STUDIA MATHEMATICA, T. LXIV. (1979)

On joint spectra

by

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Abstract. The commutation properties of some N-tuples of multiplication operators are used to show that for $N \geqslant 2$ there exist N-tuples $T = (T_1, \ldots, T_N)$ of commuting linear operators on a Banach space X with the property that there are two maximal commutative subalgebras $\mathscr{A}_1$ and $\mathscr{A}_2$ of $L(X)$ containing $T_1, \ldots, T_N$ such that the joint spectrum of T in $\mathscr{A}_1$ is different from the joint spectrum of T in $\mathscr{A}_2$. The example is closely related to an example of J. L. Taylor in [8], [9]. It is shown that, in general, Taylor's functional calculus ([9]) is richer than the analytic functional calculus (in the sense of [4]) in any closed commutative subalgebra of $L(X)$ containing $T_1, \ldots, T_N$.

1. Introduction. Let X be a complex Banach space and denote by $L(X)$ the Banach algebra of all continuous linear operators on X. For an N-tuple $T = (T_1, \ldots, T_N)$ of commuting operators in $L(X)$ we may consider the following joint spectra:

(a) $\sigma(T, X)$, the *joint spectrum of T with respect to X* in the sense of J. L. Taylor ([8]).

(b) If $\mathscr{A}$ is a closed subalgebra of $L(X)$ containing $I, T_1, \ldots, T_N$ in its center, then we denote by $\sigma_{\mathscr{A}}(T)$ the *joint spectrum of T in $\mathscr{A}$*:

$$\sigma_{\mathscr{A}}(T) := \{z \in \mathbf{C}^N\colon \sum_{j=1}^{N} (z_j I - T_j)\mathscr{A} \neq \mathscr{A}\}.$$

By [8], Lemma 1.1, the Taylor spectrum of T is always contained in $\sigma_{(T)'}(T)$, where $(T)'$ is the algebra of all continuous linear operators commuting with $T_1, \ldots, T_N$. Moreover, J. L. Taylor showed in [8] by an example of five operators $T_1, \ldots, T_5$ that the inclusion can be strict. We modify this example in order to show that the inclusion can be proper for every $N \geqslant 2$. For this example the algebra $H(\sigma(T, X))$ of germs of locally analytic functions on $\sigma(T, X)$ is strictly larger than $H(\sigma_{(T)'}(T))$. Hence, Taylor's analytic functional calculus ([9]) is, in general, richer than the analytic functional calculus (in the sense of [4]) in any commutative closed subalgebra of $L(X)$ containing $I, T_1, \ldots, T_N$. For a second (closely related example) we show that there exist maximal commutative subalgebras $\mathscr{A}_1$ and $\mathscr{A}_2$ of $L(X)$ containing $T_1, \ldots, T_N$ such that $\sigma_{\mathscr{A}_1}(T) \neq \sigma_{\mathscr{A}_2}(T)$. This gives an answer to a question raised by W. Żelazko at the Oberwolfach meeting on functional analysis in 1976.

2. The examples. We put $G := G_1 \cup G_2$ where

$$G_1 := \{z \in C^N:\ |z_j| < 1/4 \text{ for } j = 1, \ldots, N\},$$
$$G_2 := \left\{z \in C^N:\ 1/2 < \max\{|z_j|:\ j = 1, \ldots, N\} < 1\right\}.$$

Let us denote by $\mathscr{B}^0(G)$ the space of all continuous functions on G and by $\mathscr{B}^1(G)$ the space of all functions $f \in \mathscr{B}^0(G)$ such that $\frac{\partial f}{\partial \bar{z}_j}$ (in the sense of distributions) is continuous on G for $j = 1, \ldots, N$ (see [9], [10]). Let X_0 be the Banach algebra of all continuous functions on $\bar{G}$, endowed with the supremum norm $\|\cdot\|_0$, and let X_1 be the space of all functions $f \in X_0$ such that the restriction of f to G belongs to $\mathscr{B}^1(G)$ and such that for $j = 1, \ldots, N$ the functions $\frac{\partial f}{\partial \bar{z}_j}$ have continuous extensions to $\bar{G}$ (again denoted by $\frac{\partial f}{\partial \bar{z}_j}$). Endowed with the norm $\|\cdot\|_1$,

$$\|f\|_1 := \|f\|_0 + \sum_{j=1}^{N} \left\|\frac{\partial f}{\partial \bar{z}_j}\right\|_0 \qquad (f \in X_1),$$

X_1 is a Banach algebra. Spaces of this type have also been used in [1] and [2]. We consider the following multiplication operators on X_k $(k = 0, 1)$:

$$(T_j^{(k)} f)(z) := z_j f(z) \quad \text{for} \quad f \in X_k \quad \text{and} \quad z = (z_1, \ldots, z_N) \in \bar{G} \ (j = 1, \ldots, N).$$

1. Lemma. *For $k = 0, 1$ the N-tuples $T^{(k)} := (T_1^{(k)}, \ldots, T_N^{(k)})$ are X_k-scalar (in the sense of [3]) and hence decomposable in the sense of [6]. The (unique) spectral capacity for $T^{(k)}$ is given by*

$$(1) \qquad \mathscr{E}^{(k)}(F) := \{f \in X_k:\ \operatorname{supp}(f) \subset F\} \quad \textit{for} \quad F = \bar{F} \in C^N.$$

Moreover,

$$(2) \qquad \sigma(T^{(k)}, X_k) = \bar{G} = \sigma_{(T^{(k)})'}(T^{(k)}).$$

Proof. $\Phi^{(k)}: X_k \to L(X_k)$ with $\Phi^{(k)}(g)f := gf$ $(g, f \in X_k)$ is obviously an algebraic homomorphism with $\Phi^{(k)}(1) = I$ and $\Phi^{(k)}(\pi_j) = T_j^{(k)}$, where $\pi_j: G \to C$ are the coordinate functions with $\pi_j(z) = z_j$ for $z = (z_1, \ldots, z_N) \in G$. Hence $T^{(k)}$ is decomposable by [3], Theorem 4. Obviously, $\mathscr{E}^{(k)}$, as defined by (1) is a spectral capacity for $T^{(k)}$(which is unique by [6]). (2) follows by Theorem 6 in [3].

The proof of the following proposition is similar to the proof of Theorem 4.4.6 in [5] and of Proposition 2.4 in [1].

2. Proposition. *For $A \in L(X_n, X_m)$ $(n, m = 0, 1)$ the following statements are equivalent:*

(a) $A\mathscr{E}^{(n)}(F) \subset \mathscr{E}^{(m)}(F)$ for every closed $F \subset C^N$.

(b) $A = 0$ in the case $n = 0$, $m = 1$. In the case $n = m$, A is a multiplication operator

$$(3) \qquad Af = af \quad (f \in X_n)$$

with $a \in X_n$, and in the case $n = 1$, $m = 0$, A is a differential operator of the type

$$(4) \qquad Af = af + \sum_{j=1}^{N} b_j \frac{\partial f}{\partial \bar{z}_j} \quad (f \in X_1)$$

with $a, b_1, \ldots, b_N \in X_0$.

Proof. As obviously every operator of the type (b) fulfills condition (a), we have only to show that (a) implies (b). Condition (a) can also be written in the form

$$(5) \qquad \operatorname{supp}(Af) \subset \operatorname{supp}(f) \quad \text{for all } f \in X_n.$$

Therefore, for every $w \in \bar{G}$ the map $u_W: \mathscr{C}^1(C^N) \to C$ defined by $u_W(f) := \left(A(f|_{\bar{G}})\right)(w)$ is a continuous linear functional with support contained in $\{w\}$. Hence, we have in the case $n = 0$

$$(6) \qquad \left(A(f|_G)\right)(w) = a(w)f(w) \quad \text{for every } f \in \mathscr{C}^1(C^N)$$

with $a(w) \in C$, and in the case $n = 1$,

$$(7) \qquad \left(A(f|_G)\right)(w) = a(w)f(w) + \sum_{j=1}^{N} \left(b_j(w)\frac{\partial f}{\partial \bar{z}_j}(w) + c_j(w)\frac{\partial f}{\partial z_j}(w)\right)$$

for every $f \in \mathscr{C}^1(C^N)$ with $a(w), b_j(w), c_j(w) \in C$ $(j = 1, \ldots, N)$. Applying A successively to the polynomials $1, \pi_1, \ldots, \pi_N, \bar{\pi}_1, \ldots, \bar{\pi}_N$, we obtain that the functions $w \to a(w)$, $w \to b_j(w)$, $w \to c_j(w)$ $(j = 1, \ldots, N)$ are elements of X_m.

Let now w be an arbitrary point in G. There exists a function $h \in \mathscr{C}^1(C^N)$ with compact support contained in G, such that $0 \leqslant h \leqslant 1$ and such that $h \equiv 1$ in a neighbourhood U of w. Then we have (by (5)) for every $f \in X_n$: $(Af)(w) = \left(A(fh)\right)(w) + \left(A\left((1-h)f\right)\right)(w) = \left(A(hf)\right)(w)$. Now, hf can be approximated in the norm of X_n by functions in $\mathscr{C}^1(C^N)$ (by the proof of Lemma 2.6 in [9]). As A is continuous and $h \equiv 1$ in U, we obtain that (6), resp. (7), are valid for all $f \in X_n$ and $w \in G$.

In the case $n = 0$, (6) holds (by continuity of a and f) for all $w \in \bar{G}$. This proves (3) in the case $n = m = 0$.

Let us now consider the case $n = 1$. For an arbitrary $w \in G$ we can find $h \in \mathscr{C}^1(C^N)$ as above, with the additional property that the diameter of $\operatorname{supp}(h)$ is smaller than 1. For $j = 1, \ldots, N$ the functions g_j, with

$$g_j(z) := \begin{cases} h(z)(z_j - w_j)\ln\left|\ln|z_j - w_j|\right| & \text{for } z \in \operatorname{supp}(h), \\ 0 & \text{for } z \notin \operatorname{supp}(h), \end{cases}$$

belong to X_1. Then we have for $z \in U$

$$(Ag_k)(z) = a(z)g_k(z) + \sum_{j=1}^{N} \left(b_j(z) \frac{\partial g_k}{\partial \bar{z}_j}(z) + c_j(z) \frac{\partial g_k}{\partial z_j}(z) \right)$$

$$= a(z)g_k(z) + b_k(z) \frac{\partial g_k}{\partial \bar{z}_k}(z) + c_k(z) \frac{\partial g_k}{\partial z_k}(z).$$

By the continuity of the functions Ag_k, ag_k, $b_k \dfrac{\partial g_k}{\partial \bar{z}_k}$, we obtain the continuity of $c_k \dfrac{\partial g_k}{\partial z_k}$ at the point w. As for $z \in U$, $z_k \neq w_k$

$$\frac{\partial g_k}{\partial z_k}(z) = \ln\left|\ln|z_k - w_k|\right| + (2\ln|z_k - w_k|)^{-1},$$

this is only possible if $c_k(w) = 0$. As w was an arbitrary point in G and as c_k is continuous on $\bar{G}$, we have $c_k \equiv 0$ on $\bar{G}$ and (4) is proved.

In the case $n = m = 1$, we obtain for $z \in U$

$$(Ag_k)(z) = a(z)g_k(z) + b_k(z) \frac{\partial g_k}{\partial \bar{z}_k}(z)$$

with Ag_k, $ag_k \in X_1$. Hence,

$$\frac{\partial}{\partial \bar{z}_k}\left(b_k \frac{\partial g_k}{\partial \bar{z}_k} \right) = \frac{\partial b_k}{\partial \bar{z}_k} \cdot \frac{\partial g_k}{\partial \bar{z}_k} + b_k \frac{\partial^2 g_k}{\partial \bar{z}_k^2}$$

has to be continuous at w. As $\dfrac{\partial^2 g_k}{\partial \bar{z}_k^2}$ is not continuous at w, this is only possible if $b_k(w) = 0$ $(k = 1, \ldots, N)$. Consequently, $b_k \equiv 0$ $(k = 1, \ldots, N)$ and (3) holds for $n = m = 1$.

For $n = 0$, $m = 1$, $w \in G$, consider $\dfrac{\partial g_k}{\partial \bar{z}_k}$ (which belongs to X_0 because of $g_k \in X_1$). As $A\left(\dfrac{\partial g_k}{\partial \bar{z}_k}\right) = a \dfrac{\partial g_k}{\partial \bar{z}_k} \in X_1$, the function $\dfrac{\partial}{\partial \bar{z}_k}\left(A\left(\dfrac{\partial g_k}{\partial \bar{z}_k}\right)\right)$ has to be continuous at w. As above, we obtain that this is only possible if $a(w) = 0$. Hence, $a \equiv 0$ on $\bar{G}$, i.e. $A = 0$, and the proof is complete.

Let now X be the Banach space $X = X_0 \oplus X_1$. We consider the following operators in $L(X)$:

$$T_j := T_j^{(0)} \oplus T_j^{(1)} \quad (j = 1, \ldots, N),$$

$$\Phi(h) := \Phi^{(0)}(h) \oplus \Phi^{(1)}(h) \quad \text{for} \quad h \in X_1,$$

$$D_j \text{ with } D_j(f, g) := \left(\frac{\partial g}{\partial \bar{z}_j}, 0 \right) \quad (j = 1, \ldots, N) \text{ for } (f, g) \in X,$$

$$D_0 \text{ with } D_0(f, g) := (g, 0) \quad \text{for} \quad (f, g) \in X,$$

$$S_j := T_j + D_j \quad (j = 1, \ldots, N),$$

$$\Psi(k) := \Phi^{(0)}(k) \oplus 0 \quad \text{for} \quad k \in X_0.$$

3. Lemma. (a) *$T = (T_1, \ldots, T_N)$ is a X_1-scalar N-tuple and hence decomposable. The spectral capacity for T is given by*

$$\mathscr{E}(F) = \mathscr{E}^{(0)}(F) \oplus \mathscr{E}^{(1)}(F)$$
$$= \{(f, g) \in X\colon \operatorname{supp}(f) \cup \operatorname{supp}(g) \subset F\}$$

for $F = \bar{F} \subset C^N$. Moreover,

$$\sigma(T, X) = \bar{G}.$$

(b) *$S = (S_1, \ldots, S_N)$ is decomposable, the spectral capacity for S coincides with that of T, and*

$$\sigma(S, X) = \bar{G}.$$

Proof. (a) is proved in the same way as Lemma 1.

(b) The operators $S_1, \ldots, S_N, T_1, \ldots, T_N$ commute and $S_j - T_j = D_j$ is nilpotent for $j = 1, \ldots, N$. Therefore, T and S are quasinilpotent equivalent in the sense of [6], Definition 4.1, by Remark 4.3 in [6]. Theorem 4.1 in [6] implies that $\sigma(T, X) = \sigma(S, X)$. By Proposition 4.1 in [6], the N-tuple S is decomposable and the spectral capacities for T and S coincide.

4. Proposition. *For $A \in L(X)$ the following two conditions are equivalent:*

(a) *$A\mathscr{E}(F) \subset \mathscr{E}(F)$ for every closed $F \subset C^N$.*

(b) *There are functions $h \in X_1$ and $k_0, k_1, \ldots, k_N$, $k \in X_0$ such that*

$$A = \Phi(h) + \Psi(k) + \sum_{j=0}^{N} \Psi(k_j) D_j. \tag{8}$$

Proof. Obviously, (b) implies (a), so that we have to prove only the converse implication. Denote for $n = 0, 1$ by $J_n\colon X_n \to X$ the canonical injection and by $P_n\colon X \to X_n$ the canonical projection. Then, A can be written in the form

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \quad \text{where} \quad A_{ji} = P_j A J_i \text{ for } i, j = 0, 1.$$

As $J_j\mathscr{E}^{(j)}(F) \subset \mathscr{E}(F)$ and $P_j\mathscr{E}(F) \subset \mathscr{E}^{(j)}(F)$ for $j = 0, 1$ and every closed $F \subset C^N$, condition (a) implies $A_{ji}\mathscr{E}^{(i)}(F) \subset \mathscr{E}^{(j)}(F)$ for every closed $F \subset C^N$ and $i, j = 0, 1$. By Proposition 2 we obtain

$$A_{ii} = \Phi^{(i)}(h_i) \quad \text{with} \quad h_i \in X_i \ (i = 0, 1),$$
$$A_{10} = 0,$$
$$A_{01} = \Phi^{(0)}(k_0) + \sum_{j=1}^{N} \Phi^{(0)}(k_j)\frac{\partial}{\partial \bar{z}_j} \quad \text{with} \quad k_0, k_1, \ldots, k_N \in X_0.$$

Hence, A is of the type (8) (with $h := h_1$ and $k := h_0 - h_1$).

We are now able to compute the commutant algebra for T and S:

5. Proposition.

(a) $(T)' = \{\Phi(h) + \Psi(k) + \sum_{j=0}^{N} \Psi(k_j)D_j\colon\ k, k_0, \ldots, k_N \in X_0,\ h \in X_1\}$.

(b) $(S)' = \{\Phi(h) + \sum_{j=0}^{N} \Psi(k_j)D_j\colon\ k_0, \ldots, k_N \in X_0,\ h \in X_1 \cap H(G)\}$,

where $H(G)$ is the algebra of locally analytic functions on G.

Proof. (a) Obviously, every operator of the type (8) commutes with $T_1, \ldots, T_N$. On the other hand, every operator commuting with $T_1, \ldots, T_N$ fulfills condition (a) in Proposition 4 by Corollary 4.5 in [6] and is therefore of the type (8) by Proposition 4.

(b) Let A be an operator in $L(X)$ commuting with $S_1, \ldots, S_n$. As in the proof of (a), A must be of the type (8), i.e. there are functions $k, k_0, k_1, \ldots, k_N \in X_0$ and $h \in X_1$ such that

$$A = \Phi(h) + \Psi(k) + \sum_{j=0}^{N} \Psi(k_j)D_j.$$

Consequently, for $j = 1, \ldots, N$,

$$0 = S_jA - AS_j = (T_j + D_j)A - A(T_j + D_j) = D_jA - AD_j =: C_j$$

because of (a). Now, $D_jD_i = D_j\Psi(k) = 0$ for $i, j = 0, 1, \ldots, N$. Hence

$$0 = C_j = D_j\Phi(h) - \Phi(h)D_j + \Psi(k)D_j \quad \text{for} \quad j = 1, \ldots, N.$$

Therefore,

$$0 = C_j(0, 1) = \left(\frac{\partial h}{\partial \bar{z}_j}, 0\right) \quad \text{for} \quad j = 1, \ldots, N$$

and so $h \in H(G)$. This implies $\Phi(h)D_j = D_j\Phi(h)$ and therefore

$$0 = C_j(0, \bar{\pi}_j) = (k, 0), \quad \text{i.e. } k = 0,$$

and we have shown that A is of the desired type. On the other hand, if $A = \Phi(h) + \sum_{j=0}^{N} \Psi(k_j)D_j$ with $k_0, k_1, \ldots, k_N \in X_0$ and $h \in X_1 \cap H(G)$, then clearly $AS_j = S_jA$ for $j = 1, \ldots, N$.

6. Theorem. (a) *$\mathscr{A}_1 := \{\Phi(h) + \Psi(k_0)D_0\colon\ h \in X_1,\ k_0 \in X_0\}$ and $\mathscr{A}_2 := (S)'$ are maximal commutative subalgebras of $L(X)$ containing $T_1, \ldots, T_N$. If $N \geq 2$, then*

$$\sigma_{\mathscr{A}_1}(T) = \bar{G} \neq K := \{z \in C^N\colon\ |z_j| \leq 1 \text{ for } j = 1, \ldots, N\} = \sigma_{\mathscr{A}_2}(T).$$

(b) $\sigma_{(S)'}(S) = K \neq \bar{G} = \sigma(S, X)$.

Proof. (a) Obviously $\mathscr{A}_1$ and $\mathscr{A}_2$ are commutative subalgebras of $L(X)$ with $T_1, \ldots, T_N \in \mathscr{A}_j$ $(j = 1, 2)$. As $\mathscr{A}_2 = (S)'$ is the maximal subalgebra of $L(X)$ containing $S_1, \ldots, S_N$ in its center, it is a maximal commutative subalgebra of $L(X)$.

Let now $A \in L(X)$ be an operator commuting with all $B \in \mathscr{A}_1$. By Proposition 5(a) there are functions $k, k_0, k_1, \ldots, k_N \in X_0$ and $h \in X_1$ such that

$$A = \Phi(h) + \Psi(k) + \sum_{j=0}^{N} \Psi(k_j)D_j.$$

As $D_0 = \Psi(1)D_0 \in \mathscr{A}_1$, we obtain

$$0 = (AD_0 - D_0A)(0, 1) = (k, 0)$$

and therefore $k = 0$. $\Phi(\bar{\pi}_j) \in \mathscr{A}_1$ implies

$$0 = \big(A\Phi(\bar{\pi}_j) - \Phi(\bar{\pi}_j)A\big)(0, 1) = (k_j, 0),$$

i.e. $k_j = 0$ $(j = 1, \ldots, N)$. Therefore, $A \in \mathscr{A}_1$ and we have shown that $\mathscr{A}_1$ is a maximal commutative subalgebra of $L(X)$.

As $\{\Phi(h)\colon\ h \in X_1\} \subset \mathscr{A}_1$ we have $\sigma_{\mathscr{A}_1}(T) = \operatorname{supp}(\Phi) = \bar{G}$ by Theorem 6 in [3].

Let us now prove that $\sigma_{\mathscr{A}_2}(T) = K$. If $w \notin K$, i.e. $|w_p| > 1$ for some $p \in \{1, \ldots, N\}$, then we obtain with $u_p(z) := (w_p - z_p)^{-1}$ and $u_j = 0$ for $j \neq p$, $j = 1, \ldots, N$,

$$\sum_{j=1}^{N} (w_jI - T_j)\Phi(u_j) = \Phi(1) = I,$$

hence $z \notin \sigma_{\mathscr{A}_2}(T)$. If $w \notin \sigma_{\mathscr{A}_2}(T)$, then there are $U_1, \ldots, U_N \in \mathscr{A}_2$,

$$U_j = \Phi(u_j) + \sum_{p=0}^{N} \Psi(k_{p,j})D_j$$

with $k_{p,j} \in X_0$, $u_j \in H(G) \cap X_1$ $(j = 1, \ldots, N;\ p = 0, 1, \ldots, N)$, such that

$$\sum_{j=1}^{N} (w_jI - T_j)U_j = I.$$

If we apply this equation to $(1, 0) \in X$, we obtain

$$\Big(\sum_{j=1}^{N}(w_j-\pi_j)u_j\,,\,0\Big) = (1,0),$$

hence

$$\sum_{j=1}^{N}(w_j-z_j)u_j(z) = 1 \quad \text{for all } z \in \overline{G}_2. \tag{9}$$

As $N \geqslant 2$, there are unique continuous functions $v_j\colon K \to C$ which are analytic in $\mathring{K}$ and coincide with u_j on $\overline{G}_2$ $(j = 1, \ldots, N)$ (cf. [7], Theorem I.C.5). By (9) we obtain

$$\sum_{j=1}^{N}(w_j-z_j)v_j(z) = 1 \quad \text{for all } z \in K.$$

This is only possible if $w \notin K$. Thus $\sigma_{\mathscr{A}_2}(T) = K$.

(b) Let $\Delta(\mathscr{A}_2)$ be the space of all non trivial multiplicative linear functionals on $\mathscr{A}_2 = (S)'$. Then

$$\begin{aligned}\sigma_{(S)'}(S) &= \{(\varphi(S_1), \ldots, \varphi(S_N))\colon \varphi \in \Delta(\mathscr{A}_2)\}\\ &= \{(\varphi(T_1), \ldots, \varphi(T_N))\colon \varphi \in \Delta(\mathscr{A}_2)\}\\ &= \sigma_{\mathscr{A}_2}(T) = K\end{aligned}$$

by (a) and because of $\varphi(D_j) = 0$ for all $\varphi \in \Delta(\mathscr{A}_2)$ (as $D_j^2 = 0$) for $j = 1, \ldots, N$. Together with Lemma 3 (b) this proves (b).

7. Remarks. (a) Part (b) in the preceeding theorem shows that for $N \geqslant 2$ the Taylor spectrum may be strictly smaller than the commutant spectrum (cf. Theorem 4.1 in [8] for $N \geqslant 5$). Moreover, in our example $H(\sigma_{(S)'}(S)) = H(K) \subsetneqq H(\overline{G}) = H(\sigma(S, X))$. For example, the germ of the function h, which vanishes in a neighbourhood of $\overline{G}_1$ and is identical to 1 in a neighbourhood of $\overline{G}_2$, is in $H(\overline{G})$ but not in $H(K)$. The operator $\Phi(h)$ is in the algebra generated by Taylor's analytic functional calculus ([9]) but not in the algebra generated by the analytic functional calculus (in the sense of [4]) in any closed commutative subalgebra of $L(X)$ containing $I, S_1, \ldots, S_N$. This shows that, in general, Taylor's analytic functional calculus is richer than the analytic functional calculus in closed commutative subalgebras.

(b) There is no admissible algebra $\mathfrak{A}$ of functions (in the sense of [3]) such that there exists a homomorphism $\Psi_0\colon \mathfrak{A} \to L(X)$ with $\Psi_0(1) = I$ and $\Psi_0(\pi_j) = S_j$ for $j = 1, \ldots, N$. Otherwise, by Theorem 6 in [3] we would have $\sigma(S, X) = \sigma_{(S)'}(S)$ in contradiction to part (b) in the preceeding theorem. In the case $N = 1$, a corresponding example has been given in [1].

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Received January 10, 1977
in revised version April 5, 1977 (1247)