [Lu], [Sh1] and the classes $\Lambda(\varphi, C)$ were introduced in [T].

2. The K-functional for the pair $(M(X_0), M(X_1))$.

2.1. The following decomposition, introduced by Calderón in the proof of Theorem 8 of [C], will be useful in what follows. Let $h(t)$ be a non-negative function defined for $t > 0$, and let f be a μ-measurable function. Then for a fixed value t set $f = f_0+f_1$ where

$$\text{(i)}\qquad \begin{aligned} f_0 &= f - f^*(h(t)) &&\text{if}\quad f > f^*(h(t)),\\ f_0 &= f + f^*(h(t)) &&\text{if}\quad f < -f^*(h(t)),\\ f_0 &= 0 &&\text{otherwise.}\end{aligned}$$

We then have

$$\text{(ii)}\qquad f^* = f_0^* + f_1^*$$

and

$$\text{(iii)}\qquad \begin{aligned} f_0^*(s) &= 0 &&\text{for}\quad s > h(t),\\ f_1^*(s) &= f^*(h(t)) &&\text{for}\quad s < h(t).\end{aligned}$$

We will assume $I = [0, \infty)$, the reader will have no difficulty in extending the results to the case $l < \infty$.

THEOREM 2.2. *Let X_j be r.i. spaces with fundamental functions $\varphi_{X_j}(t) = \varphi_j(t)$ which satisfy 1.3(v), (vi). Let $\eta(t) = \varphi_0(t)/\varphi_1(t)$ be a monotone function taking positive values. We then have*

$$\sup_{s>0}\{f^*(2s)[\varphi_0(s) \wedge t\varphi_1(s)]\} \leqslant K\big(t, f, M(X_0), M(X_1)\big) \leqslant \sup_{s>0}\{f^*(s)[\varphi_0(s)\wedge t\varphi_1(s)]\}.$$

Proof. Let $A = \{s \in [0, \infty)\colon \eta(s) < t\}$ and put $B = [0, \infty) - A$, and denote by $\chi_A(s)$ and $\chi_B(s)$ the characteristic functions of A and B, respectively.

Let $f = f_0+f_1$ be a decomposition of f with f_j in $M(X_j)$, $j = 0, 1$. We then have for $s > 0$

$$\begin{aligned} f_0^*(s)\varphi_0(s) &= \chi_A(s)f_0^*(s)\varphi_0(s) + \chi_B(s)f_0^*(s)\varphi_0(s)\\ &\geqslant \chi_A(s)f_0^*(s)\varphi_0(s) + t\chi_B(s)f_0^*(s)\varphi_1(s).\end{aligned}$$

Similarly

$$tf_1^*(s)\varphi_1(s) \geqslant \chi_A(s)f_1^*(s)\varphi_0(s) + t\chi_B(s)f_1^*(s)\varphi_1(s).$$

Consequently

$$\begin{aligned} &\|f_0\|_{M(X_0)} + t\|f_1\|_{M(X_1)}\\ &\qquad \geqslant \sup_{s>0}\{\chi_A(s)\big(f_0^*(s)+f_1^*(s)\big)\varphi_0(s) + t\chi_B(s)\big(f_0^*(s)+f_1^*(s)\big)\varphi_1(s)\}\\ &\qquad \geqslant \sup_{s>0}\{f^*(2s)[\varphi_0(s)\wedge t\varphi_1(s)]\}\end{aligned}$$

where we have used 1.2(ii) to obtain the last inequality. Taking inf over all possible decompositions of f we obtain the first half of our theorem.

Moreover, for the decomposition in 2.1 with $h(t) = \eta^{-1}(t)$, we have

$$\begin{aligned} t\|f_1\|_{M(X_1)} &\leqslant tf^*\big(\eta^{-1}(t)\big)\varphi_1\big(\eta^{-1}(t)\big) \vee t\sup_{s>0}\{\chi_B(s)f^*(s)\varphi_1(s)\}\\ &= f^*\big(\eta^{-1}(t)\big)\varphi_0\big(\eta^{-1}(t)\big) \vee t\sup_{s>0}\{\chi_B(s)f^*(s)\varphi_1(s)\} = I_0(t) + I_1(t)\end{aligned}$$

since $t\varphi_1\big(\eta^{-1}(t)\big) = \varphi_0\big(\eta^{-1}(t)\big)$.

Further notice that

$$I_0(t),\ \|f_0\|_{M(X_0)} \leqslant \sup_{s>0}\{\chi_A(s)f^*(s)\varphi_0(s)\}.$$

Therefore

$$\begin{aligned} K\big(t, f, M(X_0), M(X_1)\big) &\leqslant \|f_0\|_{M(X_0)} + t\|f_1\|_{M(X_1)}\\ &\leqslant \sup_{s>0}\{\chi_A(s)f^*(s)\varphi_0(s)\} \vee t\sup_{s>0}\{\chi_B(s)f^*(s)\varphi_1(s)\}\\ &= \sup_{s>0}\{f^*(s)[\varphi_0(s)\wedge t\varphi_1(s)]\}.\end{aligned}$$

This completes the proof of our theorem.

3. The K-functional for the pair $(M(X_0), \Lambda(\varphi_{X_1} C))$. We begin by proving the following lemma.

LEMMA 3.1. *Let χ_A denote the characteristic function of the interval $A = [0, a)$. Further suppose that f is a non-increasing function defined on A and φ is non-decreasing and concave. Then if there is a $q > p$ such that $C(t)/t^q$ decreases we have*

$$\sup_{s>0}\{\chi_A(s)f(s)\varphi(s)\} \leqslant c\,|\chi_A f\varphi|_C$$

where $c = C^{-1}(2^q/\ln 2C(1))$.

Proof. Let $s \in A$. Then for any $\varepsilon > 0$ we have

$$\begin{aligned} f(s)\varphi(s) &\leqslant \varepsilon C^{-1}\big(C(f(s)\varphi(s)/\varepsilon)\big) \\ &\leqslant \varepsilon C^{-1}\left(\frac{1}{\ln 2}\int_{s/2}^{s} C(f(u)\varphi(2u)/\varepsilon)\,\frac{du}{u}\right) \\ &\leqslant \varepsilon C^{-1}\left(\frac{1}{\ln 2}\int C(2\chi_A(u)f(u)\varphi(u)/\varepsilon)\,\frac{du}{u}\right) \\ &\leqslant \varepsilon C^{-1}\left(\frac{2^q}{\ln 2}\int C(\chi_A(u)f(u)\varphi(u)/\varepsilon)\,\frac{du}{u}\right). \end{aligned}$$

The desired conclusion follows by letting $\varepsilon = |\chi_A f\varphi|_C$.

R. Sharpley has computed the K-functional for the pair $(M(X), \Lambda(X))$, [Sh2], we will now compute it for the pair $(M(X_0), \Lambda(\varphi_1, C))$.

THEOREM 3.2. *Let X_j, φ_j and η be as in Theorem 2.2. Further assume that $\varphi_1(s)/s^\beta$ increases for some $\beta > 0$ and that η is a monotone function increasing from 0 to ∞ and $\eta(s)/s^r$, increases for some $r > 0$.*

Given $t > 0$ let $A = \{s\colon \eta(s) < t\}$, $B = [0, \infty) - A$ and $\tilde{B} = \{s\colon \eta(s/2) > t\}$ and let χ_A, χ_B and $\chi_{\tilde{B}}$ denote the respective characteristic functions of those sets. If $C(t)/t^q$ decreases for some $q > 0$, we have

$$\begin{aligned} c_1\{\sup_{s>0}\ &(\chi_A(s)f^*(2s)\varphi_0(s)) + t\,|\chi_{\tilde{B}}f^*\varphi_1|_C\} \\ &\leqslant K(t, f, M(X_0), \Lambda(\varphi_1, C)) \\ &\leqslant c_2\{\sup_{s>0}\ (\chi_A(s)f^*(s)\varphi_0(s)) + t\,|\chi_B f^*\varphi_1|_C\}. \end{aligned}$$

Proof. Let f be in $M(X_0) + \Lambda(\varphi_1, C)$ and let $f_0 + f_1$ be a decomposition of f in that space. Then

$$\begin{aligned} &\|f_0\|_{M(X_0)} + t\,|f_1|_{\Lambda(\varphi_1, C)} \\ &\geqslant \sup_{s>0}\ \{f_0^*(s)\varphi_0(s)\}/2 + \sup_{s>0}\{f_0^*(s)\varphi_0(s)\}/2 + t\,|\chi_A f_1^*\varphi_1|_C/2 + t\,|\chi_B f_1^*\varphi_1|_C/2 \\ &= (I_1 + I_2 + I_3 + I_4)/2. \end{aligned}$$

Let us consider $I_1 + I_3$ first. Since $t\varphi_1(s) > \varphi_0(s)$ whenever $\chi_A(s) \neq 0$, by virtue of Lemma 3.1 with $f = f_1^*$, $\varphi = \varphi_1$, $a = \eta^{-1}(t)$ we have, with c the constant in that lemma,

$$\frac{1}{c}\sup_{s>0}\{\chi_A(s)f_1^*(s)\varphi_0(s)\} \leqslant \frac{t}{c}\sup_{s>0}\{\chi_A(s)f_1^*(s)\varphi_1(s)\} \leqslant t\,|\chi_A f_1^*\varphi_1|_C.$$

Whence it follows that

$$\begin{aligned} I_1 + I_3 &\geqslant \left(1 \wedge \frac{1}{c}\right)\sup_{s>0}\ \{\chi_A(s)[f_0^*(s) + f_1^*(s)]\varphi_0(s)\} \\ &\geqslant \left(1 \wedge \frac{1}{c}\right)\sup_{s>0}\{\chi_A(s)f^*(2s)\varphi_0(s)\}. \end{aligned}$$

Let us now consider $I_2 + I_4$. On account of our assumptions we have $s\eta'(s) \geqslant r\eta(s)$ and since $\varphi_1(\eta^{-1}(s)) = \varphi_0(\eta^{-1}(s))/s$, we obtain

$$\begin{aligned} I_5 &= \int C(\chi_B(s)f_0^*(s)\varphi_1(s)/\varepsilon)\,\frac{ds}{s} \\ &\leqslant \frac{1}{r}\int_{\eta^{-1}(t)}^{\infty} C(f_0^*(s)\varphi_1(s)/\varepsilon)\,\frac{s\eta'(s)}{\eta(s)}\,\frac{ds}{s} \leqslant \frac{1}{r}\int_t^\infty C\big(f_0^*(\eta^{-1}(s))\,\varphi_1(\eta^{-1}(s))/\varepsilon\big)\,\frac{ds}{s} \\ &= \frac{1}{r}\int_t^\infty C\big(f_0^*(\eta^{-1}(s)\,\varphi_0(\eta^{-1}(s))/\varepsilon s\big)\,\frac{ds}{s} \leqslant \frac{1}{r}\int_t^\infty C(\|f_0\|_{M(X_0)}/\varepsilon s)\,\frac{ds}{s} \\ &\leqslant \frac{1}{r}\,t^p C(\|f_0\|_{M(X_0)}/\varepsilon t)\int_t^\infty s^{-p}\,\frac{ds}{s} = C(\|f_0\|_{M(X_0)}/\varepsilon t)/rp, \end{aligned}$$

the last inequality being a consequence of the fact that $C(s)/s^p$ increases.

Let $\alpha = 1 \vee (1/rp)^{1/p}$. We than have

$$I_5 \leqslant C(\alpha\|f_0\|_{M(X_0)}/\varepsilon t)$$

and then it follows that

$$|\chi_B f_0^*\varphi_1|_C \leqslant \alpha\|f_0\|_{M(X_0)}/t.$$

Therefore it is readily seen that

$$\begin{aligned} I_5 + I_4 &\geqslant \left(1 \wedge \frac{1}{\alpha}\right)t\{|\chi_B f_0^*\varphi_1|_C + |\chi_B f_1^*\varphi_1|_C\} \\ &\geqslant \left(1 \wedge \frac{1}{\alpha}\right)2^{-1/p}t\,|\chi_B(f_0^* + f_1^*)\varphi_1|_C \geqslant \left(1 \wedge \frac{1}{\alpha}\right)2^{-1/p-1}t\,|\chi_{\tilde{B}}f^*\varphi_1|_C. \end{aligned}$$

Combining these estimates we get

$$\|f_0\|_{M(X_0)}+t|f_1|_{A(\varphi_1,C)} \geqslant c_1\{\sup_{s>0}(\chi_A(s)f^*(2s)\varphi_0(s))+t|\chi_{\tilde B}f^*\varphi_1|_C\}$$

where $c_1 = \frac{1}{2}\Big\{\Big[\Big(1\wedge\frac{1}{a}\Big)2^{-1/p-1}\Big]\wedge\frac{1}{c}\Big\}$.

Taking inf over all such decompositions of f we obtain the first half of our theorem.

As for the second half we use again the decomposition of f given in 2.1 with $h(t) = \eta^{-1}(t)$. From the definition of f_1 it follows that

$$\begin{aligned}|f_1^*\varphi_1|_C &= |\chi_A f^*(\eta^{-1}(t))\varphi_1+\chi_B f^*\varphi_1|_C\\ &\leqslant 2^{1/p}\{|\chi_A f^*(\eta^{-1}(t))\varphi_1|_C+|\chi_B f^*\varphi_1|_C\} = 2^{1/p}\{I_1+I_2\}.\end{aligned}$$

We compute now $|\chi_A\varphi_1|_C$. Let $\varepsilon>0$, then

$$\begin{aligned}&\int_0^\infty C(\chi_A(s)\varphi_1(s)/\varepsilon)\frac{ds}{s}\\ &= \int_0^{\eta^{-1}(t)} C(\varphi_1(s)/\varepsilon)\frac{ds}{s}\\ &\leqslant \{C(\varphi_1(\eta^{-1}(t))/\varepsilon)/\varphi_1(\eta^{-1}(t))^p\}\int_0^{\eta^{-1}(t)}\varphi_1(s)^p\frac{ds}{s}\\ &\leqslant \{C(\varphi_1(\eta^{-1}(t))/\varepsilon)/\varphi_1(\eta^{-1}(t))^p\}\{\varphi_1(\eta^{-1}(t))^p/\eta^{-1}(t)^{\beta p}\}\int_0^{\eta^{-1}(t)} s^{\beta p}\frac{ds}{s}\\ &\leqslant \frac{1}{\beta p}C(\varphi_1(\eta^{-1}(t))/\varepsilon)\leqslant C(a\varphi_1(\eta^{-1}(t))/\varepsilon)\end{aligned}$$

where $a = 1\vee\Big(\frac{1}{\beta p}\Big)^{1/p}$. Thus

$$|\chi_A\varphi_1|_C\leqslant a\varphi_1(\eta^{-1}(t))$$

and

$$I_1\leqslant af^*(\eta^{-1}(t))\varphi_1(\eta^{-1}(t)).$$

Consequently

(i) $$t|f_1|_{A(\varphi_1,C)}\leqslant 2^{1/p}a\{\sup_{s>0}[\chi_A(s)f^*(s)\varphi_0(s)]+t|\chi_B f^*\varphi_1|_C\}.$$

Also from the definition of f_0 it follows that

$$\|f_0\|_{M(X_0)}\leqslant\sup_{s>0}[\chi_A(s)f^*(s)\varphi_0(s)]$$

and consequently we obtain the second half of our theorem with $c_2 = 1+2^{1/p}[1\vee(1/\beta p)^{1/p}]$.

THEOREM 3.3. *Let X_j, φ_j, η be defined as in Theorem 3.2 except that now we assume that η decreases from ∞ to 0 and that there is a number $r>0$ such that $s^r\eta(s)$ decreases. Then the same conclusion as in Theorem 3.2 holds.*

The proof of this theorem being analogous to that of Theorem 3.2 is omitted. We only point out that in the proof of the second inequality we reverse the roles of f_0 and f_1.

Remark 3.4. When C, φ_0 and φ_1 are powers the above proofs simplify considerably. In fact, most of the assumptions on η are then automatically satisfied. In this case the result is known, see [HO].

4. The K-functional for the spaces $(\Lambda(\varphi_0, C_0, \Lambda(\varphi_1, C_1))$. In this section we will use the following lemma

LEMMA 4.1. *Let D be a positive increasing function which is concave. Further let C be as usual and let f be non-increasing and φ non-decreasing and concave. We then have*

(i) $$D\Big[\int_0^t C(f(s)\varphi(s))\frac{ds}{s}\Big]\leqslant c\int_0^t D[C(f(s)\varphi(s))]\frac{ds}{s}$$

and

(ii) $$D\Big[\int_t^\infty C(f(s)\varphi(s))\frac{ds}{s}\Big]\leqslant c\int_{t/2}^\infty D[C(f(s)\varphi(s))]\frac{ds}{s}.$$

The proof being analogous to that of [T], Lemma 3.11 is omitted here.

THEOREM 4.2. *Let X_j, φ_j and C_j be as we have considered above with C_j continuous and $\varphi_1(s)/s^\beta$ increases for some $\beta>0$ and suppose that η is increasing from 0 to ∞ and $\eta(s)/s^r$ increases for some $r>0$. Further, for $t>0$ let $A = \{s\colon \eta(s)<t\}$, $B = [0,\infty)-A$, $\tilde A = \{s\colon \eta(s/2)<t\}$ and $\tilde B = [0,\infty)-\tilde A$ and let χ_A, χ_B, $\chi_{\tilde A}$ and $\chi_{\tilde B}$ denote the corresponding characteristic functions.*

Then there are constants c_1, c_2 so that for each f in $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1)$ we have

$$\begin{aligned}c_1\{|\chi_{\tilde A}f^*\varphi_0|_{C_0}+t|\chi_{\tilde B}f^*\varphi_1|_{C_1}\}&\leqslant K(t,f,\Lambda(\varphi_0,C_0),\Lambda(\varphi_1,C_1))\\ &\leqslant c_2\{|\chi_A f^*\varphi_0|_{C_0}+t|\chi_B f^*\varphi_1|_{C_1}\}.\end{aligned}$$

Proof. We consider first the case when $C_1C_0^{-1}$ is concave. Let $f = f_0+f_1$ be a decomposition of f in $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1)$. We will first show that

(i) $$|\chi_A f_1^*\varphi_0|_{C_0}\leqslant ct|\chi_A f_1^*\varphi_1|_{C_1}$$

and

(ii) $$t|\chi_B f_0^*\varphi_1|_{C_1}\leqslant c|\chi_B f_0^*\varphi_0|_{C_0}.$$

Since $C_1C_0^{-1}$ is concave and $\varphi_0(s) < t\varphi_1(s)$ whenever $\chi_A(s) \neq 0$ from Lemma 4.1, it follows that for any $\varepsilon > 0$

$$C_1C_0^{-1}\Big[\int_0^\infty C_0\big(\chi_A(s)f_1^*(s)\varphi_0(s)/\varepsilon\big)\frac{ds}{s}\Big] \leqslant C_1C_0^{-1}\Big[\int_0^{\eta^{-1}(t)} C_0\big(tf_1^*(s)\varphi_1(s)/\varepsilon\big)\frac{ds}{s}\Big]$$

$$\leqslant c\int_0^{\eta^{-1}(t)} C_1\big(tf_1^*(s)\varphi_1(s)/\varepsilon\big)\frac{ds}{s}.$$

Whence we readily see that

$$|\chi_A f_1^*\varphi_0|_{C_0} \leqslant ct|\chi_A f_1^*\varphi_1|_{C_1}$$

and (i) holds.

Moreover, since $C_1(t)/t^p$ increase and $t\varphi_1(s) \leqslant \varphi_0(s)$ whenever $\chi_B(s) \neq 0$, we have

(iii) $$C_1\big(\chi_B(s)tf_0^*(s)\varphi_1(s)/\varepsilon\big) \leqslant \big(t\varphi_1(s)/\varphi_0(s)\big)^p C_1\big(\chi_B(s)f_0^*(s)\varphi_0(s)/\varepsilon\big).$$

Let A be the Young's conjugate of the convex function $C_0C_1^{-1}$, i.e. $A(t) = \sup_{s>0}\big(ts - C_0C_1^{-1}(s)\big)$, see [Z], Chapter 1 § 9. Then from (iii) and the definition of A we obtain

$$\int_0^\infty C_1\big(t\chi_B(s)f_0^*(s)\varphi_1(s)/\varepsilon\big)\frac{ds}{s}$$

$$\leqslant \int_{\eta^{-1}(t)}^\infty \big(t\varphi_1(s)/\varphi_0(s)\big)^p C_1\big(f_0^*(s)\varphi_0(s)/\varepsilon\big)\frac{ds}{s}$$

$$\leqslant \int_{\eta^{-1}(t)}^\infty A\Big(\big(t\varphi_1(s)/\varphi_0(s)\big)^p\Big)\frac{ds}{s} + \int_{\eta^{-1}(t)}^\infty C_0C_1^{-1}C_1\big(f_0^*(s)\varphi_0(s)/\varepsilon\big)\frac{ds}{s}$$

$$\leqslant \int_{\eta^{-1}(t)}^\infty A\big(t^p/\eta(s)^p\big)\frac{ds}{s} + \int_{\eta^{-1}(s)}^\infty C_0\big(f_0^*(s)\varphi_0(s)/\varepsilon\big)\frac{ds}{s}$$

$$\leqslant \frac{1}{r}\int_0^1 A(s^p)\frac{ds}{s} + \int_{\eta^{-1}(t)}^\infty C_0\big(f_0^*(s)\varphi_0(s)/\varepsilon\big)\frac{ds}{s}.$$

From this we readily see that

$$t|\chi_B f_0^*\varphi_1|_{C_1} \leqslant c|\chi_B f_0^*\varphi_0|_{C_0}$$

and (ii) holds as well.

Now for our decomposition $f = f_0 + f_1$ we have

$$\begin{aligned}|f_0|_{A(\varphi_0,C_0)} + t|f_1|_{A(\varphi_1,C_1)} &= |(\chi_A+\chi_B)f_0^*\varphi_0|_{C_0} + t|(\chi_A+\chi_B)f_1^*\varphi_1|_{C_1}\\ &\geqslant \tfrac{1}{2}\{|\chi_A f_0^*\varphi_0|_{C_0} + |\chi_B f_0^*\varphi_0|_{C_0} + t|\chi_A f_1^*\varphi_1|_{C_1} + t|\chi_B f_1^*\varphi_1|_{C_1}\}\\ &\geqslant c\{|\chi_A f_0^*\varphi_0|_{C_0} + t|\chi_B f_0^*\varphi_1|_{C_1} + |\chi_A f_1^*\varphi_1|_{C_0} + t|\chi_B f_1^*\varphi_1|_{C_1}\}\\ &\geqslant c\{|\chi_A(f_0^*+f_1^*)\varphi_0|_{C_0} + t|\chi_B(f_0^*+f_1^*)\varphi_1|_{C_1}\}\\ &\geqslant c\{|\chi_{\tilde A} f^*\varphi_0|_{C_0} + t|\chi_{\tilde B} f^*\varphi_1|_{C_1}\}.\end{aligned}$$

By taking the inf over all possible decompositions of f in $A(\varphi_0, C_0) + A(\varphi_1, C_1)$ we obtain the first half of our theorem in this case. The case when $C_1C_0^{-1}$ is convex is treated in an entirely analogous fashion and the proof is omitted here. To complete the proof of our theorem we consider the decomposition of f given in 2.1 with $h(t) = \eta^{-1}(t)$.

As in Theorem 3.2(i) we see that

(i) $$t|f_1|_{A(\varphi_1,C_1)} \leqslant c\{f^*\big(\eta^{-1}(t)\big)t\varphi_1\big(\eta^{-1}(t)\big) + t|\chi_B f^*\varphi_1|_{C_1}\}.$$

Now from Lemma 3.1 it follows that

(ii) $$f^*\big(\eta^{-1}(t)\big)t\varphi_1\big(\eta^{-1}(t)\big) = f^*\big(\eta^{-1}(t)\big)\varphi_0\big(\eta^{-1}(t)\big) \leqslant c|\chi_A f^*\varphi_0|_{C_0}.$$

Also from the definition of f_0 we have

(iii) $$|f_0|_{A(\varphi_0,C_0)} \leqslant |\chi_A f^*\varphi_0|_{C_0},$$

where our second inequality follows combining (i), (ii) and (iii).

THEOREM 4.3. *Let X_j, φ_j, C_j be as in Theorem 4.2 except that we now assume that η is decreasing from ∞ to 0 and $s^r\eta(s)$ decreases for some $r > 0$. Then the some conclusion as in Theorem 4.2 holds.*

Proof. The proof is immediate once we observe that

$$K\big(t, f, A(\varphi_0, C_0), A)\varphi_1, C_1\big) = tK\big(1/t, f, A(\varphi_1, C_1), A(\varphi_0, C_0)\big)$$

and we apply Theorem 4.2.

5. Admissible maps from $A(\varphi_0, C_0) + A(\varphi_1, C_1)$ into $M(Y_0) + M(Y_1)$.

5.1. An operation $g = Tf$ of a class of functions f in a measure space (M, μ) into a class of functions g on (N, ν) is called a *sublinear operation* if it satisfies the following properties:

(i) $f = f_0 + f_1$ and Tf_i $(i = 0, 1)$ are defined, then Tf is defined;

(ii) $|T(f_0+f_1)| \leqslant |Tf_0| + |Tf_1|$ ν-almost everywhere;

(iii) For any scalar λ we have $|T(\lambda f)| = |\lambda||Tf|$ ν-almost everywhere.

5.2. Given two interpolation pairs (B_0, B_1) and $(\tilde B_0, \tilde B_1)$ and a sublinear mapping $T\colon B_0 + B_1 \to \tilde B_0 + \tilde B_1$ we say that T is *admissible* if T maps B_i

nto $\tilde{B}_i$ continuously, $i = 0, 1$, i.e. there exists constants c_i such that

$$\|Tf\|_{\tilde{B}_i} \leqslant c_i \|f\|_{B_i}, \quad f \in B_i,\ i = 0, 1.$$

LEMMA 5.3. *Let T be a sublinear admissible mapping for the interpolation pairs (B_0, B_1) and $(\tilde{B}_0, \tilde{B}_1)$. Then for f in B_0+B_1 and $t > 0$ we have*

$$K(t, Tf, \tilde{B}_0, \tilde{B}_1) \leqslant cK(t, f, B_0, B_1), \quad c = c_0 \vee c_1.$$

Proof. It is an immediate consequence of the definition of the K-functional, see for instance [Ho], Theorem 1.1.

We will assume now that we have an interpolation pair $(M(Y_0), M(Y_1))$ where the fundamental functions $\psi_i(t) = \|\chi_{[0,t]}\|_{Y_i}$, $i = 0, 1$, satisfy the hypothesis of Theorem 2.2, and an interpolation pair $(\Lambda(\varphi_0, C_0), \Lambda(\varphi_1, C_1))$ where the fundamental functions φ_i satisfy the conditions of either Theorem 4.2 or Theorem 4.3. As usual $\eta(t) = \varphi_0(t)/\varphi_1(t)$ and we shall denote $\psi_0(t)/\psi_1(t) = \xi(t)$.

THEOREM 5.4. *Let the interpolation pairs $(\Lambda(\varphi_0, C_0), \Lambda(\varphi_1, C_1))$ and $(M(Y_0), M(Y_1))$ be as above. Then a sublinear operator T: $\Lambda(\varphi_0, C_0) + \Lambda(\varphi_1, C_1) \to M(Y_0)+M(Y_1)$ is admissible if and only if for each f in $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1)$ and $t > 0$,*

(i) $$(Tf)^*(2\xi^{-1}(t))\varphi_0(2\xi^{-1}(t)) \leqslant c\{|\chi_A f^* \varphi_0|_{C_0} + t|\chi_B f^* \varphi_1|_{C_1}\},$$

where $A = \{s\colon \eta(s) < t\}$ and $B = [0, \infty) - A$.

Proof. Since $\psi_0(s)/s$ decreases we have $\psi_0(2s) \leqslant 2\psi_0(s)$. Now if T is admissible then from Lemma 5.3 it follows that

$$K(t, Tf, M(Y_0), M(Y_1) \leqslant cK(t, f, \Lambda(\varphi_0, C_0), \Lambda(\varphi_1, C_1))$$

and (i) follows from the remark above and Theorems 2.2 and 4.2 or 4.3 as the case may be.

Conversely, assume that (i) holds and T is sublinear. In 4.2(ii) we have seen that

$$t|\chi_B f^* \varphi_1|_{C_1} \leqslant c|\chi_B f^* \varphi_0|_{C_0}.$$

Consequently from (i) we have that for any $t > 0$,

$$(Tf)^*(2\xi^{-1}(t))\psi_0(2\xi^{-1}(t)) \leqslant c\{|\chi_A f^* \varphi_0|_{C_0} + |\chi_B f^* \varphi_0|_{C_0}\} \leqslant 2c|f|_{\Lambda(\varphi_0, C_0}.$$

where the last inequality follows from Lemma 1.1.

Therefore we obtain that

$$|Tf|_{M(Y_0)} \leqslant 2c|f|_{\Lambda(\varphi_0, C_0)}.$$

Also in 4.2(i) we have observed that

$$|\chi_A f^* \varphi_0|_{C_0} \leqslant ct|\chi_A f^* \varphi_1|_{C_1}$$

and from (i) we get

$$(Tf)^*(2\xi^{-1}(t))\psi_0(\xi^{-1}(t)) \leqslant ct\{|\chi_A f^* \varphi_1|_{C_1} + |\chi_B f^* \varphi_1|_{C_1}\} \leqslant |f|_{\Lambda(\varphi_1, C_1)}.$$

Since $\psi_0(\xi^{-1}(t))/t = \psi_1(\xi^{-1}(t))$ and $\psi_1(s)/s$ decreases, we then have

$$(Tf)^*(2\xi^{-1}(t))\psi_1(2\xi^{-1}(t)) \leqslant 4c|f|_{\Lambda(\varphi_1, C_1)}$$

and consequently

$$|Tf|_{M(Y_1)} \leqslant 4c|f|_{\Lambda(\varphi_1, C_1)}.$$

This completes the proof that T is admissible.

Remark 5.5. The case $\varphi_i(s) = s^{1/p_i}$ and $\psi_i(s) = s^{1/q_i}$, $i = 0, 1$, is Calderón's theorem alluded to above. A particularly interesting example is that of the operators discussed in [T], Theorem 4.16.

6. Admissible maps from $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1)$ into $M(Y_0)+\Lambda(\psi_1, \tilde{C}_1)$.

THEOREM 6.1. *Let φ_j, C_j, η be as in Theorems 4.2 or 4.3 and let ψ_j, $\tilde{C}_j$, ξ be as in Theorems 3.2 or 3.3. Then a sublinear mapping T: $\Lambda(\varphi_0, C_0) + \Lambda(\varphi_1, C_1) \to M(Y_0)+\Lambda(\psi_1, \tilde{C}_1)$ is admissible if and only if for each f in $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1)$ and $t > 0$ we have*

(i) $$\sup_{s>0}\{\chi_A(s)(Tf)^*(2s)\psi_0(s)\} + t|\chi_{\tilde{B}}(Tf)^*\psi_1|_{\tilde{C}_1} \leqslant c\{|\chi_A f^*\varphi_0|_{C_0} + t|\chi_B f^* \varphi_1|_{C_1}$$

where $A = \{s\colon \xi(s) < t\}$, $\tilde{B} = \{s\colon \xi(s/2) > t\}$, $A' = \{s\colon \eta(s) < t\}$ and $B' = [0, \infty) - A'$.

Proof. That (i) holds when T is admissible follows from Lemma 5.3 and Theorems 3.2 or 3.3 and 4.2 or 4.3 as the case may be. Conversely, if (i) holds then as in Theorem 5.4 we see that for $t > 0$

$$(Tf)^*(2\xi^{-1}(t))\psi_0(\xi^{-1}(t)) \leqslant c|f|_{\Lambda(\varphi_0, C_0)}$$

and consequently

$$|Tf|_{M(Y_0)} \leqslant c|f|_{\Lambda(\varphi_0, C_0)}.$$

On the other hand, from (i) and 4.2(i) it also readily follows that

$$t|\chi_{\tilde{B}}(Tf)^*\psi_1|_{\tilde{C}_1} \leqslant ct\{|\chi_{A'} f^*\varphi_1|_{C_1} + |\chi_{B'} f^*\varphi_1|_{C_1}\} \leqslant 2ct|f|_{\Lambda(\varphi_1, C_1)}$$

and so

(ii) $$|\chi_{\tilde{B}}(Tf)^*\psi_1|_{\tilde{C}_1} \leqslant 2c|f|_{\Lambda(\varphi_1, C_1)}.$$

Now if we let t approach 0 in (ii) we obtain

$$|Tf|_{\Lambda(\psi_1, \tilde{C}_1)} \leqslant c|f|_{\Lambda(\varphi_1, C_1)}.$$

This shows that T is admissible.

Remark 6.2. For $\varphi_i(s) = \psi_i(s) = s^{1/p_i} = C_i^{-1}(s)$, $i = 0, 1$, $\tilde{C}_1 = C_1$, Theorem 6.1 says that a sublinear mapping T from $L^{p_0}+L^{p_1}$ into $L^{p_0}+L^{p_1}$ is admissible if and only if for each f in $L^{p_0}+L^{p_1}$, $t > 0$ and $1/p_0 - 1/p_1 = 1/\sigma > 0$ we have

$$\sup_{0<s<t^\sigma}\{(Tf)^*(2s)s^{1/p_0}\} + t\Big\{\int_{t^\sigma}^{\infty}(Tf)^*(s)^{p_1}ds\Big\}^{1/p_1} \leqslant c\Big\{\Big(\int_0^{t^\sigma} f^*(s)^{p_0}ds\Big)^{1/p_0} + t\Big(\int_{t^\sigma}^{\infty} f^*(s)^{p_1}ds\Big)^{1/p_1}.$$

This bound applies to the Hilbert transform with $p_0 = 1$, $p_1 > 1$, $\sigma = p_1' = p_1/(p_1-1)$ for instance, although in that case a better result is known, see [C], Appendix.

A result of the general charater of Theorem 6.1 was shown to hold in a particular case in [JT], Theorem 4.6.

7. Admissible maps from $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1)$ into $\Lambda(\psi_0, \tilde{C}_0)+\Lambda(\psi_1, \tilde{C}_1)$.

THEOREM 7.1. *Let φ_j, C_j, η, ψ_j, $\tilde{C}_j$, ξ be in Theorems 4.2 or 4.3. Then a sublinear map T: $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1, C_1) \to \Lambda(\psi_0, \tilde{C}_0)+\Lambda(\psi_1, \tilde{C}_1)$ is admissible if and only if for each f in $\Lambda(\varphi_0, C_0)+\Lambda(\varphi_1 C_1)$ and $t > 0$*

$$|\chi_{\tilde{A}}(Tf)^* \psi_0|_{\tilde{C}_0} + t|\chi_{\tilde{B}}(Tf)^* \psi_1|_{\tilde{C}_1} \leqslant c\{|\chi_A f^* \varphi_0|_{\tilde{C}_0} + t|\chi_B f^* \varphi_1|_{C_1}\},$$

where $\tilde{A} = \{s: \xi(s/2) > t\}$, $\tilde{B} = [0, \infty) - \tilde{A}$, $A\{s: \eta(s) < t\}$ and $B = [0, \infty) - A$.

The proof of this theorem requires no new ideas and is therefore omitted.

8. Additional comments. The main ideas in the proof of § 1,2 can be used to obtain a considerable extension of the results discussed here. Indeed, suppose that we have a mapping $f \to l(f)$ with the following properties.

(i) If f is (M, μ) measurable, then $l(f)$ is (Lebesgue) measurable in $[0, \infty)$;

(ii) $l(f)$ is positive and non-increasing;

(iii) $l(\lambda f) = |\lambda|^s l(f)$ for some $0 < s \leqslant 1$ and all real λ;

(iv) $l(f+g, 2t) \leqslant c[l(f)(t) + l(g)(t)]$.

Examples of $l(f)$ are $f^*(t)$, $K(t, f, B_0, B_1)/t$ and $f^{**}(t, r)^r$ where if $|f|^r$ is locally summable we set

$$f^{**}(t, r) = \left[\frac{1}{t}\int_0^t (f^*(u))^r\, du\right]^{1/r}.$$

It is then possible to obtain some of the results of [Ho] and [S], including reiteration. Also similar ideas can be used to obtain interpolation of weighted L^p classes, these results were first arrived at in [G], see also [P2]. All these considerations would take us far from our original discussion and are therefore left for another occasion.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

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